CHAPTER II

THEORY OF COULOMB EXCITATION

The nuclear states excited in the Coulomb excitation process are the low lying collective excitations which are induced by the electric quadrupole field of the impinging particles. It is assumed that the initial kinetic energy of the impinging particle is sufficiently low, so that mutual electrostatic repulsion between projectile and target ensures that nuclear forces play no role. This condition on the bombarding energy is given by the usual expression for the Coulomb barrier, \( E_B \)

\[
E < E_B = \frac{Z_1 Z_2 (1 + A_1/A_2)}{(A_1^{1/3} + A_2^{1/3})} \quad \ldots (1)
\]

where the bombarding energy, \( E \), is given in (Lab. system) MeV, \( Z_1 \) and \( Z_2 \) are the charge numbers of the projectile and the target, \( A_1, A_2 \) are the respective mass numbers of the projectile and target. For bombarding energy, \( E < E_B \) the orbit of the projectile can be described by a hyperbola and the differential cross-section for elastic scattering is given by the Rutherford law, which in the centre of mass (CM) system is
\[
\left( \frac{d\sigma}{d\Omega} \right)_R = \frac{1}{4} a^2 \sin^{-4}(\theta/2)
\]

\ ...(2)

where \( \theta \) is the scattering angle and \( a \) is half the distance of closest approach for \( 180^\circ \) scattering which is given by

\[
a = z_1 z_2 e^2 / m_0 v^2
\]

\ ...(3)

with \( m_0 \), the reduced mass of the projectile and the target nucleus.

The dynamics of the Coulomb excitation process is viewed as a classical Rutherford's scattering; i.e. the semiclassical approximation. In this approximation the cross section for Coulomb excitation for a given level, \( f \), from the ground state can be written in terms of the probability for excitation along the orbit, times the differential elastic scattering cross section:

\[
\left( \frac{d\sigma}{d\Omega} \right)_f = P_{i \rightarrow f} \left( \frac{d\sigma}{d\Omega} \right)_R
\]

\ ...(4)

If one is interested in the nuclear probability for excitation, \( P \) regardless of the orientation of the initial and final nuclear states, then one can express \( P \) in terms of the amplitudes \( b_{if} \) for a transition from the initial nuclear state \( i \) to various final steps \( f \) as:
\[
\mathcal{P}_i \rightarrow f = (2I_i + 1)^{-1} \sum_{M_i,M_f} b_{if}^2 \quad \cdots(5)
\]

where \( I_i \) is the spin of the initial nuclear state, \( M_i \) and \( M_f \) are the magnetic quantum numbers of the initial and final states.

As the projectile travels along its orbit, it exerts a time varying electric field at the target nucleus. Using first order time dependent perturbation theory, one finds that the transition amplitude \( b_{if} \) is given by

\[
b_{if} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \langle f | H(t) | i \rangle e^{i\omega t} dt \quad \cdots(6)
\]

where \( H(t) \) is the interaction energy of the target nucleus and incident particle and

\[
\omega = \frac{\Delta E}{\hbar} = E_f - E_i / \hbar \quad \cdots(7)
\]

is the nuclear frequency associated with the excitation energy \( \Delta E \). \( E_i \) and \( E_f \) refer to the initial and final energies of the projectile.

The time dependent Hamiltonian is

\[
H(t) = \int \rho_n (\vec{r}) \phi (\vec{r},t) d\tau \quad \cdots(8)
\]
where \( f_n(\vec{r}) \) is a nuclear charge density operator given by

\[
f_n(\vec{r}) = \sum_k e_k \delta(\vec{r} - \vec{r}_k)
\]

...(9)

(the sum extending over the protons) and \( \phi(\vec{r}, t) \) is the Coulomb potential of the projectile:

\[
\phi(\vec{r}, t) = \frac{Z_1 e}{|\vec{r} - \vec{R}(t)|} - \frac{Z_1 e}{\vec{R}(t)}
\]

...(10)

The vectors are referred to an origin at the center of the nucleus, the position of the projectile is denoted by \( \vec{R}(\theta_1, \phi_1) \), and a volume element of the target nucleus \( d\vec{r} \) by \( \vec{r}(\theta_2, \phi_2) \). The second term in equation (10) arises mainly due to scattering and does not contribute to the excitation, is substracted.

In order to evaluate the matrix element in (8) one expands the potential (10) in multipole moments. This is done by noting that \( \frac{1}{|\vec{r} - \vec{R}|} \) can be expanded in multipole moments:

\[
\frac{1}{|\vec{r} - \vec{R}|} = \frac{1}{R(t)} \left[ 1 + \sum_{\lambda} \left( \frac{\vec{r}}{R} \right)^{\lambda} P^{\lambda}_l(\cos \alpha) \right]
\]

...(11)
where $\alpha$ is the angle between the vectors $\vec{F}$ and $\vec{R}$.

Multiplying equation (11) by $Z_1 e$ throughout and with some adjustments it can be shown that

$$\phi(\vec{F}, t) = \frac{Z_1 e}{|\vec{F} - \vec{R}|} - \frac{Z_1 e}{\vec{R}} = \frac{Z_1 e}{\vec{R}} \sum_{\lambda} \frac{1}{\lambda} p_{\lambda}(\cos \alpha) \quad \ldots(12)$$

Expressing $p_{\lambda}(\cos \alpha)$ in terms of spherical harmonics addition theorem:

$$p_{\lambda}(\cos \alpha) = \frac{4\pi}{2\lambda + 1} \sum_{\mu=-\lambda}^{\lambda} y_{\lambda\mu}^*(\theta_2, \phi_2) y_{\mu}(\theta_1, \phi_1) \quad \ldots(13)$$

using equation (13) in equation (12)

$$\phi(\vec{F}, t) = 4\pi Z_1 e \sum_{\lambda=1}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \frac{(2\lambda + 1)^{-1}}{(2\lambda + 1)^{-1}} R \frac{1}{\lambda} \frac{1}{\lambda} y_{\lambda\mu}^*(\theta_2, \phi_2)$$

$$\times y_{\lambda\mu}(\theta_1, \phi_1) \quad \ldots(14)$$

with the aid of equation (14) the Hamiltonian equation (9) can be written as

$$H(t) = 4\pi Z_1 e \sum_{\lambda=1}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \frac{(2\lambda + 1)^{-1}}{(2\lambda + 1)^{-1}} R \frac{1}{\lambda} \frac{1}{\lambda} y_{\lambda\mu}(\theta_1, \phi_1) m^*(E_\lambda, m) \quad \ldots(15)$$

where $m(E_\lambda, m) = \int \frac{1}{\lambda} y_{\lambda\mu}(\theta_2, \phi_2) \int m(\vec{r}) d\vec{r} \quad \ldots(16)$

The latter quantity is known as nuclear electric multipole.
operator of order $\lambda$. The transition amplitude $b_{1f}$ equation (6) can be written with the help of equation (15) as

$$b_{1f} = \frac{\hbar^2 Z_1 e}{\hbar} \sum_{\lambda, \mu} (2\lambda + 1) -1 \langle I_1 M_1 | M(E_\lambda, \mu) | I_f M_f \rangle S_{E_{\lambda, \mu}} \ldots (17)$$

$$S_{E_{\lambda, \mu}} = \int_{-\infty}^{\infty} e^{i\omega t} y_{\lambda, \mu}(\theta(t), \phi(t)) \left[ r_p(t) \right]^{-\lambda-1} dt \ldots (18)$$

where (18) is orbital integral and the nuclear states are defined by the total angular momentum $I$ and magnetic quantum number $M$.

The multipole moments or the nuclear matrix elements are tensor operators, which are evaluated in terms of the reduced matrix element, $\langle I_1 | M(E_\lambda) | I_f \rangle$, and a vector addition or Wigner coefficient using Wigner-Eckart theorem:

$$\langle I_1 M_1 | M(E_\lambda, \mu) | I_f M_f \rangle = (-1)^{I_1 - M_1} \left( \begin{array}{cc} I_1 & I_f \\ \lambda & I_f \end{array} \right) \langle I_1 | M(E_\lambda) | I_f \rangle$$

$$-M_1 \quad \mu M_f \ldots (19)$$

The orbital integrals $S_{E_{\lambda, \mu}}$ that enter the transition amplitude $b_{1f}$ have been evaluated by Alder et al. (10) which are quoted below:
\[ S_{E, \lambda, \mu} = \sqrt{v}^{-1} a_{\lambda \mu} \left( \frac{2}{\pi} \right) \int _{0}^{\infty} \frac{\sinh \omega + i \varepsilon}{\sinh \omega + 1} \left[ \cosh \omega \right]^{\lambda + \mu} d \omega \]

where

\[ y_{\lambda \mu}(\frac{\pi}{2}, 0) = \begin{cases} \left( \frac{2^{\lambda + 1}}{4\pi} \right)^{\frac{1}{2}} \frac{[\lambda - \mu]!(\lambda + \mu)!}{(\lambda - \mu)!!(\lambda + \mu)!!} \left( -1 \right)^{\frac{(\lambda + \mu)}{2}} & \text{for even } (\lambda + \mu) \\ 0 & \text{for odd } (\lambda + \mu) \end{cases} \]

...\(21\)

and

\[ I_{\lambda \mu}(\theta, \xi) = \int_{-\infty}^{\infty} e^{i\xi(\varepsilon \sinh \omega + \omega)} \left[ \cosh \omega + i(\varepsilon - 1) \right]^{\lambda + \mu} \left( \varepsilon \cosh \omega + 1 \right)^{\lambda + \mu} d \omega \]

...\(22\)

where the eccentricity \( \varepsilon \) of hyperbola is related to the deflection angle \( \theta \) by \( \varepsilon = 1/\sin(\theta/2) \). The parameter introduced here is frequently referred to as the "adiabaticity parameter" and it is defined as

\[ \xi = \frac{\Delta E}{h v} = \frac{Z_{1}Z_{2}e^{2}}{h v} \]

...\(23\)

with \( E = \frac{1}{2} m_{0}v^{2} \) = kinetic energy of projectile.

In terms of these quantities, the transition amplitude is

...
\[ b_{1f} = \frac{4\pi^2 e}{i \hbar} \sum_{\lambda \mu} (2\lambda + 1)^{-1} (-1)^{I_1 - M_1} \begin{pmatrix} I_1 & I_f \\ -M_1 & M_f \end{pmatrix} \left\langle I_1 \mid M(\ell, \lambda) \mid I_f \right\rangle \]

\[ \chi v^{-1} a^{-\lambda} Y_{\lambda \mu} \left( \frac{\pi}{2}, 0 \right) I_{\lambda \mu}(\theta, \xi) \]

...(24)

The differential cross section for the transition, equation (4), with the aid of equation (2) and (5) can be written as:

\[ \left( \frac{d\sigma}{d\Omega} \right)_f = (2I_1 + 1)^{-1} \sum_{M_1 M_f} b_{1f} \chi \frac{1}{4} a^2 \sin^4(\theta/2) \]

...(25)

Inserting (24) in (25) the inelastic excitation cross section becomes:

\[ \left( \frac{d\sigma}{d\Omega} \right)_f = (2I_1 + 1)^{-1} \frac{16\pi^2 e^2}{h^2} \chi \frac{1}{4} a^2 \sum_{M_1 M_f} \sum_{\lambda \mu} (2\lambda + 1)^{-2} \begin{pmatrix} I_1 & I_f \\ -M_1 & M_f \end{pmatrix} \]

\[ \begin{pmatrix} I_1 & I_f \\ -M_1 & M_f \end{pmatrix} \chi \left\langle I_1 \mid M(\ell, \lambda) \mid I_f \right\rangle \chi v^{-2} a^{-2\lambda} Y_{\lambda \mu} \left( \frac{\pi}{2}, 0 \right) \chi I_{\lambda \mu}(\theta, \xi) \]

\[ \chi \sin^4(\theta/2) \]

using orthogonality relation for vector addition coefficients
\[
\frac{d\sigma}{d\Omega} = \sum_{\lambda=1}^{\infty} \frac{4\pi^2 \alpha^2}{\hbar^2 v^2} \frac{(2I_f+1)^{-1} |\langle I_i | M(E_\lambda) | I_f \rangle|^2}{(2\lambda+1)^3} \\
\times \sum_{\mu} |Y_{\lambda\mu}(\mathbf{\hat{r}},0)|^2 \times |I_{\lambda\mu}(\Theta,\xi)|^2 \times \sin^2(\Theta/2) \quad \ldots(26)
\]

Introducing the reduced transition probability \(B(E_\lambda)\) for the radiative transition \(E_\lambda\) between the nuclear states \(I_i\) and \(I_f\),

\[
B(E_\lambda; I_i \rightarrow I_f) = \sum_{\mu} |\langle I_i M_1 | M(E_\lambda, \mu) | I_f M_\mu \rangle|^2 \\
= (2I_i+1)^{-1} |\langle I_i | M(E_\lambda) | I_f \rangle|^2 \quad \ldots(27)
\]

the inelastic differential cross-section (26) becomes

\[
d\sigma = \sum_{\lambda=1}^{\infty} d\sigma_{E_\lambda} \quad \ldots(28)
\]

with

\[
d\sigma_{E_\lambda} = \frac{\gamma^2}{(2\alpha)^2} a^{-2\lambda+2} \frac{B(E_\lambda)}{(Z_2 e)^2} d_f(E_\lambda(\Theta,\xi)) \quad \ldots(29)
\]

where
\[
\frac{d \sigma_{E\lambda}(\theta, \xi)}{(2\lambda+1)^3} \sum_{\mu} |Y_{\lambda \mu}(\xi, 0)|^2 \times |I_{\lambda \mu}(\theta, \xi)|^2 \times \sin^{-\frac{1}{4}}(\theta/2) d\Omega
\]

...(30)

The dimensionless quantity \( \gamma \), which measures the effective strength of the interaction in equation (29) is defined by

\[
\gamma = z_1 z_2 e^2 / h \nu
\]

For Coulomb excitation, \( \gamma >> 1 \) where \( \nu \) is the velocity of the incident particle.

The total excitation cross section of order \( E_{\lambda} \), obtained by integrating over all scattering angles, is given by

\[
\sigma_{E\lambda} = \gamma^2 a^{-2\lambda+2} \frac{B(E_{\lambda})}{(z_2 e)^2} \int \sigma_{E\lambda}(\xi)
\]

...(31)

with

\[
\sigma_{E\lambda}(\xi) = 2 \frac{(2\pi)}{2\lambda+1} \sum_{\mu} |Y_{\lambda \mu}(\xi, 0)|^2 \int_0^{\pi} |I_{\lambda \mu}(\theta, \xi)|^2 \frac{\cos(\theta/2)}{\sin^{\frac{1}{4}}(\theta/2)} d\theta
\]

...(32)

**SYMMETRIZATION OF SEMI-CLASSICAL RESULTS**

In deriving the above results (equation 31 and 32) a
classical description of the projectile orbit was used, which makes no distinction between the initial and final parameters of the particle since the radiative energy vanishes classically: $\Delta E = \hbar \omega \rightarrow 0$. To improve the accuracy of the results an adhoc approximation that does distinguish initial and final parameters, must, however, in agreement with the general reciprocity principle, be introduced.

The reciprocity relation

$$(2I_i + 1) V_i^2 \sigma_i \rightarrow f = (2I_f + 1) V_f^2 \sigma_f \rightarrow i$$

yields a general symmetry requirement for the cross section.

$$\sigma_i \rightarrow f = (V_f / V_i)^2 \frac{2(2I_f + 1)}{2I_i + 1} \times \text{(function symmetric in initial and final states)}$$

...(33)

The cross section involves the square of a matrix element which is symmetrical in the initial and final state.

The simplest symmetrization procedure obeying this requirement, consists of replacing $V$ by a geometrical average over the initial and final values $V_i$ and $V_f$. Thus one replaces $V$ in the classical equations by

$$V = (V_i V_f)^{\frac{1}{2}}$$

...(34)
Introducing symmetrized parameters $a$ and $\xi$, given by

$$a = \frac{z_1 z_2 e^2}{m_0 v_1 v_f} \quad \ldots(35)$$

$$\xi = \frac{z_1 z_2 e^2}{\hbar} (\frac{v_1}{v_f} - \frac{v_1}{v_1}) = \gamma_f - \gamma_i \quad \ldots(36)$$

where

$$\gamma_i = \frac{z_1 z_2 e^2}{\hbar v_1} \quad \text{and} \quad \gamma_f = \frac{z_1 z_2 e^2}{\hbar v_f} \quad \ldots(37)$$

The symmetrized expression for the excitation cross section is obtained by replacing $\gamma$, $a$, and $\xi$ in equation (31) with symmetrized expressions:

$$\sigma_{E_\lambda} = \gamma_i^2 a^{2\lambda+2} \frac{B(E_\lambda)}{(z_2 e)^2} \int_{E_\lambda} f_{E_\lambda}(\xi) \quad \ldots(38)$$

**NUMERICAL RESULTS**

In order to obtain an expression for $\sigma_{E_\lambda}$ numerically, it is convenient to express all parameters involved as functions of the energy of the incident projectile,

$$E = \frac{1}{2} m_1 v_1^2 \quad \ldots(39)$$

and the energy independent quantities such as the charge,
mass numbers and the excitation energy. From equation (39)

$$V_1 = \left(\frac{2E}{m_1} \right)^{1/2} = \left(\frac{2E}{A_1 M} \right)^{1/2}$$  \hspace{1cm} \ldots(40)

where $A_1$ is the projectile mass in units of the proton mass $M$. The final velocity $V_f$ is expressed in terms of $E$ and excitation energy $\Delta E'$ as

$$\frac{1}{2} m_1 V_f^2 = E - \Delta E'$$  \hspace{1cm} \ldots(41)

where

$$\Delta E' = (1 + A_4/A_2) \Delta E$$  \hspace{1cm} \ldots(42)

with $\Delta E$ the excitation energy. Introducing the parameter $\zeta$ for convenience as

$$\zeta = \frac{\Delta E'}{E}$$  \hspace{1cm} \ldots(43)

The equation (41) becomes as

$$V_f = \left(\frac{2E}{A_1 M} \right)^{1/2} \left(1 - \zeta \right)^{1/2}$$  \hspace{1cm} \ldots(44)

The symmetrized parameter $a$ given by (35) can be written in terms of (42) and (44) as:

$$a = \frac{Z_1 Z_2 e^2}{m_0 V_1 V_f}$$
\[
\frac{1}{M} \frac{1}{(1 + \frac{A_1}{A_2})} \frac{Z_1Z_2e^2}{(2E/A_1M)^{1/2}} \frac{1}{(2E/A_2M)^{1/2}} (1 - \frac{1}{\sqrt{E}})^{-1/2}
\]

Because \( m_0 = \frac{\dot{A}_1}{(1 + \frac{A_1}{A_2})} \)

\[
a = \frac{1}{2}(1 + \frac{A_1}{A_2}) \frac{Z_1Z_2e^2}{E} (1 - \frac{1}{\sqrt{E}})^{-1/2}
\]

\[
= \frac{1}{2} \frac{e^2}{\hbar c} (1 + \frac{A_1}{A_2}) \frac{Z_1Z_2}{E_MeV} (1 - \frac{1}{\sqrt{E}})^{-1/2}
\]

where, \( \frac{e^2}{\hbar c} = 1/137 \) and \( \hbar c = 1.9732 \times 10^{-11} \text{ MeV} \cdot \text{cm} \).

so \[ \frac{1}{2} \frac{e^2}{\hbar c} \cdot \hbar c = 0.07199 \times 10^{-12} \text{ MeV} \cdot \text{cm} . \]

Thus one gets

\[
a = 0.07199 (1 + \frac{A_1}{A_2}) \frac{Z_1Z_2}{E_MeV} (1 - \frac{1}{\sqrt{E}})^{-1/2} \times 10^{-12} \text{ cm} \quad \ldots (45)
\]

where \( E_MeV \) is the initial energy expressed in MeV. The parameter \( \eta \) for the initial and final states may be written as

\[
\eta_1 = \frac{\frac{1}{2} \frac{Z_1Z_2}{e}}{A_1/10.008 E_MeV}^{1/2} \quad \ldots (46)
\]

and
\[ \gamma_{\ell} = \frac{z_1 z_2 e^2}{\hbar} \gamma_{\ell} = \gamma_1 (1 - \xi)^{-1/2} \]

The quantum mechanical excitation and angular distribution functions are also expressed in terms of \( \gamma_1 \) and \( \xi = \gamma_\ell - \gamma_1 \).

Introducing these modified definitions of \( \gamma_4 \), \( \gamma_4^{(46)} \), \( \xi \) in equation (38)

\[
C_{E_\lambda} = \left[ \frac{z_1^2 A_1 A_2}{40.03} \left( 0.07199 \left( 1 + \frac{A_1}{A_2} \right) \right) \right]^{2\lambda+2} \text{ barns} \times 
\]

\[
E_{\text{MeV}}^{\lambda-2} (E_{\text{MeV}} - \Delta E_{\text{MeV}})^{\lambda-1} \times B(E_\lambda) \int_{E_\lambda} B(E_\lambda; \gamma_1, \gamma_\ell) \]

The reduced nuclear transition probability, \( B(E_\lambda) \), is measured in units of \( e^2 \left( 10^{-24} \text{ cm}^2 \right)^\lambda = e^2 \text{ (barn)}^\lambda \).

Since \( E_2 \) excitation is by far the most commonly encountered case, the final expression for the total cross section for \( E_2 \) Coulomb excitation in first order theory is written from equation (48) with \( \lambda = 2 \) as:

\[
\sigma_{E_2} = 4.819(1 + \frac{A_1}{A_2})^{-2} \frac{A_4}{z_2^2} (E - \Delta E') \int_{E_2} (\xi)^\lambda B(E_2; I_1 \rightarrow I_2) \text{ barns}
\]

where \( B(E_2) \) is in units of \( e^2 \text{ (barn)}^2 \).

The classic review paper by Alder et al. (10) gives in detail the derivations of electric and magnetic excitation cross
sections using semi-classical and quantum treatments. He has also tabulated \( f(\xi) \) values. It is evident that the quantum treatment of the non-relativistic Coulomb excitation is a problem that really involves only technical difficulties. It is surprising, however, that the semi-classical calculations for the total cross sections turned out to be quite accurate.

The matrix elements \( m(E_\lambda; I_1 \rightarrow I_f) \) in equation (15) and (27) which govern the response of the nucleus to the electric field are very simply related to the reduced transition probabilities encountered in electromagnetic decays. The processes of Coulomb excitation and electromagnetic decay have an intimate connection. For Coulomb excitation, in first order perturbation theory, the probability for excitation by multipolarity, \( \lambda \) is proportional to the reduced transition probability.

\[
P_{I_1 \rightarrow I_f} \propto B(E_\lambda; I_1 \rightarrow I_f)
\]

...(50)

The probability for electromagnetic decay from the same level to the ground state is given by Blatt and Weisskopf (11).

\[
\frac{1}{\tau_\lambda} = \tau = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \frac{1}{\hbar} \left( \frac{\Delta E}{\hbar c} \right)^{2\lambda+1} B(E_\lambda; I_f \rightarrow I_1)
\]

...(51)
where $\tau$ is the life time of the level and $\Delta E$ excitation energy. The relationship between the reduced matrix elements and reduced transition probabilities from equation (27) is

$$B(E_\lambda; I_1 \rightarrow I_f) = (2I_1+1)^{-1} \left| \langle I_1 | M(E_\lambda) | I_f \rangle \right|^2.$$ 

The reduced matrix elements are symmetrical i.e.

$$\langle I_1 | M(E_\lambda) | I_f \rangle = (-1)^{I_f-I_1+\lambda} \langle I_f | M(E_\lambda) | I_1 \rangle \quad \ldots(52)$$

where as one can easily see from equation (52) that $B(E_\lambda; I_f \rightarrow I_1) = B(E_\lambda)\downarrow$ is related to $B(E_\lambda; I_1 \rightarrow I_f) = B(E_\lambda)\uparrow$ by spin factor $2I_1+1/2I_f+1$. That is

$$B(E_\lambda; I_f \rightarrow I_1) = \frac{2I_1+1}{2I_f+1} B(E_\lambda; I_1 \rightarrow I_f) \quad \ldots(53).$$

The two processes obey the same selection rules, which for electric and magnetic multipolarities are

$$|I_1 - I_f| \leq \lambda \leq I_1 + I_f$$

$$\pi_1 \pi_f = \begin{cases} (-1)^\lambda \text{ for } E_\lambda \\ (-1)^{\lambda+1} \text{ for } M_\lambda \end{cases} \quad \ldots(54)$$

where $\pi_1$ and $\pi_f$ are the parities of the initial and final nuclear states.
There are two significant differences between excitation and decay. Firstly, magnetic excitations are exceedingly weak, being of order \((V/c)^2\) relative to the corresponding electric multipole. Secondly, the decrease in excitation probability per unit increase in multipolarity is about a factor of 100, whereas in decay the corresponding factor is around \(10^6\). For these reasons, the electric quadrupole transitions are of special importance in the Coulomb excitation, and in fact, the majority of the excitations observed so far are of \(E_2\) type. The cross sections for magnetic excitations are very much smaller than that of electric excitations. Even in cases where the radiative de-excitation process takes place by a mixed \(M_1+E_2\) transition, the excitation will always occur through \(E_2\).

**THEORY OF ANGULAR DISTRIBUTION OF DE-EXCITATION GAMMA RAYS**

The properties of nuclear states populated by Coulomb excitation can be examined in a variety of ways: Direct observation of the inelastically scattered incident particles; study of the intensity and angular distribution of the de-excitation gamma-rays; and study of the conversion electrons of the de-excitation.

The Coulomb excitation process initiated by an incident
particle beam defines a fixed direction in space, hence the de-excited gamma radiation emitted will not be isotropic, but will display preferential decay directions. The directional properties of a Coulomb plane wave (i.e. a charged particle beam with an asymptotically defined direction) can be conveniently expressed in terms of tensor parameters introduced by Fano(12). The angular distribution of the de-excited gamma rays following Coulomb excitation can be expressed in terms of the tensor parameters of the particle beam and of the gamma ray.

The process of Coulomb excitation is closely related to electromagnetic decay. This close relationship is basic to the possibility of introducing particle parameters. By using the particle parameters, the angular distribution of the de-excited gamma radiation with respect to the incident particle may be expressed in terms of a gamma-gamma correlation.

Assuming a hypothetical case, the similarity between the angular distribution of the gamma rays and the gamma-gamma correlation can be explained as follows. If $I_1$, $I_f$, $I_{ff}$, respectively, correspond to the ground state spin, the spin of the Coulomb excited state, and spin of the final state populated by the gamma decay, then fig.a represents the excitation of a state from $I_1$ to $I_f$ and its decay by
gamma emission to $I_{ff}$ while fig. b shows the hypothetical gamma-gamma cascade employed in obtaining the angular distribution of the gamma ray following Coulomb excitation.

![Diagram](image)

**Fig. a.**

**Fig. b.**

Alder and Winther (13), using semiclassical treatment, have derived explicit expressions for the angular distribution of the gamma radiation following Coulomb excitation. According to Biedenharn and Rose (14) the gamma-gamma correlation functions for the correlation in which one gamma ray is not pure, i.e., mixed $E_2$ and $M_1$ transition, can be written as

$$W = W_I + \delta^2 W_{II} + 2\delta W_{III} \quad \ldots(55)$$

where $\delta^2$ is the ratio of reduced matrix elements, and
\[ W_I = \sum_k B_k(L_1 L_2) P_k(\cos \theta), \]
\[ W_{II} = \sum_k B_k(L_1' L_2) P_k(\cos \theta), \]
\[ W_{III} = \sum_k F_k(L_1 L_1' I_{ff} I_f) P_k(L_2 I_1 I_f) P_k(\cos \theta), \]

with \( B_k(L_1 L_2) = F_k(L_1 I_{ff} I_f) P_k(L_2 I_1 I_f) \)

and \( B_k(L_1' L_2) = F_k(L_1' I_{ff} I_f) P_k(L_2 I_1 I_f) \).

Here \( k \) is a parameter which is always positive even number.

Therefore,

\[ W(\theta) = \sum_k \left[ F_k(L_1 I_{ff} I_f) P_k(L_2 I_1 I_f) + \delta^2 F_k(L_1' I_{ff} I_f) P_k(L_2 I_1 I_f) \right. \]
\[ \left. + 2\delta F_k(L_1 L_1' I_{ff} I_f) P_k(L_2 I_1 I_f) \right] P_k(\cos \theta) \]

For \( k = 0 \)

\[ W(0) = [1 + \delta^2] = N - \text{normalisation constant}. \]

Defining \( A_k \) such that

\[ A_k = F_k(L_2 I_1 I_f) \left[ \frac{F_k(L_1 I_{ff} I_f) + 2\delta F_k(L_1' I_{ff} I_f) + \delta^2 F_k(L_1' I_{ff} I_f)}{1 + \delta^2} \right] \]

...(56)
one can express the gamma-gamma angular correlation function for mixed multipoles terms of $A_k$ as:

$$\mathcal{W}(\theta) = \sum_{k} A_k P_k(\cos \theta) \quad \ldots (57)$$

The angular distribution of the de-excited gamma rays following Coulomb excitation by a charged particle may be written in terms of gamma-gamma correlation function equation (55).

$$\mathcal{W}(\theta_r) = \sum_{k} a_k^\lambda(\xi) A_k P_k(\cos \theta_r) \quad \ldots (58)$$

where $a_k^\lambda(\xi)$ are known as the particle parameters, which depend on excitation process through the parameter $\xi$ have been evaluated by numerical methods for quadrupole excitation by Alder et al. (10). The gamma-gamma directional angular correlation coefficients, $A_k$ are tabulated by Biedenharn and Rose (14). The $\theta_r$ is the angle between the direction of incident beam and the gamma quantum.

The Coulomb excitation process is predominantly $E_2$ and even for odd- $A$ target element where the nuclear transition favours an $M_1$ mixture the excitation process to all practical purposes is pure $E_2$. Thus for Coulomb excited gamma ray angular correlation, the effective angular
momentum scheme can be

$$I_1 \left( \frac{L_2=2}{E_2} \right) = \frac{L_1=1 - M_1}{L_1'=L_1+1=E_2} I_{ff} .$$

In this scheme, the $A_k$ becomes

$$A_k = F_k(2I_1I_f) \left[ \frac{F_k(1I_{ff}I_f)+2sF_k(12I_{ff}I_f)+s^2F_k(2I_{ff}I_f)}{1+s^2} \right] \ldots (59)$$

The $s^2$ in above equation is defined as

$$s^2 = \left| \frac{\langle I_{ff} | | E_2 || I_f \rangle}{\langle I_{ff} | | M_1 || I_f \rangle} \right|^2 \ldots (60)$$

**EXTRACTION OF $B(E_2)$**

The total cross section for the $E_2$ excitation is given by equation (49)

$$C_{E_2}(E) = 0.819 \left( \frac{A_2}{A_1+A_2} \right)^2 \frac{A_1}{Z_2^2}(E - \Delta E') B(E_2) \int_{E_2}^c |f_{E_2}(\xi)|^2 \text{ barns} \ldots (61)$$

where the notations are that of Alder et al. (10). The thick target, gamma ray yield of a given transition $Y_{th}$ is obtained by integrating equation (43) over the trajectory of
a projectile in the target and multiplying the decay fraction $\varepsilon$ of this mode,

$$Y_{th} = \frac{N_A a I \varepsilon}{A} \int_{0}^{E_1} \sigma_{E_2}(E) \frac{dE}{dx}\ dx + \text{(contributions from higher excited levels)} \quad \ldots(62)$$

Here, $N_A$ is Avogadro's number, $A$ the number of target nuclei per molecule, $A$ the molecular weight of the target and $I$ the number of projectiles incident on the target. The stopping power $(dE/dx)$ is in units of MeV cm$^2$/g. The second term in equation (62) is the contribution from cascade decays of higher levels.

Neglecting contributions from higher excited levels to $Y_{th}$, an expression for the reduced transition probability $\varepsilon B(E_2)$ can be written with the aid of equation (61) and (62) as:

$$\varepsilon B(E_2) = \frac{Y}{I} \frac{K}{I_{E_2}} \quad \ldots(63)$$

where 

$$I_{E_2} = \int_{0}^{E_1} (E - \Delta E') \int_{E_2}^{E_1} (\xi) \frac{dE}{dx}\ dx \quad \ldots(64)$$
and \( k = \left[ \frac{4.813}{A} N A \alpha \left( \frac{A_2}{A_1 + A_2} \right)^2 \frac{A_1}{z^2_2} \cdot 10^{-24} \text{ cm}^2 \right]^{-1} \) ...(65)

Thus, the reduced transition probability \( \varepsilon B(E_\gamma) \) for the excitation which leads to a particular gamma ray transition is obtained by determining \( (Y/I) \) experimentally and calculating numerically the integration \( I_{E_\gamma} \) and the value of \( k \) as in equation (65).

The numerical evaluation of \( I_{E_\gamma} \) was done using \( \int_{E_\gamma} (\eta \xi) \) values tabulated by Alder et al. (10), and calculating stopping power \( dE/dx \). Above the energies of 2 MeV/a.m.u. Bethe’s formula for the stopping power holds very well (12) and that of a compound medium may be written as

\[
- \left( \frac{dE}{dx} \right) = (\gamma z_i)^2 \frac{0.307}{A_i^2} \sum \alpha_{i} z_i \ln \left[ \frac{1.022 \times 10^6 \beta_i^2}{I_i} \right] \text{ MeV cm}^2 \text{ g}^{-1} \]  

(66)

where \( (\gamma z_i) \) is the effective charge of the incident ions.

\( A \), the molecular weight of the absorbing material i.e. target nucleus, \( a_i \) the number of atoms of type \( i \) per molecule;

\( I_i \) the mean ionization potential of atom \( i \) in units of eV.
\[ I_1 = 9.1 \times 10^2 z_2 (1 + 1.9 z_2^{-2/3}). \]

\( z_2 \) is target proton number and \( \beta = V/c \), \( V \) is the velocity of incident ions.

In order to facilitate the evaluation of \( I_{E_2} \), the stopping powers (\( dE/dx \)) were calculated using equation (66) for particular incident ions and target as a function of incident ion energy or velocity. Assuming an energy dependence for

\[ \frac{dE}{dx} = S(E) = bE^n, \]

a plot of \( \log S(E) \) against \( \log (E) \) was made to obtain the exact values of \( b \) and exponent \( n \).