Exponential moments and quark-antiquark potential

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Exponential moments have been calculated using a rather general Hamiltonian for the $Q\bar{Q}$ system and compared with the moments calculated from quantum chromodynamics (QCD). The Hamiltonian contains momentum-dependent perturbations. Only such perturbations when coupled with a confining potential rising slower than a quartic power can be made compatible with the moments obtained from QCD.

To study the physics of quarks at large distances Shifman, Vanishtein, and Zakharov\textsuperscript{1} (SVZ) have recently tried to use the results of quantum chromodynamics (QCD) at short distances. Duality\textsuperscript{2-5} between resonance physics and the QCD perturbation theory is invoked to study the confinement mechanism. An operator-product expansion\textsuperscript{6} of the vacuum expectation value of two heavy-quark current operators is made as follows:

$$\langle \Psi | T(j_{\mu}^{Q}(x)j_{\nu}^{Q}(0)) | \Psi \rangle = C_{\text{ren}}(q^2) + C_{G}(q^2) \langle \Omega | G_{\mu\nu} G_{\mu\nu}^{*} | \Omega \rangle + \cdots .$$

The vacuum-polarization function \(\pi(q^2)\) can also be analyzed in terms of moments\textsuperscript{1,7} given by

$$M_n(Q^2) = \int \frac{\text{Im}\pi(s) ds}{(s+Q^2)^{n+1}}, \quad s = q^2, \quad Q^2 = -q^2 .$$

SVZ further showed\textsuperscript{1} that the ratio of moments \(r_n = M_n/M_{n-1}\) is more stable in \(n\) than \(M_n\) in practical computations. One can also introduce a set of exponential moments:

$$M(\sigma) = \lim_{Q^2 \rightarrow \infty} (Q^2)^{\sigma+1} \text{Im} M_n(Q^2) = \int ds e^{-s} \text{Im} \pi(s)$$

with

$$Q^2 = \frac{n}{\sigma} .$$

A stabler ratio \(R\) defined as

$$R(\sigma) = -\frac{d}{d\sigma} \ln M(\sigma)$$

is actually used for analyzing the exponential moments. The QCD result for \(M(\sigma)\) has been given by Bertlmann.\textsuperscript{4} The result for \(R(\sigma)\) is

$$R(\sigma) = 4m^2 - \frac{d}{d\sigma} \ln A(\sigma) + a_0 a(\sigma) + b(\sigma) ,$$

where

$$A(\sigma) = \frac{3}{16} \frac{1}{\sqrt{\pi}} \frac{1}{G(\frac{3}{2},w)} .$$

In the limit \(\sigma \rightarrow 0\), which we shall later refer to, the expression for the ratio is

$$R(\sigma) = 4m^2 + \frac{1}{\sigma} + 3\phi_1^2 \sigma .$$

The inverse-power moments are

$$M_N = \frac{4\pi}{f} |\psi_f(0)|^2 .$$

With \(N \rightarrow \infty\) and \(N/(N)^{1/2} \tau\) finite,

$$\langle E_j - E\rangle - Ne^{-E/\tau} ,$$

and the inverse moments generalize to

$$\langle -E\rangle M_N \rightarrow M(\tau) = 4\pi (K=0) e^{-H(t)} \langle \bar{\psi} = 0 \rangle .$$

If the Hamiltonian \(H\) has a small perturbation term, the ra-
The ratio of moments is perturbatively,
\[ R(\tau) = \frac{M_o(\tau)}{M_o(\tau)} - \frac{d}{d\tau} \left( \frac{\delta M(\tau)}{M_o(\tau)} \right) . \]  
(14)

If the nonrelativistic Hamiltonian is
\[ H = \frac{k^2}{2\mu} + V(r) = \frac{k^2}{2\mu} + \sum \lambda_i r^i \]
the ratio of moments can be easily obtained as \(^4\)
\[ R(\tau) = \frac{\tau}{\tau} + \sum \lambda_i \Gamma \left[ 2 \cdot \frac{1}{2} \cdot \left( \frac{\tau}{2\mu} \right)^{\tau/2} \right] . \]  
(15)

When the nonrelativistic expression of \( R(\sigma) \) of Eq. (11) is compared with Eq. (15), one obtains the potential
\[ V(r) = -i 2. \frac{\alpha_s}{r^2} + \frac{1}{64} \bar{m}^4 . \]  
(16)

When the nonrelativistic expression of \( R(\sigma) \) of Eq. (11) is compared with Eq. (15), one obtains the potential
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However, the quartic part is indicative of problems for duality. All known charmonium spectra \(^8\) can be explained by much weaker \( r \) dependence.

The operator-product-expansion result of \( R(\tau) \) is presumably accurate. To study changes in the Hamiltonian formulation of Bertlmann, let us examine the results of a relativistic kinetic-energy term for \( H \), namely,
\[ H_0 = (k^2 + \mu^2)^{1/2} - \mu . \]  
(17)

\( M_o(\tau) \) is readily evaluated:
\[ M_o(\tau) = 4\pi (x = 0|e^{-H_0}|y = 0) \]
\[ = - \frac{e^{\mu^2}}{\pi} \frac{\partial}{\partial \tau} \left[ \frac{1}{\tau} K_i(\mu \tau) \right] . \]  
(18)

\( K_i(\mu \tau) \) is a modified Bessel function. The perturbed value for the moment is
\[ \delta M(\tau) = \int e^{-\mu^2} |e^{-H_0}|y = 0 \]
\[ = e^{\mu^2} \frac{1}{\pi} \sum \lambda_i \Gamma \left[ \frac{3}{2} \cdot \frac{1}{2} \cdot \left( \frac{\tau}{2\mu} \right)^{\tau/2} \right] 
+ e^{\mu^2} \frac{8}{\pi} \sum \lambda_i \int_0^\infty dr \int_0^\infty dk' \left( k'^2 + \mu^2 \right)^{1/2} k' \sin k'r \int_0^\infty dk \frac{\nu}{k'^2 + k'^2} \]  
(19)

The last term in the above equation is the cut contribution. For both large and small \( \mu \) this additional term appears to be small. We neglect it altogether to obtain
\[ \delta M(\tau) = e^{\mu^2} \frac{1}{\pi} \sum \lambda_i \Gamma \left[ \frac{3}{2} \cdot \frac{1}{2} \cdot \left( \frac{\tau}{2\mu} \right)^{\tau/2} \right] 
+ e^{\mu^2} \frac{8}{\pi} \sum \lambda_i \int_0^\infty dr \int_0^\infty dk' \left( k'^2 + \mu^2 \right)^{1/2} k' \sin k'r \int_0^\infty dk \frac{\nu}{k'^2 + k'^2} \sin (k'^2 - \mu^2)^{1/2} . \]  
(20)

and the perturbed ratio of moments is
\[ R(\tau) = \frac{\frac{d}{d\tau} \left[ \frac{1}{\tau} K_i(\mu \tau) \right]}{\frac{d}{d\tau} \left[ \frac{1}{\tau} K_i(\mu \tau) \right]} + \frac{\Delta \lambda}{\tau} \left[ \frac{\tau}{2\mu} \right]^{\tau/2} . \]  
(21)

For \( \mu \to \infty \),
\[ R(\tau) = \frac{3}{2\tau} + \sum \lambda_i \Gamma \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \left( \frac{\tau}{2\mu} \right)^{\tau/2} \right] . \]  
(15)

One gets back the nonrelativistic result of Bertlmann, which has produced the quartic confining potential.

In the limit \( \mu \to 0 \), which is like a relativistic limit with the average quark mass being negligibly small,
\[ R(\tau) = \frac{3}{2\tau} + \sum \lambda_i \Gamma \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \left( \frac{\tau}{2\mu} \right)^{\tau/2} \right] . \]  
(22)

To compare this \( R(\tau) \) [given in Eq. (22)] with QCD \( R(\sigma) \) [given in Eq. (12)] one first substitutes
\[ \omega = \bar{m} \tau, \quad 2\mu = \bar{m} \]
\[ M(\sigma) = 4\bar{m} M(\tau) \]
\[ = \int e^{-\mu^2} \left[ R(\sigma) - 4\bar{m}^4 \right] = R(\tau) \]  
(23)

Now comparing Eqs. (22) and (23) one obtains a confining potential which is linear in \( r \). However, the strength of the confining part is infinite and the Coulomb part is not at all reproduced.

What we realize from this analysis is that the results can be drastically altered with changes in the free Hamiltonian. When the quarks are confined it is difficult to know the exact energy-momentum relationship for them. In particular, there could be a momentum-dependent potential as well.

Thus we are led to examine the case of a general Hamil-
tonian given by
\[ H = \tilde{\alpha}\mu^{1-}\kappa^{+} + \tilde{\beta}\mu^{1-}\kappa^{+} + \psi(r) \] (24)
where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are dimensionless constants. The term \( \tilde{\beta}\mu^{1-}\kappa^{+} \) will be treated as the small momentum-dependent perturbation. To deduce the necessary equations for \( M(\tau) \) for the problem, let us write the Hamiltonian as
\[ H = T + \sum_{i}\lambda_{i}\tau^{i} \] (25)
To the first order in \( \lambda \),
\[ e^{-H}=e^{-Tr}+\sum_{i}\lambda_{i}\frac{\partial}{\partial \lambda_{i}}e^{-H\tau} \] (26)
Using the operator identity established by Betts, we obtain
\[ \delta M(\tau) = -\frac{2\tilde{\beta}\mu^{1-}}{\pi\nu} \left\{ \beta^{(s+1)}(\mu) - \beta^{(s+3)}(1-s)v \right\} \left\{ \frac{3+\beta}{\nu} \right\} \tau^{-(s+3)/2} \]
\[ -\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ +\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ \times \left\{ \frac{3-2\nu-s+\beta}{\nu} + \Gamma \left\{ \frac{3-s}{\nu} \right\} \right\} \tau^{s+2s-3}/2 \] (29)
Thus the modified \( R(\tau) \) may be written as
\[ R(\tau) = \frac{3}{\nu\tau} + \frac{\beta_{1}^{(s+1)}(\mu)}{\nu^{2}} \left\{ \frac{3-s}{\nu} \right\} \tau^{s+2s-3}/2 \]
\[ +\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ -\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ \times \left\{ \frac{3-2\nu-s+\beta}{\nu} + \Gamma \left\{ \frac{3-s}{\nu} \right\} \right\} \tau^{s+2s-3}/2 \] (30)
Substituting
\[ \frac{1}{2\nu m} \left\{ R(\tau) - 4\tilde{m}^{2} \right\} = R(\tau) \]
\[ M(\tau) = 4\tilde{m}M(\tau) \]
in Eq. (11), we get the QCD expression
\[ R(\tau) = 3\nu\tau + \frac{2}{\nu} \left\{ -\sigma_{1} \frac{2\nu m^{2}}{3} \right\} \right\} \tau^{s+2s-3}/2 \]
\[ +\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ -\sum_{i}\beta_{i}^{(s+1)}(\mu) \frac{\Gamma(1+s/2)}{\Gamma(1-s/2)} \left\{ \frac{3-s}{\nu} \right\} \beta^{(s+1)/(s+s/2)} \]
\[ \times \left\{ \frac{3-2\nu-s+\beta}{\nu} + \Gamma \left\{ \frac{3-s}{\nu} \right\} \right\} \tau^{s+2s-3}/2 \] (31)
This is to be compared with the result, Eq. (30), for the general Hamiltonian.

The first terms in Eqs. (30) and (31) are identical. Neglecting the second term in Eq. (30) altogether for our consideration, we compare the next term with the second term in Eq. (31). These are due to the Coulomb potential and should yield flavor-independent results. This implies that we must have
\[ \frac{s}{\nu} = \frac{1}{2} \]
\[ \frac{s-\nu e}{\nu} = \frac{1}{2} \]
We get
\[ \nu = 2 \] and \[ s = -1 \] (32)
This is an interesting result for duality, insofar as the vacuum-polarization contribution fixes the nature of the main kinetic-energy term to be of the form
\[ -\mu^{1-}\kappa^{+} = \frac{\kappa^{2}}{\mu} \] (33)
This proves that the problem is essentially nonrelativistic. Furthermore, the coefficient
\[ \lambda = -\frac{4\alpha_{s}}{3} \left( \tilde{\alpha} \right)^{1/2} 2^{1/2} \]
leads to
\[ \alpha = \frac{1}{2} \, . \]

So the first term in the Hamiltonian is \( k^2/2\mu \). Next we compare the interference term in Eq. (30) to the fourth term (leading term in \( r \)) in Eq. (31). The interference term reproduces the \( r^2 \) behavior if
\[ \frac{s + \nu - \beta}{\nu} = 2 \, . \]
\[ s = \beta + 2 \, . \] (34)

Let us examine the results for some \( s \) values. The case \( \beta = 0 \) is of considerable interest, because in that case one obtains \( s = 2 \), that is, \( V_c(r) \) is proportional to \( r^2 \), where \( V_c(r) \) represents the confining part of the potential:
\[ V_c(r) = \frac{\phi_1}{16\rho^2} \frac{1}{m} r^2 \, . \] (35)

The second term in Eq. (30) resulting from such a \( V_c(r) \) goes as \( r \). One also gets an equivalent term in the field-theoretic expression for \( R(r) \).

Thus a constant perturbative term gives a harmonic-oscillator potential. If the perturbed momentum-dependent potential is linear in \( k \), the potential is a cubic in \( r \). Again, to obtain a linear potential the perturbed momentum-dependent potential has to be of the form \( \rho \mu/k \).

Summarizing, we have been able to show that the main term in the expression for the Hamiltonian is necessarily of the nonrelativistic form \( H_0 = k^2/2\mu \) and it is possible to obtain lower powers than quartic for the confining potential with a momentum-dependent or a constant perturbation. In the absence of such perturbations, there could be serious disagreement with duality.

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1M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979); B147, 519 (1979).
Bosonic loops and gluon condensate

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Abstract. Contributions to the vacuum polarisation in $\text{QCD}$ are calculated separately with fermion as well as boson loops to have an idea of results expected for possible supersymmetric extension. It is found that the results are not altered in any significant way.

Keywords. Bosonic loops; gluon condensate; supersymmetry.

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1. Introduction

Supersymmetry treats both bosons and fermions on the same footing. As virtual objects in closed loops, both fermions and bosons are likely to be equally important. Vacuum polarisation and non-perturbative effects for quantum chromodynamics with quark loops have been calculated by Shifman et al (1981). Bell and Bertlmann (1981) have also analysed the exponential moments of vacuum polarisation function and have obtained a non-relativistic potential which is confining in nature. However, non-perturbative analysis with spin zero objects like $s$-quarks in the loops has not yet been carried out.

Since the lowest order vacuum polarisation function with spin zero particles has already been calculated by several authors (Akhiezer and Berestetskii 1965), in the present work we shall calculate the non-perturbative higher order contribution with scalar quark loops or bosonic loops. Our calculation shows that in the non-relativistic limit the potential is still the superposition of a coulomb and a confining term with coefficients which do not differ appreciably from those calculated with fermionic loops alone.

One begins by calculating the vacuum expectation value of two heavy, quark or $s$-quark currents, $j_{\mu}(x)$. Its Fourier transform, the polarisation tensor in the true $\text{QCD}$ vacuum is

$$i \int d^4 x \exp(iq \cdot x) \langle \Omega | T(j_{\mu}(x)j_{\nu}(0)) | \Omega \rangle = (q_{\mu}q_{\nu} - q^2 g_{\mu\nu})\pi(Q^2)$$

$$Q^2 = -q^2$$

(1)

$\pi(Q^2)$ has an operator product expansion (Wilson 1969)

$$\pi(Q^2) = \sum_x C_x O_x \simeq C_{\text{pert}}(Q^2) I + C_0(Q^2) \langle \Omega | G^a_{\mu\nu} G^a_{\nu\mu} | \Omega \rangle + \ldots$$

(2)

where $I$ is the identity operator and $G^2 = \langle G^a_{\mu\nu} G^a_{\nu\mu} \rangle$ is the local operator constructed from gluon fields. The functions $C_{\text{pert}}(Q^2)$ and $C_0(Q^2)$ can be calculated perturbatively. The gluon condensate $\langle GG \rangle$ represents the confinement mechanism and is important.
in determining the properties of the bound states. The polarisation function \( \pi(Q^2) \) satisfies a dispersion relation

\[
\pi(Q^2) = \frac{1}{\pi} \int \frac{\text{Im } \pi(s) \, ds}{s + Q^2},
\]

where the resonances in the physical \( \text{Im } \pi(s) \) are related to the QCD calculations in (2). Differentiating (3) \( n \) times with respect to \( (-Q^2) \) one obtains the power moments (Shifman et al 1981),

\[
M_n(Q^2) = \frac{1}{\pi} \int \frac{\text{Im } \pi(s) \, ds}{(s + Q^2)^{n+1}}.
\]

The increase in \( n \) emphasizes the low energy region and probes large distances. Moments at arbitrary \( Q^2 \) have been calculated for various currents by Reinders et al (1981). Their application to charmonium produced beautiful results.

The exponential moment

\[
M(\sigma) = \int ds \exp (-\sigma s) \text{Im } \pi(s)
\]

\[Q^2 = \frac{n}{\sigma}, \quad n \to \infty\]

analysed by Bell and Bertlmann (1981) was an improvement on the inverse power moments. The exponential weight cuts off the large \( S \) contribution in \( \text{Im } \pi(s) \) more sharply than a power weight and enhance the low energy contribution relative to the high energy one. Moreover they are directly related to imaginary time Green's function in the non-relativistic limit. Calculated within potential models, the results have always been closer to the exact results than the corresponding power moments (Bell and Bertlmann 1981).

2. Calculation of exponential moments

2.1 Quark loops

As noted earlier Bell and Bertlmann (1981) have calculated the exponential moments for quarks. We denote the moment by

\[
M_f(\sigma) = \int ds \exp (-\sigma s) \text{Im } \pi_f(s).
\]

This exponential moment can be written as a sum of several terms

\[
M_f(\sigma) = \exp (-4m^2\sigma) \left[ \pi_A(\sigma) \left\{ 1 + \sigma A_f(\sigma) + \sigma b_f(\sigma) \right\} \right].
\]

The first term is the simple single loop result with

\[
\pi_A(\sigma) = \frac{3}{16} \frac{1}{\sqrt{\pi}} \sigma G(\frac{1}{2}, \frac{5}{2}, w)
\]

the Whittaker function

\[
G(b, c, w) = \int dt \exp (-t) t^{c-1} (w + t)^{-b}
\]

The second term is the contribution from the single fermion loop corrected by one virtual gluon exchanges. This has been calculated by Schwinger (1973). The ex-
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The extrapolated term is
\[
\text{Im} \frac{1}{F}(s) = \text{Im} \pi^0(s) \left[ 1 + \frac{4 \alpha_s}{3} \left( \frac{\pi}{2} - \frac{3 + \nu}{4} \left( \frac{\pi}{2} - \frac{3}{4 \pi} \right) \right) \right]
\]
\[\nu = \left(1 - \frac{4m^2}{s}\right)^{1/2}\]

where \(\nu\) is the velocity of the quark of mass \(m\). Substituting (10) in (6) one obtains the quantity \(a_\nu(\sigma)\) of the second term of (7) as
\[
a_\nu(\sigma) = \frac{4}{3\sqrt{\pi}} C \left( \frac{\pi}{12} - \frac{3}{8\pi} \right) G(1, 2, w) + \left( \frac{\pi}{6} - \frac{1}{4\pi} \right) G(2, 3, w) - C.
\]

where \(C = \frac{\pi}{2} - \frac{3}{4\pi}\).

The third term \(b_\nu(\sigma)\) is the non-perturbative contribution. This is obtained by calculating the box diagrams, with fermions in the loop, which yields the term \(G_\nu(\sigma), G_\nu(\sigma)\) in the operator expansion. Here the quantity \(b_\nu(\sigma)\) is
\[
b_\nu(\sigma) = - \frac{w^2 G(-\frac{3}{4}, \frac{3}{4}, w)}{2 G(\frac{3}{4}, \frac{3}{4}, w)},
\]

where \(w = 4m^2\) is a dimensionless variable and
\[
\phi = \frac{4\pi}{9} \langle \alpha_s G^2 \rangle \quad \text{is the gluon condensate parameter.}
\]

The ratio of moments calculated by Bertlmann (1981) for the above case is
\[
R_\nu(\sigma) = - \frac{d}{d\sigma} \log M_\nu(\sigma),
\]

with
\[
R_\nu(\sigma) = 4m^2 - \frac{d}{d\sigma} \left[ \log A_\nu(\sigma) + \alpha_\nu a_\nu(\sigma) + \phi b_\nu(\sigma) \right].
\]

With a power potential \(V(r) = \Sigma \lambda_i r_i\), the ratio of moments calculated from the inverse moments of \(svz\) (Bertlmann 1981) is
\[
R(\tau) = \frac{3}{2\tau} + \sum \lambda_i \Gamma \left( 2 + \frac{S}{2} \right) \left( \frac{\tau}{2\mu} \right)^{5/2}
\]
\[\tau = m\sigma; \quad 2\mu = m\]

Taking the large \(\sigma\) (or \(\tau\)) limit of \(R_\nu(\sigma)\) and comparing it with \(R(\tau)\) (equation (14)) one obtains the following non-relativistic potential
\[
V(r) = - \frac{4 \alpha_s}{3} r + \frac{\phi_1}{64} m r^4
\]
\[\phi_1 = (4m^2)^2 \phi\]
This is the superposition of a coulomb and a quartic potential. The strength of the confining part, being proportional to quark mass, has a strong flavour dependence.

2.2 Scalar quark loops

Similar equations as well as expressions for the potential can also be obtained with scalar quarks in the vacuum loops. The scalar quarks are assumed to have similar quantum numbers as ordinary quarks. We define, for a particular flavour but summed over all colours, a polarisation tensor

\[ \pi^a_{\mu
u}(q) = i \int d^4x \exp(iqx) \langle \Omega | T(j^a_{\mu}\bar{\phi} \mu) | \Omega \rangle \]

\[ = (q_{\mu}q_{\nu} - q^2g_{\mu\nu}) \pi^a(Q^2) \]  

(16)

where the electromagnetic current \( j^a_{\mu} \) are due to the S-quarks or bosons

\[ j^a_{\mu}(x) = -i \phi^a(x) \overline{\partial}_{\mu} \phi(x) \]

As before we can write

\[ \pi^a(Q^2) = \langle \Omega | C^a_{\mu}(Q^2) I + C^a_{\mu}(Q^2) J_1 G^2 + \ldots | \Omega \rangle, \]

(17)

and the exponential moment as

\[ M_a(s) = \int ds \exp\left(-\alpha s\right) \text{Im} \pi_a(s), \]

(18)

which can also be written as

\[ M_a(s) = \exp(-4m^2s) \pi A_a(s) \left[ 1 + \alpha_s a_g(s) + \phi \beta(s) \right]. \]

(19)

The first term in (19) is obtained from the simple S-quark loop given in figure 1. The imaginary part of the vacuum polarisation function to the lowest order is (Akhiezer and Berestetskii 1965)

\[ \text{Im} \pi^a_{\mu
u}(s) = \frac{1}{16\pi} \left[ \frac{(s - 4m^2)^3}{s} \right]^{1/2} \theta(s - 4m^2). \]

(20)

Substituting this in (18) one obtains

\[ \pi A_a(s) = \frac{3}{64} \frac{1}{\sqrt{\pi}} \frac{1}{s} G(\frac{1}{2}, \frac{1}{2}, w). \]

(21)

The virtual gluon corrections to the simple S-quark loop is given in figure 1b. The corresponding imaginary part has also been calculated by Schwinger for bosons. He has shown that an adequate interpolation for large and small \( \sigma \) is obtained by the

*Figure 1. a. Simple S-quark loop diagram. b. Simple S-quark loops with virtual gluon correction to order \( \alpha_s \).*
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interpolation formula

\[ \text{Im } \frac{1}{B(s)} = \text{Im } \pi^B_0(s) \left[ 1 + \frac{4\pi}{3} \left\{ \frac{\pi}{2\nu} \left( \frac{1}{2} + \frac{\nu}{2} \right) \right\} \right] \]  

(22)

This along with (8) gives

\[ a_B(0) = \frac{8}{9\sqrt{\pi}} G(1, 2, w) - 3C_1 G(2, 3, w) - C_1, \]

(23)

where

\[ C_1 = \frac{\pi}{3} - \frac{2}{\pi}. \]

The third term \( b_B(0) \) in (19) is the non-perturbative contribution of the scalar quarks which we calculate below.

3. Calculation of fourth order diagrams with boson loops

With respect to heavy quarks the vacuum gluon field can be considered as an external field. Thus any gauge condition is admissible. One thus uses the Schwinger gauge (Dubovikov and Smilga 1981)

\[ x \cdot A^\mu(x) = 0. \]

(24)

Here \( A^\mu(x) \) can be expressed directly in terms of \( G^\mu_\nu(0) \), the gluon field tensor, and its covariant derivatives, namely,

\[ A^\mu_\nu(x) = \frac{1}{2\nu!} x \cdot G^\mu_\nu(0) + \frac{1}{3\nu!} x_\nu x_\nu D_\mu G^\mu_\nu(0) + \ldots \]

(25)

Fourier-transformed this gives

\[ A^\mu_\nu(k) = -i \frac{(2\pi)^4}{2} G^\mu_\nu(0) \frac{\partial}{\partial k_\mu} \delta^4(k) \]

\[ + \left( -i \right)^2 \frac{(2\pi)^4}{3} (D_\mu G^\mu_\nu(0)) \frac{\partial^2}{\partial k_\mu \partial k_\nu} \delta^4(k) + \ldots \]

(26)

To obtain the \( G^2 \) contribution it suffices to keep only the first term in the expansion of \( A(k) \) and insert \( A(k) \) twice. The diagrams which contribute to \( G^2 \) in case of spin zero quarks are given in figures 2a–e. Figure 2e comes from the usual contact terms in the Lagrangian like \( \phi^* \phi A^\mu_\nu \) and \( \phi^* \phi A^\mu_\nu A^\mu_\nu \).

Using the fact

\[ \text{vac} | G^\mu_\nu(0) G^\mu_\nu(0) | \text{vac} \rangle = \frac{1}{96} \delta^{ab} (g_{\mu a} g_{\nu b} - g_{\mu b} g_{\nu a}) \langle \text{vac} | G^2 | \text{vac} \rangle. \]

(27)

One obtains

\[ \left[ \pi^B_0(q^2) \right] a = \frac{1}{3} \times 96 \langle \phi^* G^2 \phi \rangle (g_{\mu a} g_{\nu b} - g_{\mu b} g_{\nu a}) \left[ \frac{\partial^4 p}{(2\pi)^4} \frac{\partial}{\partial k_1} \frac{\partial}{\partial k_2} \right] \left[ \frac{(p^2 - m^2)(p^2 + k_2^2 - m^2)}{(p^2 + k_2^2)(p^2 + k_1^2)} \right] \]

(28)
Figures 2a–e. Diagrams to order $\alpha_s$ to determine the coefficient function $C_{\lambda}(Q^2)$ with $S$-quark loops.

Contracting $\mu$ and $\nu$ and differentiating with respect to $k_1$ and $k_2$ this reduces to zero. Proceeding in the same way

\begin{equation}
[n_{\mu}(q^2)]_{b+\epsilon} = -\frac{i}{12} \left\langle q^2 G^2 \right\rangle \int \frac{d^4 p}{(2\pi)^4} \frac{p^2 + p'p'}{(p^2 - m^2)^3 (p'^2 - m^2)},
\end{equation}

(29)
Bosonic loops and gluon condensate

\[ [\pi^p_{\mu}(q^2)]_x = \frac{i}{6} \langle g^4 G^2 \rangle \int \frac{d^4 p}{(2\pi)^4} \left[ \frac{2}{(p^2 - m^2)^3(p'^2 - m^2)} \right. \]
\[ + \frac{2p^2}{(p^2 - m^2)^3(p'^2 - m^2)} \left. + \frac{8m^2}{(p^2 - m^2)^3(p'^2 - m^2)^3} \right] \]
\[ + \frac{2m^2(p^2 + 2p'p' + p'^2)}{(p^2 - m^2)^2(p'^2 - m^2)^2} \left( p^2 - m^2 \right)^2 \]  

(30)

Thus the net contribution to \( \pi^p_{\mu}(q^2) \) with scalar quarks or internal boson state reduces to the following expression

\[ \pi^p_{\mu}(Q^2) = \frac{\alpha_s G^2}{8\pi} \int_0^1 dx \left[ -\frac{8}{3m^2} + \frac{5x}{2Q^2x(1-x) + m^2} \right. \]
\[ - \frac{m^2x(1-2x)}{2[Q^2x(1-x) + m^2]^2} + \frac{16m^4x^3}{3[Q^2x(1-x) + m^2]^3} \]
\[ \left. + \frac{4m^2Q^2x^3}{3[Q^2x(1-x) + m^2]^3} \right] \]

(32)

Hence

\[ C^p_b(Q^2) = \frac{\alpha_s G^2}{288\pi Q^2} \left[ \frac{15a - 3a^2 - 2}{2a^{3/2}} \ln \frac{\sqrt{a + 1}}{\sqrt{a + 1} + a(a - 1)} \right] \]

(33)

where \( a = 1 - \frac{4m^2}{S} \); \( S = q^2 = -Q^2 \).

The non-perturbative part of the exponential moment is then given by

\[ M_B^{nepert}(s) = -\frac{\alpha_s G^2 \sigma \sqrt{\pi}}{576} \left[ 4G(\frac{1}{2}, \frac{3}{2}, w) + 19G(\frac{1}{2}, \frac{3}{2}, w) \right. \]
\[ + 24G(\frac{3}{2}, \frac{3}{2}, w) \left. \right] \]

(34)

and

\[ b_b(s) = \frac{w^2 G(\frac{1}{2}, \frac{3}{2}, w) + 18G(\frac{1}{2}, \frac{3}{2}, w) + 6G(\frac{3}{2}, \frac{3}{2}, w)}{G(\frac{3}{2}, \frac{3}{2}, w)} \]

(35)

4. Non-perturbative moments and the potential with scalar quarks

To obtain an equivalent non-relativistic potential in this case we proceed in the same way as Bertlmann and compute the ratio of moment

\[ R_b(s) = 4m^2 - \frac{d}{ds} \left[ \log A_b(s) + \alpha_s a_b(s) + \phi b_b(s) \right] \]

(36)

For heavy quarks one considers the non-relativistic limit, which is obtained for \( s \to \infty \), or \( w \to \infty \).
In this limit
\[ \pi A_\mu (\sigma) = \frac{3}{64 \sqrt{\pi}} \frac{1}{(2m)^3} \sigma^{-5/2} \]
\[ a_\mu (\sigma) = \frac{8 \sqrt{\pi}}{9} w^{1/2} - C_1 \]  
\[ b_\mu (\sigma) = -\frac{w^3}{3} \]  
Thus
\[ R_\mu (\sigma) = 4m^2 + \frac{5}{2\sigma} + 4m^2 \left( -\alpha_s \frac{4 \sqrt{\pi}}{9} w^{-1/2} + \phi w^2 \right) \]  

To compare this with (14) one makes the following substitution
\[ w = m, \quad 2\mu = m \exp (4m^2 \sigma) M(\sigma) = 4m M(t) \]  
\[ \frac{1}{2} (4m)^{-1} [R(\sigma) - 4m^2] = R(t) \]  
With these (38) reduces to
\[ R(t) = \frac{3}{2e} + \left[ -\alpha_s \frac{4 \sqrt{\pi}}{15} (t/m)^{-1/2} + \frac{3\phi_1}{80} m (t/m)^2 \right] \]  
Comparing (40) and (14) one obtains the equivalent non-relativistic potential
\[ V(r) = -\frac{8 \alpha_s}{15} r + \frac{\phi_1}{160} m r^4. \]  
Equation (41), for the potential with scalar quarks in the loops is new and has not been published before. It is also the superposition of a coulombic and a quartic term, the coefficients of both the terms being smaller than those in (15). The strength of the confining part has flavour dependence also.

5. Possible supersymmetric calculation

We consider here the possibility that both quark and scalar quark currents co-exist. This may for instance happen in a supersymmetric phase. When supersymmetry is broken at high temperatures the scalar becomes heavier and need not be considered any further. In such a case one may consider the calculation of this section to be more of a hypothetical one. Normal quarks at low temperature contribute as usual and there will be no change in the present world, and the values of \( R \) and the leptonic width of \( J/\psi \) will not be affected.

So for completeness we consider the system in which the fermion and bosons of nearly equal mass are both assumed to be present in the loop diagrams. The exponential moment to first order in \( \alpha_s \) and \( \phi \) is then
\[ M_\mu (\sigma) = \exp (-4m^2 \sigma) \pi A_\mu (\sigma) \left[ 1 + \alpha_s A_\mu (\sigma) + \phi B_\mu (\sigma) \right], \]
where the free term is
\[ \pi A_\alpha(\sigma) = \pi A_\alpha(\sigma) + \pi A_\nu(\sigma) \]
\[ = \frac{3}{16} \frac{1}{\sqrt{\pi} \sigma} G(\frac{1}{2}, \frac{5}{2}, w) + \frac{3}{16} \frac{1}{\sqrt{\pi} \sigma} G(\frac{3}{2}, \frac{5}{2}, w). \]

Using the integral representation of \( G \) function
\[ G(\frac{1}{2}, \frac{3}{2}, w) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty dt \exp(-t) t^{3/2} (w + t)^{-3/2}, \]
\[ = -2 G(\frac{1}{2}, \frac{3}{2}, w) + 2 G(\frac{1}{2}, \frac{3}{2}, w), \]
\[ \pi A_\alpha(\sigma) = \frac{3}{32} \frac{1}{\sqrt{\pi} \sigma} \left[ G(\frac{1}{2}, \frac{3}{2}, w) + G(\frac{1}{2}, \frac{3}{2}, w) \right]. \] (43)

and the perturbative gluon correction
\[ a_\alpha(\sigma) = \frac{8}{3 \sqrt{\pi} \left[ G(\frac{1}{2}, \frac{3}{2}, w) + G(\frac{1}{2}, \frac{3}{2}, w) \right]} \mathcal{G} \left( \frac{3}{2}, \frac{5}{2}, w \right) + \mathcal{G} \left( \frac{1}{2}, \frac{5}{2}, w \right) \]
\[ = \left[ \frac{2\pi}{3} + \frac{1}{2\pi} \right] G \left( \frac{1}{2}, \frac{5}{2}, w \right) + \mathcal{G} \left( \frac{1}{2}, \frac{5}{2}, w \right) \]
\[ = \frac{1}{\sqrt{\pi} \sigma} \left[ G(\frac{1}{2}, \frac{3}{2}, w) + G(\frac{3}{2}, \frac{5}{2}, w) \right]. \] (44)

The non-perturbative contribution to the momenta in this case is
\[ b_\alpha(\sigma) = \frac{w^2}{24} \frac{4G(\frac{1}{2}, \frac{3}{2}, w) + 19G(\frac{1}{2}, \frac{3}{2}, w) + 24G(\frac{3}{2}, \frac{3}{2}, w)}{G(\frac{1}{2}, \frac{3}{2}, w) + G(\frac{3}{2}, \frac{3}{2}, w)} \]
\[ - w^2 \left[ G(-\frac{1}{2}, \frac{3}{2}, w) + G(\frac{1}{2}, \frac{3}{2}, w) \right]. \] (45)

In the non-relativistic limit \( w \to \infty \)
\[ \pi A_\alpha(\sigma) = \frac{3}{16} \frac{1}{\sqrt{\pi} \sigma} \sigma^{-3/2} \]
\[ a_\alpha(\sigma) = \frac{4}{3} \frac{1}{\sqrt{\pi} \sigma} w^{3/2} - C_2 \]
\[ b_\alpha(\sigma) = - \frac{11w^2}{24} - \frac{w^3}{2}. \] (46)

We have retained the contribution of both bosons and fermions in the non-perturbative part.

Hence we have for the ratio of moment
\[ R_\alpha(\sigma) = 4m^2 + \frac{3}{2\sigma} + 4m^2 \left( -a_\sigma \frac{2\sqrt{\pi}}{3} w^{-1/2} + \frac{11 \phi}{12} w + \frac{3 \phi}{2} w^2 \right). \] (47)
To compare (47) with (14) we make the following substitution
\[ w = mx, \quad 2\mu = m \]
\[ (4m)^{-1} [R(\sigma) - 4m^2] = R(\tau) \]
Equation (47) then reduces to
\[ R(\tau) = \frac{3}{2\pi} - \frac{2}{3} \sqrt{\frac{\pi}{\tau}} \left( \frac{\tau}{m} \right)^{-1/2} + \frac{11\phi_1}{192} \frac{1}{m} (\tau/m) + \frac{3\phi_1}{32} + m(\tau/m)^2. \] (48)
Comparing (14) and (48) one gets the following non-relativistic potential.
\[ V(r) = -\frac{4a_s}{3} + \frac{11\phi_1}{384} \frac{1}{m} r^2 + \frac{\phi_1}{64} m r^4. \]
This potential contains the usual coulombic term, and the leading term in the confining part is the quartic term. Further the confining part also has the same flavour dependence (equation (15)).
Thus we conclude that including both spin \( \frac{1}{2} \) and spin zero object loops contributing to the vacuum polarisation function does not alter the quartic confining nature of the potential.

References

Akhiezer and Berestetskii 1965 Quantum Electrodynamics Vol. 1 848
Schwinger J 1973 Particles, sources and fields (Addison-Wesley) Vol. 2
Wilson K 1969 Phys. Rev. 179, 499
Calculation of nonperturbative QCD parameters

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To calculate the values of quark and gluon condensates, the nonperturbative part of the vacuum-polarization tensor is extrapolated in an optimal manner from low to fairly high values of negative $s$. The extrapolation is made using a variable obtained by conformally mapping the cut plane of analyticity of a polarization function into the inside of a suitable ellipse. Taking the canonical value of the quark condensate, the gluon condensate is found to have the value $G = (0.0115 \pm 0.0015) \text{GeV}^4$.

The $e^+e^-$ data combined with the known analyticity property of the vacuum-polarization function can be used to determine the nonperturbative QCD parameters. Such calculations were first reported by Shifman, Vainshtein, and Zakharov (SVZ).

The vacuum-polarization function $\Pi(s)$ is a real analytic function in the complex $s$ plane with a cut along the real axis from $4m_q^2$ to $\infty$. It satisfies a once-subtracted dispersion relation of the form

$$ \Pi(s) = \frac{s}{\pi} \int_{4m_q^2}^\infty \frac{\text{Im} \Pi(s') ds'}{s'(s' - s)} \quad (1) $$

The imaginary part is

$$ \text{Im} \Pi(s) \bigg|_{s > 4m_q^2} = R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \text{had})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \quad (2) $$

and $R(s)$ (Ref. 2) has been measured up to a c.m. energy of about 38 GeV with $s_{\text{max}} \approx (38 \text{ GeV})^2$. If $R(s)$ were known for all $s$ one could easily calculate $\Pi(s)$ for all $s$. In particular, for negative $s$ one can use an optimal extrapolation technique using conformal mapping. This novel idea is presented below.

In a region of moderately high spacelike four-momenta $\Pi(s)$ can be described by a Wilson-type operator expansion technique where the order parameters appear as coefficients of $1/s^1$, $1/s^2$, etc. These constitute the nonperturbative part of the polarization function $\Pi(s)$ and are responsible for explaining the confinement mechanism.

To isolate the nonperturbative part, one needs the perturbative part

$$ [\Pi^{\text{pert}}] = \frac{s}{\pi} \int_0^s \frac{\text{Im} \Pi^{\text{pert}}(s') ds'}{s'(s' - s)} + K \quad (3) $$

where $K$ is an unspecified real constant. To a fairly high degree of reliability

$$ \text{Im} \Pi^{\text{pert}}(s) = \frac{1}{s} \sum_{i=1}^f Q_i \bar{\rho}_i v_i \alpha_i(s-4m_i^2) \left[ 1 + \frac{s}{4} f_2(v_i, \alpha_i(s)) + C \left( \frac{\alpha_i(s)}{\pi} \right)^2 \right] \quad (4) $$

where $Q_i$ is the charge and $m_i$ is the mass of the $i$th quark, $v_i$ is its velocity.

The strong coupling is

$$ \alpha(s) = \frac{12\pi}{(33 - 2n_f) \ln(s/\Lambda^2)} \left[ 1 - \frac{462 \ln\ln(s/\Lambda^2)}{625 \ln(s/\Lambda^2)} \right] \quad (7) $$

and

$$ C = 1.986 - 0.175 n_f \quad (8) $$

where $n_f$ is the number of quark flavors.

Thus the nonperturbative part

$$ [\Pi^{\text{nonpert}}] = \frac{s}{\pi} \int_0^s \frac{R(s' - \text{Im} \Pi^{\text{pert}}(s')) ds'}{s'(s' - s)} - K \quad (9) $$

is to be extrapolated to sufficiently large negative values of $s$ to obtain the parameters of Wilson expansion.

$$ [\Pi^{\text{nonpert}}] = \frac{B}{s^2} + \frac{C}{s} + \cdots \quad (10) $$

for large $s$. Following SVZ (Ref. 1) and Caprini and Verzegnassi we write the QCD-calculated value of $B$ as

$$ B = Q + \gamma G \quad (11) $$

where $Q$ is the contribution of the light-quark vacuum condensate

$$ Q = 24\pi \sum_{i=1}^3 m_i \langle 0 | q_i \bar{q}_i | 0 \rangle \quad (12) $$

containing the renormalization-group-invariant quantities $m_i \langle 0 | q_i \bar{q}_i | 0 \rangle$ and $G$ is the renormalization-group-invariant gluon condensate

$$ G = \left( \langle 0 | G_{\mu\nu}^a G_{\rho\sigma \mu\nu}^a | 0 \rangle \right) \quad (13) $$

The coefficient $\gamma$ which multiplies $G$ is written as

$$ \gamma = \left( \frac{s}{4\pi^2} \right) \left( \frac{\alpha(s)}{\pi} \right)^2 \quad (14) $$

and

$$ f_2(v_i, \alpha_i(s)) = \frac{\pi^2}{4} \left[ 1 + \frac{\alpha_i(s)}{2 \pi} \right] \quad (15) $$

with

$$ f_1(v_i, \alpha_i(s)) = \frac{\pi^2}{4} \left[ 1 + \frac{\alpha_i(s)}{2 \pi} \right] \quad (16) $$

The coefficient $\gamma$ which multiplies $G$ is written as
\[ \gamma = \sum_{i=1}^{3} 2q_i^2 + \pi \sum_{i=1}^{3} Q_i \mathcal{F}_i(s), \]

where the sum \( i = 1,3 \) in the first term is for the light quarks and the sum is for heavy quarks in the second term

\[ \mathcal{F}_i(a_i) = \frac{3(a_i + 1)(a_i - 1)}{a_i^2} - \frac{1}{2\sqrt{a_i}} \ln \sqrt{a_i} + 1 \]

\[ - \frac{3a_i^2 - 2a_i + 3}{a_i^2}, \]

\[ a_i = 1 - \frac{4m_i^2}{s}. \]

The integral representation for this function which is useful for calculating derivatives, is

\[ \frac{m_i^2}{3} \left[ \frac{1}{6} - \frac{4x(1-x)}{3} \right] + \frac{m_i^4}{3} \left[ \frac{1}{6} + \frac{x(1-x)}{3} \right] + \frac{m_i^2 - x(1-x)l}{m_i^2 - x(1-x)s}. \]

From Eqs. (9) and (10) one can also construct the moment equations given by

\[ M_n(s_0) = \frac{1}{n!} \frac{d^n \Pi(s)}{ds^n} \bigg|_{s = s_0}, \]

so that

\[ M_n(s_0) = \frac{1}{n!} \frac{d^n \Pi(s)}{ds^n} \bigg|_{s = s_0}, \]

for sufficiently large negative \( s \).

Consider Eq. (9) for \( \Pi^{\text{open}}(s) \). Here \( s/s_0 \) can be taken as an expansion parameter. It has already been shown by SVZ (Ref. 1), Bertlmann, \( B \) and Reinders, Rubenstein, and Yazaki \( R \) (RRY) that the asymptotic limit is reached, by about 1 GeV. Therefore \( s_0 \) is greater than 1 GeV\(^2\). One can expand \( \Pi^{\text{open}}(s) \) or any of its moment in a Taylor series for small \( s \) as

\[ \Pi^{\text{open}}(s) = \sum_{n} a_n \left| \frac{s}{s_0} \right|^n \text{ for } s < s_0 \].

Thus there are two representations for \( \Pi^{\text{open}}(s) \) valid in different regions: Eq. (10) for large \( s \) and Eq. (20) for small \( s \). One wishes to set the corresponding equations for moments equal, evaluate Eq. (20) using experimental input \( R \), and thus determine the QCD parameters in Eq. (10). This requires that there should be a common region of validity. However in normal analysis the range of \( s \) in which both expressions for \( \Pi^{\text{open}}(s) \) are valid is very limited. By mapping the cut plane into an ellipse and expanding in the mapped variable the range of applicability of a series expansion is considerably increased. As a result an extrapolation can "see" the terms containing the QCD parameters of Eq. (10).

To be explicit consider the difference function

\[ F(s) = \Pi^{\text{open}}(s) - \frac{B}{s^2} - \frac{C}{s} \].

This function or its derivatives will have a smooth changeover from low \( s \) to large \( s \) provided the value of \( B \) and \( C \) are correctly subtracted. What we mean is that the goodness of the low-energy fit to the function in Eq. (21), or its derivatives, will be reflected in a correct subtraction of the coefficients \( B \) and \( C \). The low-energy fit in general will contain several terms depending on the rate of convergence of the Taylor series. However, there exists a method first proposed by Cutkosky and Deo and by Cui, \( J \) by which such series can be made optimally convergent.

The rate of convergence of the polynomial expansion is determined by the singularity structure of the function. In our case the function has only a cut from 0 to \( \infty \) in the complex \( s \) plane [see Fig. 1(a)]. We are interested in extrapolating the values of the dispersion integrals to negative values of \( s \), say from \(-0.2 \) to \(-1.2 \) GeV\(^2\). This range will cover the low-energy region as well as overlap with the asymptotic region. The mapping

\[ W = 1.4 + 2s \]

will bring this region to lie between \( +1 \) to \(-1 \) with the cut starting from 1.4 GeV\(^2\) to \( \infty \). However to optimize the rate of convergence of the polynomial function the entire singularity-free \( s \) plane is mapped into the interior of a unifocal ellipse, the cut lying on the boundary of the ellipse of the mapped \( z \) plane [Fig. 1(b)]. The mapping is such that the region of interest \(-0.2 \) to \(-1.2 \) GeV\(^2\) still lies between \( +1 \) to \(-1 \). The ellipse is given by

\[ z = \sin \frac{1}{k} F(\arcsin k) \]

where \( k = 1/\omega \) and \( K(k) \) and \( F(\phi,k) \) are the complete and incomplete elliptic integrals of the first kind. The functions of interest are now expanded in a complete set of orthogonal polynomials \( P_n(z) \). This expansion is expected to be valid for fairly large values of \( s \) if \( B \) and \( C \)
If $Q$ and $G$ of $B$ and $C$ were not exact, the high-energy tail will not be correctly subtracted. As a result of non-convergence $\chi^2$ will be large for any order of truncation of the expansion.

To begin with the actual calculations, the term $C/s^3$ was assumed to be small. Both the contents $G$ and $Q$ of $B/s^2$ were first varied. As we change the values of $g$ and $G$ in $B$, $\chi_1^2$ changes. For the correct values of $Q$ and $G$, the $\chi^2$ value should reduce to a minimum exhibiting a pronounced dip in the $\chi^2$-$G$ plot, for different $Q$ values. This is found to be the case as can be seen from Fig. 2.

The actual errors reported by different groups for experimental data on $R(s)$ vary widely from experiment to experiment, especially in the low positive-$s$ region, which is the region of interest. For calculation of the errors to the polarization function and its moments it is the normal practice to give a uniform error to $R(s)$ and compute the error by evaluating the dispersion integral. In our calculation we have used an error varying from 2% to 5%. Our results are not very sensitive to the above variation.

The orthogonal polynomials are then constructed with the experimental errors $W(z)$ as their weights by Schmidt's orthogonal procedure

$$\int_{-1}^{1} dz W(z) P_n(z) P_m(z) = \delta_{mn}. \quad (24)$$

The functions $F(s)$ are approximately written as a finite series.

$$F(s) = \sum \alpha_n P_n(z). \quad (25)$$

From a fit to the data with a series of $L$ terms the $\chi_L^2$ is obtained as

$$\chi_L^2 = \left| \sum_i \frac{F_i(s) - \sum \alpha_n P_n(z)}{W_i} \right|^2. \quad (26)$$

If $Q$ and $G$ of $B$ and $C$ were not exact, the high-energy tail will not be correctly subtracted. As a result of non-convergence $\chi^2$ will be large for any order of truncation of the expansion.

To begin with the actual calculations, the term $C/s^3$ was assumed to be small. Both the contents $G$ and $Q$ of $B/s^2$ were first varied. As we change the values of $Q$ and $G$ in $B$, $\chi^2$ changes. For the correct values of $Q$ and $G$, the $\chi^2$ value should reduce to a minimum exhibiting a pronounced dip in the $\chi^2$-$G$ plot, for different $Q$ values. This is found to be the case as can be seen from Fig. 2.
We find that the first few moments do not show the minimum as the number of the required polynomial fit is larger. Starting from the seventh moment the dip is quite pronounced. For such values of \( N > 7 \) the 'cutoff' \( \Lambda \) is varied in a wide range (50—1000 MeV). There was no appreciable change; the values of \( G \) obtained for different values of \( Q \) remained within experimental error which is about \( \pm 10\% \).

The calculation of \( \chi^2 \) made for the twelfth moment for different values of \( G \) and \( Q \) are shown in the \( \chi^2 - G \) plots of Fig. 2. A plot, Fig. 3, between \( Q \) and \( G \) for minimum \( \chi^2 \) turns out to be a straight line indicating that the coefficient \( \gamma \) does not exhibit a significant \( s \) dependence, its value being constant at 2.08. One determines \( B = Q + \gamma G \) to be \( -1 \times 10^{-3} \text{ GeV}^4 \), so it is not possible to pinpoint a unique value of \( Q \) and \( G \) separately. This point has also been noted by Caprini and Verzegnassi. Following Gell-Mann, Oakes, and Renner if we take

\[
Q = -0.024 \text{ GeV}^4
\]

we obtain

\[
G = (0.0115 \pm 0.0001) \text{ GeV}^4.
\]

This value of \( G \) is in agreement with the results obtained by previous workers. To have an idea of the admissible values of \( C \) of Eq. (10), we kept the value of \( G \) and \( Q \) fixed at the minimum \( \chi^2 \) point and changed \( C \). Fortunately we did find a minimum for \( C \) close to

\[
C = -0.5 \times 10^{-4} \text{ GeV}^4.
\]

Thus the series given by Eq. (10) is really converging. This value of \( C \) is even smaller than the value reported by Caprini and Verzegnassi.

\^\cite{1} M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979); B147, 519 (1979).
\^\cite{2} D. H. Saxon, Hadron. J. 6, 455 (1983).
\^\cite{4} K. Wilson, Phys. Rev. 179, 1499 (1969).
\^\cite{7} E. D. Bloom, Report No. SLAC-PUB-3573, 1985 (unpublished).
\^\cite{11} M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
Nonperturbative QCD order parameters from $e^+e^-$ data by analytic continuation

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We suggest a method of finding an analytic function which has an imaginary part directly related to the $e^+e^-$ annihilation data and which leads to the determination of Wilson-operator-product-expansion coefficients of vector-current-current correlation functions.

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Recently, we have calculated the gluon condensate parameter using the conformal-mapping technique to extrapolate the vacuum-polarization tensor in an optimal manner to negative values of $s$. The value of the parameter so obtained was in agreement with other reported results. In this Brief Report we want to calculate the order parameters of QCD by another method, improving on the suggestion made by Caprini and Verzegnassi. The Fourier transform of the vector-current-current correlation function in QCD can be written as

$$
\int d^4x e^{i k \cdot x} \langle 0 | J_{\mu}(x) J_{\nu}(0) | 0 \rangle = \langle q_{\mu} q_{\nu} - q^2 g_{\mu\nu} \rangle \pi(s),
$$

where $s = -q^2$.

$
\pi(s)
$

has well-known analytic properties. We shall work with the first derivative of the polarization function $\pi'(s)$, having the same analytic properties and which satisfies the dispersion relation

$$
\pi'(s) = \frac{1}{\pi} \int_{4m_f^2}^{s} \frac{\text{Im} \pi(s') ds'}{(s'-s)^2},
$$

where

$$
\text{Im} \pi(s) \bigg|_{s > 4m_f^2} = R(s) \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)},
$$

is assumed to be known. The problem is to extrapolate the function $\pi'(s)$ to negative values of $s$ with the constraint that the perturbative and nonperturbative vacuum-polarization functions satisfy the following known equations:

$$
\pi'(s) = \pi', \quad \pi' = \int_{4m_f^2}^{s} \frac{\text{Im} \pi(s') ds'}{(s'-s)^2},
$$

$\pi'$ being the number of quark flavors.

What is more important is that QCD order parameters $B$ and $C$ are given by

$$
u_i = \left(1 - \frac{4m_f^2}{s} \right)^{1/2},
$$

and

$$
\alpha_i = \left( \frac{12\pi}{(33-2n_f) \ln(s/\Lambda^2)} \right) \left(1 - \frac{462 \ln(s/\Lambda^2)}{625 \ln(s/\Lambda^2)} \right).
$$

The strong coupling is

$$
\alpha_i(s) = \frac{12\pi}{(33-2n_f) \ln(s/\Lambda^2)} \left(1 - \frac{462 \ln(s/\Lambda^2)}{625 \ln(s/\Lambda^2)} \right),
$$

where $n_f$ is the number of quark flavors.

$B$ and $C$ vs. $n_f$.

FIG. 1. Plot of $B$ and $C$ vs. $n_f$.
for large negative values of \( s \) as predicted by Wilson-operator product expansion.\(^5,6\) The asymptotics is supposed to reach quite early by about 0.8 GeV\(^2\). For larger values of \( s \) much greater than 1 GeV\(^2\), one has to consider higher-terms in inverse powers of \( s \) in the Wilson expansion.\(^7\)

The polarization function and its derivative are analytic in the entire complex \( s \) plane except for a cut from \( 4m_\pi^2 \) to \( \infty \). Information known from the significant portion of this cut is to be used to find the value of the functions in the region \(-0 \rightarrow -\infty\). For our purpose it will be enough if we extrapolate to the region \(-0 \rightarrow -s_+\), where by assigning values near 1 GeV\(^2\) to \( s_+\), we shall cover both low and asymptotic regions for Eq. (6). We conformally\(^2,8\) map the \( s \) plane into a \( w(s) \) plane, where the entire region of analyticity is within a unifocal ellipse and the cut \( 4m_\pi^2 \) to \( \infty \) forming the boundary of the ellipse and the portion \(-0 \rightarrow -s_+\) of the real negative axis is mapped from \(-1 \rightarrow -1\) in the \( w(s) \) plane. Details of the mapping are given in Ref. 1.

We shall consider minimizing a quantity \( d^2 \) which involves a contour integral over the unifocal ellipse of the square of a function \( F(w;B,C) \), depending on \( B \) and \( C \),

\[
d_2 = \left| \frac{\partial u}{\partial w} \right|^{1/2} \left| F(w;B,C) \right|^2 \right|^{1/2} \tag{10}
\]

The contour integral is difficult to evaluate. However, if the function \( F \) is analytic within the ellipse, then denoting the real part of \( w \) as \( x \), we can also write

\[
d_2 = \left| \int_{-1}^{-1} \left( 1 - x^2 \right)^{-1/2} \left| F(x;B,C) \right|^2 \right|^{1/2} \tag{11}
\]

This function \( F \) can be expanded in an optimally converging series in Chebyshev polynomials. If, for instance,

\[
F = \sum_n f_n T_n(x),
\]

then

\[
d_2 = \left| \sum_n f_n^2 \right|^{1/2} \tag{13}
\]

The coefficients \( f_n \) will become smaller for larger \( n \). Let us examine the function

\[
\varphi(s_\omega) = \frac{1}{\pi} \int_{s_{\text{max}}}^\infty \frac{\text{Im} \, \text{part} \, s} {\left( s - s_\omega \right)^2} \, ds',
\]

where \( s_\omega \) is a point on the ellipse and \( s_{\text{max}} \) is equal to the semimajor axis of the ellipse. Since \( s \) on the ellipse has a magnitude less than \( s' \), this function can be expanded in a power series in \( s \). However the asymptotic behavior is governed by

\[
\text{Table showing the values of the coefficients } f_n, \quad \text{with increasing values of } a, \quad \text{for increasing values of } n.
\]

\begin{tabular}{|c|c|c|}
\hline
\( n \) & \( f_n \) & \( s_n \) \\
\hline
1 & 0.3 & 0.51 \times 10^{-4} \\
2 & 0.51 \times 10^{-4} & 0.55 \times 10^{-4} \\
3 & 0.6 \times 10^{-4} & 0.5 \times 10^{-4} \\
4 & 0.5 \times 10^{-4} & 0.4 \times 10^{-4} \\
5 & 0.4 \times 10^{-4} & 0.3 \times 10^{-4} \\
6 & 0.3 \times 10^{-4} & 0.2 \times 10^{-4} \\
7 & 0.2 \times 10^{-4} & 0.1 \times 10^{-4} \\
8 & 0.1 \times 10^{-4} & 0.0 \times 10^{-4} \\
\hline
\end{tabular}
It is to be noted that $s$ in this equation will be varying from $-s$ to 0, and the $x$'s are the corresponding values on the real axis of the $s$ plane going from 1 to +1.

The function $F$ can now be expanded in an optimally convergent way in a series of Chebyshev polynomials. Let

\[ \frac{2}{s}(x-1)^n = \sum_{n=0}^{\infty} a_n T_n(x) , \]

\[ -\frac{1}{s}(x-1)^n = \sum_{n=0}^{\infty} \beta_n T_n(x) , \]

\[ (x-1)^n \int_0^{s_{\text{max}}} ds' \frac{\text{Im} \pi_{n,m}^{\text{non-pert}}(s')}{(s'-s)^2} = \sum_{n=0}^{\infty} \gamma_n T_n(x) . \]

Then

\[ d_2^2 = \sum_{n=0}^{\infty} (a_n B + \beta_n C - \gamma_n)^2 . \]

To minimize $d_2$, we differentiate the Eq. (22) with respect to $B$ and $C$ and equate the resulting expressions to zero. We then obtain the two equations

\[ B \sum a_n^2 + C \sum \alpha_n \beta_n = \sum \gamma_n a_n , \]

\[ B \sum a_n \beta_n + C \sum \alpha_n \beta_n = \sum \gamma_n b_n , \]

from which $B$ and $C$ can be easily calculated

\[ \sum \gamma_n a_n \sum \beta_n - \sum \gamma_n \beta_n \sum a_n \beta_n \]

\[ = \sum a_n \sum \beta_n - \left( \sum \alpha_n \beta_n \right) ^2 \]

\[ \sum \gamma_n \beta_n \sum a_n - \sum \gamma_n a_n \sum a_n \beta_n \]

\[ = \sum a_n \sum a_n \beta_n - \left( \sum \alpha_n \beta_n \right) ^2 \]

To calculate $\gamma_n$ we need the $e^+ e^-$ annihilation data which we have taken from Ref. 9. The value of $s_{\text{max}}$ as given by the mapping is about 3 GeV$^2$ covering all the important resonances.

The coefficients $\alpha_n$, $\beta_n$, and $\gamma_n$ become smaller with increasing $n$ as can be seen from the Table I. We also exhibit the values of $B$ and $C$ as they vary with a larger number of coefficients ($n$) in Fig. I. We have also estimated the error in $B$ and $C$ due to 10% error in the $e^+ e^-$ data. The results of the computation are

\[ B = -(10.6 \pm 1.4) \times 10^{-4} \text{ GeV}^4 , \]

\[ C = -(0.3 \pm 0.1) \times 10^{-4} \text{ GeV}^4 . \]

These agree with our previously reported values. 1

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