Chapter 3

TOPOLOGICAL
SUPERCONFORMAL
SYMMETRIES

In this chapter we study the algebraic aspects of some topological symmetries. As described in the introduction the topological theories have interesting properties like independence of the correlations on the local structure of the manifold and truncation of the spectrum to a finite number of fields. In this chapter we shall address the problem of finding higher superconformal symmetries in the theories with topological symmetry. A step in this direction was taken in Ref.[5] where an $N = 1$ topological superconformal symmetry was found by twisting the $N = 3$ superconformal algebra presented in last chapter. We further generalize this procedure and obtain a $N = 2$ topological superconformal field theory in Section 3.1. In Section 3.2 we present a topological Kac-Moody symmetry using the Wakimoto realization.
for the \( SU(2) \) Kac-Moody algebra. In Section 3.3, following closely the description in Section 3.1, we present a brief discussion of topological \( N = 2 \) superconformal algebra on higher genus Riemann surfaces using the Krichever-Novikov formalism.

3.1 \( N = 2 \) Topological Superconformal Algebra

In this Section we construct \( N=2 \) topological superconformal field theory (TSCFT). Since it was found Ref[5] that twisting of \( N = 3 \) superalgebra gives an \( N = 1 \) TSCFT we must start with \( N=4 \) superconformal theory to get an \( N = 2 \) TSCFT.

As we have seen in Chapter 2, there exist several versions of \( N=4 \) superconformal algebras[6, 7]. It is important to note that not all of them can be twisted to give an \( N=2 \) TSCFT. For example, the \( N=4 \) superconformal algebra of Ademollo et al [6] has only one underlying \( SU(2) \) Kac-Moody symmetry and therefore only a single \( U(1) \) with respect to which the energy-momentum tensor can be twisted. Since all the four superconformal generators have nonzero charge under this \( U(1) \), weights of all the generators shift in the twisted theory and hence, there is no residual supersymmetry in the resulting topological field theory. Thus for our purpose we have to start with the Sevrin et. al.’s superalgebra.

Now since we already know that twisting of \( N = 3 \) superconformal theory gives \( N = 1 \) supersymmetry for the topological theory therefore if we can get an \( N = 3 \) substructure of the Sevrin et. al.’s algebra then we, at least, will get an \( N = 1 \) structure in our topological theory. Therefore we start by showing that the \( N = 4 \)
superconformal algebra of Ref.[7] has an underlying N=3 superconformal symmetry. The N=3 subalgebra is generated for arbitrary $\gamma$ by eight of the generators of the original N=4 superconformal algebra. They are the diagonal SU(2) generators $(J_L)_m^i$, a spin-$\frac{1}{2}$ generator $\Gamma_r$, three spin-3/2 generators $\tilde{G}_r^i$ and the modified energy-momentum operator $\tilde{L}_m$ defined by

\begin{equation}
(J_L)_m^i \equiv i(A_m^{+i} + A_m^{-i}), \quad \Gamma_r \equiv -iQ_r^i, \quad (3.1.1)
\end{equation}

\begin{equation}
\tilde{G}_r^i \equiv G_r^i + (2\gamma - 1)(r + \frac{1}{2})Q_r^i, \quad (3.1.2)
\end{equation}

\begin{equation}
\tilde{L}_m \equiv L_m + \frac{1}{2}(2\gamma - 1)(m + 1)U_m. \quad (3.1.3)
\end{equation}

The remaining eight generators of the N=4 algebra form a weight zero superfield with respect to this N=3 subalgebra. The components of this superfield are $\tilde{G}_r^i, (J_R)_m^i, \tilde{Q}_r^i$, and $2iU_m$ with spin $3/2, 1, 1/2$ and $0$ respectively defined by

\begin{equation}
\tilde{G}_r^4 \equiv G_r^4 + (2\gamma - 1)(r + \frac{1}{2})Q_r^4, \quad (3.1.4)
\end{equation}

\begin{equation}
(J_R)_m^i \equiv -(A_m^{+i} - A_m^{-i}), \quad (3.1.5)
\end{equation}

\begin{equation}
\tilde{Q}_r^i \equiv iQ_r^i, \quad 2iU_m \equiv -mu_m. \quad (3.1.6)
\end{equation}

The generators $\tilde{L}_m$ of the modified Virasoro algebra have a modified central charge $\frac{c}{4\gamma(1-\gamma)}$. It is easily verified that the newly defined generators have above mentioned weights except a central term in $[\tilde{L}_m, u_n]$. We also find the following (anti)commutators:

\begin{equation}
\{\tilde{G}_r^i, \tilde{G}_s^j\} = 2\tilde{L}_{r+s} + \frac{1}{3} \frac{c}{4\gamma(1-\gamma)}(r^2 - \frac{1}{4})\delta_{r+s,0} \quad (no \ sum \ on \ i)(3.1.7)
\end{equation}

\begin{equation}
\{\tilde{G}_r^i, \tilde{G}_s^j\} = i(r-s)c^{ijk}(J_L)_m^k, \quad [\tilde{G}_r^i, (J_L)_m^j] = i\epsilon^{ijk}\tilde{G}_m^{+r}, \quad (3.1.8)
\end{equation}
\[(J_L)^i_m, \Gamma_r = 0, \quad [(J_L)^i_m, \tilde{G}_r^i] = m\Gamma_{m+r} \quad \text{(no sum on } i)\), \quad (3.1.9)\]

\[[(J_L)^i_m, (J_L)^j_n] = i\epsilon^{ijk}(J_L)^k_{m+n} + \frac{c}{4\gamma(1-\gamma)} \frac{m}{3}\delta_{ij}\delta_{m+n,0}, \quad (3.1.10)\]

\[{\Gamma_r, \Gamma_s} = \frac{1}{34\gamma(1-\gamma)}\delta_{r+s,0}, \quad \{\Gamma_r, \tilde{G}_s^i\} = (J_L)^i_{r+s}. \quad (3.1.11)\]

Which are the (anti)commutators of the \(N = 3\) superconformal algebra as can be seen by comparing with the (anti)commutators of \(N = 3\) superalgebra in Section 2.1. Thus the generators defined in eqs.(3.1.1)-(3.1.3) satisfy an \(N=3\) superconformal algebra with central charge \(\frac{c}{4\gamma(1-\gamma)}\). Next to prove that the remaining \(N=4\) generators \(u_m, \tilde{Q}_r^i\) and \((J_R)^i_m\), defined eqs.(3.1.4)-(3.1.6) transform as spin-0 “primary” superfield with respect to the \(N=3\) generators \(\tilde{G}_r^i, (J_L)^i_m, \Gamma_r\), modulo the central terms in some of the (anti)commutators, we compute:

\[\{\tilde{G}_r^i, \tilde{G}_s^4\} = (r-s)(J_R)^i_{r+s}, \quad [\tilde{G}_r^i, (J_R)^i_m] = \tilde{G}_r^{i+m}, \quad (3.1.12)\]

\[\{\tilde{G}_r^i, \tilde{Q}_s^i\} = -\frac{1}{2}(r+s)u_{r+s} - (2\gamma - 1)(r + \frac{i}{2})\frac{c}{34\gamma(1-\gamma)}\delta_{r+s,0}, \quad (3.1.13)\]

\[[\tilde{G}_r^i, (J_R)^i_m] = -i\epsilon^{ijk}\tilde{Q}_{r+m}^k, \quad \{\tilde{G}_r^i, \tilde{Q}_s^j\} = i\epsilon^{ijk}(J_R)^k_{r+s}, \quad (3.1.14)\]

\[[\tilde{G}_r^i, u_m] = 2\tilde{Q}_{r+m}^i, \quad [(J_L)^i_m, \tilde{G}_r^i] = m\tilde{Q}_m^i, \quad (3.1.15)\]

\[[(J_L)^i_m, (J_R)^i_n] = -(2\gamma - 1)m\frac{i}{3}\frac{c}{4\gamma(1-\gamma)}\delta_{{m+n},0} \quad \text{(no sum on } i)\), \quad (3.1.16)\]

\[[(J_L)^i_m, (J_R)^i_n] = i\epsilon^{ijk}(J_R)^k_{m+n}, \quad [(J_L)^i_m, \tilde{Q}_r^i] = i\epsilon^{ijk}\tilde{Q}_m^k, \quad i \neq i, \quad (3.1.17)\]

\[[(J_L)^i_m, u_n] = = [(J_L)^i_m, \tilde{Q}_r^i] = 0 \quad \text{(no sum on } i)\), \quad (3.1.18)\]

\[\{\Gamma_r, \tilde{G}_s^4\} = \frac{1}{2}(r+s)u_{r+s} + \frac{i}{3}(s + \frac{1}{2})\frac{c}{4\gamma(1-\gamma)}\delta_{r+s,0}, \quad (3.1.19)\]

\[[\Gamma_r, (J_R)^i_n] = \tilde{Q}_{r+n}^i, \quad [\Gamma_r, \tilde{Q}_s^i] = [\Gamma_r, u_m] = 0. \quad (3.1.20)\]

which are consistent with those of \(N = 3\) superfield[8] except a central term in
eqs. (3.1.13), (3.1.15) and (3.1.19) as can be seen by comparing with the transformation properties of $N = 3$ superfields, given in Appendix 2.A, for $\Delta = 0$. Thus we have identified an $N=3$ superconformal structure in the $N=4$ superconformal theory.

The TSCFT is now constructed following Ref.[5]. We begin by redefining some of the generators of the $N=3$ algebra as,

$$G_r^\pm \equiv \frac{\tilde{G}_r^1 \pm i\tilde{G}_r^2}{\sqrt{2}}, \quad J_m^{gh} \equiv (J_{L,m})^3$$  \hspace{1cm} (3.1.21)

It is then easy to check that the operators $\tilde{L}_m$, $G_r^\pm$ and $J_m^{gh}$ satisfy an $N=2$ algebra. The twisting of this $N=2$ algebra, to construct a TCFT, is done in the usual way by defining

$$\mathcal{L}_m \equiv \tilde{L}_m - \frac{1}{2}(m + 1)J_m^{gh}.$$  \hspace{1cm} (3.1.22)

$\mathcal{L}_m$ satisfies the Virasoro algebra without the central term. Due to such modifications, the weight of the supersymmetry generator $G^-$ becomes 2 and that of $G^+$ becomes 1 and therefore we define

$$G_n \equiv G_{n+\frac{1}{2}}, \quad Q_n \equiv G_{n-\frac{1}{2}}.$$  \hspace{1cm} (3.1.23)

The BRST charge is the zeroth component $Q_0$ which satisfies the nilpotency condition. $J_m^{gh}$ is the ghost number current in the TCFT. This is the usual algebraic structure of a topological field theory[1]. Now we show that this algebraic structure is further extended and in addition we also have an $N=2$ superconformal symmetry.

As in Ref.[5], one finds that a modified supercharge

$$G_r = \tilde{G}_r^3 + i(r + \frac{1}{2})Q_r^4$$  \hspace{1cm} (3.1.24)
remains a weight-$\frac{3}{2}$ operator in the TCFT. However the weights of the generators $(J_L)^\pm$ defined by

$$(J_L^\pm)_m \equiv \pm \frac{(J_L)_m^1 \pm i(J_L)_m^2}{\sqrt{2}}$$

(3.1.25)

in the twisted theory are shifted to $\frac{1}{2}$ and $\frac{3}{2}$ respectively. Therefore again as in eqs.(3.1.23) we shift the modes and define

$$(J_L^+)_r \equiv (J_L)_r^{1/2}, \quad (J_L^-)_r \equiv (J_L)_r^{-1/2}.$$  

(3.1.26)

Similar shift of weights occurs for the various fields of the spin-0 N=3 supermultiplet of eqs.(3.1.4)-(3.1.6). The weights of the operators $\tilde{Q}^3$, $u_m$ and

\begin{align*}
G_r^4 &= \tilde{G}_r^4 - i(r + \frac{1}{2})Q_r^3, \\
(J_R^3)_m &= [i(A_m^3 - A_m^{-3}) + U_m], \\
(J_R^\pm)_r &\equiv \frac{1}{\sqrt{2}}[(J_R^+)_r^{\mp1/2} \pm i(J_R^-)_r^{\mp1/2}], \\
(Q^\pm)_m &\equiv \frac{1}{\sqrt{2}}[\tilde{Q}_m^{\mp1/2} \pm i\tilde{Q}_m^{\mp1/2}]
\end{align*}

(3.1.27) - (3.1.30)

as well as other generators of the twisted theory are given in Table-1.

<p>| Table 1 |
|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>Operator</th>
<th>Weight</th>
<th>Operator</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L, G$</td>
<td>2</td>
<td>$G^4, J_R^-$</td>
<td>$3/2$</td>
</tr>
<tr>
<td>$G^3, J_R^-$</td>
<td>$3/2$</td>
<td>$\tilde{Q}^-, J_R^3$</td>
<td>1</td>
</tr>
<tr>
<td>$Q, J^{sh}$</td>
<td>1</td>
<td>$\tilde{J}_R^+, \tilde{Q}^3$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$\tilde{J}_L^+, \Gamma$</td>
<td>$1/2$</td>
<td>$\hat{Q}^+, u$</td>
<td>0</td>
</tr>
</tbody>
</table>

| Table 2 |
|-----------------|-----------------|
| I | $L$ | $G^3$ | $G^4$ | $\tilde{J}_R^3$ |
| II | $G$ | $\tilde{J}_L^-$ | $\tilde{J}_R^-$ | $\tilde{Q}^-$ |
| III | $J^{sh}$ | $\Gamma$ | $\tilde{Q}^3$ | $u$ |
| IV | $Q$ | $\tilde{J}_L^+$ | $\tilde{J}_R^+$ | $\tilde{Q}^+$ |

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There are four weight-\(\frac{3}{2}\) fields in the topological theory we have just constructed. Next we show that two of them, i.e. \(G^3_r\) and \(G^4_r\), the spin-1 current \((\tilde{J}_R)_m^3\), and the conformal generator \(\mathcal{L}_m\) satisfy \(N=2\) superconformal algebra without a central term. Other two spin-\(\frac{3}{2}\) fields become the BRST partners of \(G^3\) and \(G^4\) in the TCFT. For example, it can be checked that

\[
\{G^i_r, G^j_s\} = 2\mathcal{L}_{r+s}, \quad \text{(no sum on } i, i = 3, 4). \tag{3.1.31}
\]

Similarly, one also finds that,

\[
\{G^3_r, G^4_r\} = i(r - s)(\tilde{J}_R)_r^3, \tag{3.1.32}
\]

\[
[(\tilde{J}_R)_m^3, G^3_r] = iG^4_{m+r}, \tag{3.1.33}
\]

\[
[(\tilde{J}_R)_m^3, G^4_r] = -iG^3_{m+r}, \tag{3.1.34}
\]

which establishes the \(N=2\) structure of the topological theory.

After identifying the symmetry generators of the residual \(N=2\) superconformal algebra in the TCFT we now proceed to show that four fields \(u, \Gamma, \tilde{Q}^3\) and \(J^\phi\) form a spin-0 multiplet of this residual \(N=2\) supersymmetry algebra. The "primary" nature of these fields with respect to the twisted conformal generator \(\mathcal{L}_m\) is obvious except a central term in \([\mathcal{L}_m, u_n]\). \(N=2\) "superprimary" properties are seen from the (anti)commutators:

\[
[G^3_r, J^\phi_m] = -m\Gamma_{m+r}, \quad [G^4_r, J^\phi_m] = -m\tilde{Q}^3_{m+r}, \tag{3.1.35}
\]

\[
\{G^3_r, \Gamma_s\} = J^\phi_{r+s} - (r + \frac{1}{2})\frac{1}{34\gamma(1 - \gamma)} \delta_{r+s,0},
\]
These relations are precisely the transformation rules for spin-0 primary superfield of the N=2 superconformal algebra except central terms in eqs. (3.1.36) and (3.1.37) as can be verified with properties of a \( N = 2 \) superfield obtained by appropriate truncation of the properties of \( N = 3 \) superfield given in Appendix 2.A (see also [9]). The transformations with respect to the SU(2) generator, \( (J^3_R)_m \), of the N=2 superconformal algebra are given as:

\[
[(J^3_R)_m, J^{th}_n] = (2\gamma - 1)\frac{1}{3}n\frac{c}{4\gamma(1 - \gamma)}\delta_{m+n,0},
\]

\[
[(J^3_R)_m, \Gamma_s] = i\tilde{Q}^3_{m+s},
\]

\[
[(J^3_R)_m, u_n] = -\frac{2i}{3}\frac{c}{4\gamma(1 - \gamma)}\delta_{m+n,0}.
\]

and are the same as expected from an N=2 superfield[9].

In Table-2 we rearrange all the 16 generators of the original N=4 algebra into four N=2 multiplets. We have already shown the N=2 structure of the multiplets I and III of Table-2. Similar algebra can be done to show the N=2 structure of the multiplets II and IV as well. We now show that in Table-2 multiplets I and IV are the BRST derivatives of II and III respectively. The BRST partnership for the second column of the Table-2 is standard in any TCFT. Explicitly we have the following commutators:
\[ [Q_0, J^3_m] = -Q_m, \quad (3.1.42) \]
\[ [Q_0, G_m] = 2 \mathcal{L}_m. \quad (3.1.43) \]

For the third column of the Table-2 the proof is similar to the case of the Ref. [5] and one obtains it from the following relations

\[ [Q_n, (\bar{J}_L)^r] = n \Gamma_{n+r} - G_{n+r}^3, \quad \{Q_n, G^3_r\} = -n(\bar{J}_L^+)_{r+n}, \quad (3.1.44) \]
\[ \{Q_n, \Gamma_r\} = (\bar{J}_L^+)_{r+n}, \quad [Q_n, (\bar{J}_L^+)_{r}] = 0. \quad (3.1.45) \]

by setting \( n = 0 \). The BRST structure for the other eight operators follows from the following (anti)commutators

\[ [Q_m, \dot{Q}^3_r] = -(\bar{J}_R^+)_{m+r}, \quad [Q_m, u_n] = 2\dot{Q}^+_{m+n}, \quad (3.1.46) \]
\[ [Q_m, (\bar{J}_R^+)_{n}] = \{Q_m, \dot{Q}^+_{n}\} = 0, \quad (3.1.47) \]
\[ [Q_m, (\dot{J}_R^-)_{n}] = G^4_{m+n} + m\dot{Q}^3_{m+n}, \quad (3.1.48) \]
\[ \{Q_m, \dot{Q}^-_{n}\} = i(\bar{J}_R^3)_{m+n} - \frac{i}{3}(2\gamma - 1)\frac{c}{4\gamma(1 - \gamma)} m\delta_{m+n,0}, \quad (3.1.49) \]
\[ \{Q_m, G^4_r\} = m(\bar{J}_R^+)_{m+r}, \quad [Q_m, (\bar{J}_R^3)_{n}] = 0. \quad (3.1.50) \]

Now putting \( m = 0 \) in above (anti)commutators one establishes the fact that the operators \( \dot{Q}^+, \bar{J}_R^+, \bar{J}_L^3, \) and \( G^4 \) are the BRST derivatives of the remaining operators, i.e., \( u, \dot{Q}^3, \dot{Q}^- \) and \( \bar{J}_R^- \) respectively.

## 3.2 Topological Kac-Moody Symmetry

In [1] an explicit construction for TCFT’s were done using the free field realizations. In this Section, we extend the results of [1] to level zero \( SU(2) \) Kac-Moody
Algebra. We show, using the Wakimoto realization of $SU(2)$ Kac-Moody and associated conformal algebra, that this theory is topological at level zero. We now start by recapitulating the essential features of [1] in terms of free field realization. In Feigin-Fuchs construction, the expression for the energy momentum tensor $T(z)$ is

$$T(z) = -\frac{1}{2} (\partial \phi(z))^2 + \alpha_0 \partial^2 \phi(z) \quad (3.2.51)$$

where $\phi(z)$ is a free boson with the two point function:

$$\langle \phi(z)\phi(\omega) \rangle = -\log(z - \omega). \quad (3.2.52)$$

The central charge of the theory, $c = 1 - 12\alpha_0^2$, vanishes for $\alpha_0 = \frac{1}{2\sqrt{3}}$. The screening charge of the Feigin-Fuchs construction, $Q = \oint G(z)dz = \oint e^{i\sqrt{3} \phi(z)}dz$, satisfies the nilpotency condition $Q^2 = 0$. Moreover, using the operator product expansions (OPE's) it can be shown that for $\alpha_0 = \frac{1}{2\sqrt{3}}$

$$\{Q, \tilde{G}(z)\} = T(z), \quad \tilde{G}(z) = e^{-i\sqrt{3} \phi(z)}. \quad (3.2.53)$$

Therefore if $Q$ is identified as the BRST charge, the energy-momentum tensor is BRST exact and the theory is topological. This topological field theory is in fact a twisted $N = 2$ superconformal theory. To see this, we write the energy-momentum tensor as

$$T(z) = T_{N=2}(z) + \frac{1}{2} \partial J(z) \quad (3.2.54)$$

where $T_{N=2}(z) = -\frac{1}{2} (\partial \phi(z))^2$ and $J(z) = \frac{1}{\sqrt{3}} \partial \phi(z)$. It is straightforward to show that the operators $T_{N=2}(z), J(z), G(z)$ and $\tilde{G}(z)$ satisfy the $N = 2$ superconformal
algebra with central charge $c = 1$. We now extend these results to the SU(2) Kac-Moody CFT using the Wakimoto realization [14, 15]. In this realization[15], the SU(2) generators are written as

\begin{align}
J^+(z) &= \omega^+(z), \quad J^0(z) = -i(\omega(z)\omega^+(z) + \frac{1}{2\alpha_0} \partial \phi(z)), \\
J^- &= \omega(z)\omega(z)\omega^+(z) + ik\partial\omega(z) + \frac{1}{\alpha_0} \partial \phi(z)\omega(z)
\end{align}

(3.2.55)

where $\omega(z)$, $\omega^+(z)$ and $\phi(z)$ are free boson fields with the two point functions:

\begin{align}
\langle \omega(z_1)\omega^+(z_2) \rangle &= -(\omega^+(z_1)\omega(z_2)) = \frac{i}{(z_1 - z_2)}, \\
\langle \phi(z_1)\phi(z_2) \rangle &= -\log(z_1 - z_2).
\end{align}

(3.2.56)(3.2.57)

It can be checked explicitly that in this realization the currents (3.2.55) satisfy the SU(2) Kac-Moody algebra with level $k = -2 + \frac{1}{2\alpha_0}$. The Sugawara construction leads to the following expression for energy-momentum:

\begin{align}
T(z) = -\frac{1}{2}(\partial \phi(z))^2 + i\alpha_0 \partial^2 \phi(z) + i\omega^+(z)\partial\omega(z).
\end{align}

(3.2.58)

The corresponding central charge is $c = 3 - 12\alpha_0^2 = \frac{3k}{k+2}$. We specialize to the case of level zero Kac-Moody algebras. In this case we get $\alpha_0 = \frac{1}{2}$ and $c = 0$. Also, the eqs.(3.2.55) and (3.2.58) become

\begin{align}
J^+(z) &= \omega^+(z), \quad J^0(z) = -i(\omega(z)\omega^+(z) + \partial \phi(z)), \\
J^- &= \omega(z)\omega(z)\omega^+(z) + 2\partial \phi(z)\omega(z)
\end{align}

(3.2.59)

and

\begin{align}
T(z) = -\frac{1}{2}(\partial \phi(z))^2 + i\frac{1}{2} \partial^2 \phi(z) + i\omega^+(z)\partial\omega(z)
\end{align}

(3.2.60)
respectively. Here we would like to remark that although the algebra satisfied by the generators (3.2.59)-(3.2.60) has level \( k = 0 \), but it is not a classical algebra. This is because OPE's of the generators involve multi-contractions.

Now we show that the Kac-Moody CFT represented by eqs. (3.2.59)-(3.2.60) is topological and is a twisted version of an \( N = 2 \) superconformal theory. To illustrate the \( N = 2 \) structure we rewrite

\[
T(z) = T_{N=2}(z) + \frac{1}{2} \partial J^{(1)}(z), \quad J^{(1)}(z) = i \partial \phi(z), \tag{3.2.61}
\]

\[
T_{N=2}(z) = -\frac{1}{2} (\partial \phi(z))^2 + i\omega^+(z) \partial \omega(z). \tag{3.2.62}
\]

The supercharges of the \( N = 2 \) theory can also be obtained from the knowledge of the operator content of the level zero Kac-Moody algebra. The two supercharges of the algebra are

\[
G(z) = \omega^+(z) e^{i\phi(z)}, \tag{3.2.63}
\]

\[
\tilde{G}(z) = 2i \partial \omega(z) e^{-i\phi(z)}. \tag{3.2.64}
\]

First of these is the screening operator \( \Phi_+(z) = \omega^+(z) e^{i\phi(z)} \) of the \( SU(2) \) Kac-Moody algebra [15]. The operators (3.2.62)-(3.2.64) satisfy an \( N = 2 \) superconformal algebra with central charge \( c = 3 \).

We now show the topological nature of the original \( k = 0 \) Kac-Moody theory. We define the BRST charge as

\[
Q = \oint G(z) dz \tag{3.2.65}
\]
Now using the usual OPE of $G(z)$ and $\bar{G}(z)$ we find that $T(z)$ is BRST exact, ie, 

$$\{Q, \bar{G}(z)\} = T(z)$$

and the theory is topological. Next by defining the operators,

$$j^+(z) = e^{-i\phi(z)}, \quad j^0(z) = -i\omega(z)e^{-i\phi(z)}, \quad (3.2.66)$$

$$j^-(z) = \omega^2(z)e^{-i\phi(z)} \quad (3.2.67)$$

one obtains,

$$\{Q, j^{\pm,0}(z)\} = J^{\pm,0}(z). \quad (3.2.68)$$

Therefore $J^{\pm,0}(z)$ are BRST exact and $j^{\pm,0}(z)$ are their BRST partners. The primary fields of the $SU(2)$ Kac-Moody CFT in the Wakimoto representation are given by

$$\Phi^j_m(z) = (\omega(z))^{j-m}e^{-2i\omega j\phi(z)}, m = -j, ..., j \quad [15].$$

The conformal weight of the primary fields $\Phi^j_m(z)$ is given by $\Delta^j_m = \frac{j(j+1)}{k+2}$. It is known that due to unitarity considerations [16], $j$ is restricted to $0 \leq j \leq \frac{k}{2}$ and hence in the present case only $j = 0$ primary state survives and has weight zero. Moreover, due to BRST exactness of $T(z)$ and $J^{\pm,0}(z)$, Virasoro as well as Kac-Moody secondary states are absent from the BRST cohomology.

### 3.3 Topological Symmetries on Higher Genus

In this Section we give brief summary of a formulation of topological field theories of Section 3.1 on higher genus compact Riemann surfaces. We start with a review of Krichever-Novikov(KN) approach [18, 19, 20]. The essential features of the K-N formalism are as follows. Let us consider a Riemann surface $\Sigma_g$ with genus
$g > 1$ having two distinguished punctures $P_{\pm}$ corresponding to the points $z = 0$ and $z = \infty$ on the complex plane. It is possible to construct local complex coordinate system $z_{\pm}$ in the neighbourhood of these punctures such that $P_{\pm}(z_{\pm}) = 0$. Using Riemann Roch Theorem Krichever and Novikov (K-N) obtained a set of complete orthonormal bases for the space of meromorphic forms of weight $\lambda$ on $\Sigma_g$. Details of this construction can be found in [18, 20] and [12]. In case of half integral weight $\lambda$ there are two different cases: for Ramond sector the space of forms is holomorphic outside $P_{\pm}$ and a slit from $P_+$ to $P_-$ along a Jordan curve and for Neveu-Schwarz sector it is holomorphic outside $P_{\pm}$. In our discussion we restrict ourself to N-S sector only which is the relevant sector for the TCFT obtained after twisting.

We now apply the K-N formalism and derive a generalization of the Eguchi-Yang prescription to twist an N=2 SCFT to obtain a TCFT[12]. The $N = 2$ superconformal generators $T(P)$, $G_{\pm}(P)$ and $J(P)$ on $\Sigma_g$ are now meromorphic forms of weights 2, 3/2, and 1 respectively, where $P$ is a generic point on $\Sigma_g$. Using the K-N bases of meromorphic forms, these operators can be expanded in terms of their modes $L_n$, $G_{\pm}^s$ and $J_n$ respectively. To derive the topological theory we have the following higher genus analog of the twisting operation.

$$\tilde{T}(P) = T(P) + \frac{1}{2} d_P J(P)$$

(3.3.69)

where $\tilde{T}(P)$ is the new energy-momentum tensor and $d_P$ is a global derivative on $\Sigma_g$ having the local form $dz_{\pm} \partial_{z_{\pm}}$ in the coordinate system $z_{\pm}$ mentioned earlier. We note that the new Virasoro algebra satisfied by the modes of $\tilde{T}(P)$ is centerless.
Also the generators $\tilde{G}(P)$ and $\tilde{Q}(P)$ are weight two and one tensors w.r.t. the new energy momentum tensor.

To obtain the topological version of K-N algebra we use the Fourier projections $\hat{L}_m, \hat{G}_r, \hat{Q}_r$ and $J_n$ of the operators $\hat{T}(P), \hat{G}(P), \hat{Q}(P)$ and $J(P)$. The algebra is obtained by constructing the commutators as

$$[\hat{L}_m, \hat{L}_n] = \frac{1}{(2\pi i)^2} \oint_{C_r} dw e_n(w) \oint_C dz e_m(z) \hat{T}(w)$$

(3.3.70)

where the contour $C$ envelopes the point $w$. The algebra is found to be

$$[\hat{L}_m, \hat{L}_n] = \sum_{s=g/2}^{s=-g/2} A_{mn}^s \hat{L}_{m+n-s}$$

(3.3.71)

where the full details of the algebra including expression for the structure constants $A_{mn}^s$ and the information about the meromorphic forms $e_m(z)$ can be found in [12].

The BRST charge in the twisted version of the theory is defined as an integral of the new dimension one operator $\tilde{Q}(P)$ over a closed cycle on $\Sigma_g$. This may be chosen as one of the level curves $C_r$. With the use of the appropriate mode expansion and the asymptotic forms for the basis in the local complex co-ordinate system $z_{\pm}$ around the punctures $P_{\pm}$ the BRST charge on $\Sigma_g$ is $Q_B = \tilde{Q}_{g/2}$. The anticommutator of $\tilde{G}_m$ with $Q_B$ gives a linear combination of the modes of the energy momentum tensor with appropriate coefficients. This is unlike the case on the complex plane. Thus we conclude that the condition that the energy momentum tensor is a BRST derivative of the other weight two generator is modified on the Riemann surface. We state that now we have a generalized BRST derivative condition for the energy momentum tensor. Above expressions reduce to their familiar forms[1] on the complex plane.

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when evaluated in a suitable local coordinate system. In [12] it was observed that the physical subspace of the Hilbert space of the theory on the Riemann surface consists of the chiral primary states of the original theory upto a BRST exact state. All these were shown to be of dimension zero w.r.t. the twisted energy momentum tensor.

It is straightforward to obtain $N = 2$ Toplogical Superconformal field theories on $\Sigma_2$. One starts with K-N formulation of $N=4$ SCFT on $\Sigma_g$. As for the complex plane case the algebra consists of the energy momentum tensor $T(P)$, four weight $3/2$ supercurrents $G^a(P)$, six Kac-Moody currents $D^{\pm i}(P)$ of $su(2) \otimes su(2)$ algebra, four weight $1/2$ fields $Q^a(P)$ and a $u(1)$ current $U(P)$. The modes of these generators may be obtained using the appropriate meromorphic forms for the expansion and recalling the orthogonality conditions. Now it is straightforward to obtain the $N=4$ K-N superalgebra through the double complex contour representation as shown earlier in eqn.(3.3.70). Thereafter, following closely the procedure in Section 3.1, one can obtain the $N=3$ K-N subalgebra. For example the $N = 3$ energy momentum tensor is defined as $\tilde{T}(P) = T(P) - \frac{1}{2}\gamma d_P U(P)$. With similar suitable definitions of $(J_L)^i(P)$, $\Gamma(P)$, and $\tilde{G}^i(P)$ in terms of original $N = 4$ opepators one can show that these operators with weights $1$, $1/2$ and $3/2$ respectively, satisfy an $N=3$ K-N superalgebra.

As in Section 3.1, the central charge of $N=3$ algebra is $\tilde{c} = \frac{c}{4\gamma(1-\gamma)}$. Following Section 3.1, one can also verify that the rest of the operators, i.e., $u(P)$, $\tilde{Q}^i(P)$, $J_R^i(P)$ and $\tilde{G}^i(P)$ and having weights $0,1/2, 1$ and $3/2$ respectively form an $N = 3$ superfield.

Finally with a new definition of the energy momentum tensor
\[ T(P) = \bar{T}(P) + \frac{1}{2} d_P J^{gh}(P), \quad J^{gh}(P) = (J_L)^3(P) \] (3.3.72)

One can show that \( T(P) \) satisfies centerless Virasoro algebra. With a suitable operator \( G^\pm(P) \) and \( J^{gh}(P) \) it forms an N=2 algebra. This set of operators form the topological K-N algebra obtained in [12]. As in Section 3.1 \( T(P) \) and \( J^{gh}(P) \) transform as weight 2 and 1 fields respectively w.r.t. \( T(P) \). The BRST charge is again given as \( Q_B = \frac{1}{2\pi i} \oint_{C_\tau} G^+ \). If we rename \( G^+(P) = Q(P) \) and \( G^-(P) = \bar{Q}(P) \) then \( Q(P) \) is the BRST operator. From the rest of the generators we can define superpartners of these N=2 operators \( T(P) \), \( G(P) \), \( Q(P) \), and \( J^{gh}(P) \). Thus, as in the case of complex plane, we have a residual N=2 superconformal symmetry. With suitable redefinitions of the rest of the operators we can also find the two pairs of BRST multiplets. The details of these constructions can be found in [21].
References


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