Chapter 2

Symmetries of String Effective Action with Cosmological Constant

Let us recall; if we consider a compactified string on a circle of radius $R$, the spectrum of the string remains invariant under the transformation $R \rightarrow \frac{1}{R}$. In a more general scenario, when we envisage evolution of a string in the background of its massless excitations and compactify $d$ of its coordinates on a torus, a richer symmetry structure emerges if all these backgrounds are treated independent those of compactified coordinates. The reduced string effective action has an invariance under global symmetry transformations, $O(d, d)$. The T-duality belongs to discrete subgroup of $O(d, d)$ transformations.

Our purpose in this chapter is to derive classical solutions of four-dimensional string effective actions with cosmological constant. In section-I, we generate new solutions through $O(d, d)$ transformations. First we consider a Minkowskian action where solution was derived for $N = 2$ supergravity action which was designated as “electrovac” solution [44, 45]. However, the dilaton field appears in the massless multiplet of the string theory and solutions of the string effective action are required to satisfy equations of motion associated with each massless mode of the string. Therefore, the presence of the dilaton field in the action imposes additional constraints on the background fields. We shall present solutions in the presence of constant dilaton background. However, this effective action also exhibits an invariance under noncompact symmetries. Thus we shall obtain new background fields from the known classical
solution by suitably implementing noncompact symmetry transformations. We recall that T-duality and $O(d,d)$ transformations have been applied earlier to obtain new solutions in the context of string cosmological solutions [46, 47, 48, 49, 40], black holes [50, 51, 52, 53, 54, 55, 56] and topology changing processes [57, 58, 59]. The second problem we consider is the Euclidean effective action in four dimensions in the presence of Abelian gauge fields. This action admits classical solutions such that Weyl curvature tensor and the gauge field strength satisfy self-duality condition and the solution has the interpretation of gravitational instanton [60, 61]. Furthermore, we show that the new metric and gauge field configurations can be generated by a global noncompact transformation. However, the new backgrounds are such that they do not satisfy self-duality condition.

It is recognized that S-duality is an important symmetry of string theory which relates the strong and weak coupling domains. The consequences of this symmetry [17, 18] are interesting and surprising. In the recent past, a variety of novel results have been derived for supersymmetric gauge theories [15, 16] in sequel to the new developments in string theory. In section-II, we explore some interesting consequences of S-duality specially when cosmological constant is present in the theory.

2.1 $O(d,d)$ symmetry with nonzero cosmological constant

Let us recall from chapter-1 the most salient aspects of dimensional reduction of the string effective action. We write the low energy heterotic string effective action in $D$ dimensions with Abelian gauge fields,

$$S = \int d^Dx \sqrt{-\hat{g}} e^{-\hat{\Phi}} \left( R_{\hat{g}} + \hat{g}^{\hat{\mu} \hat{\nu}} \partial_{\hat{\mu}} \hat{\Phi} \partial_{\hat{\nu}} \hat{\Phi} - \frac{1}{4} \hat{F}^{\hat{I}}_{\hat{\mu} \hat{\nu}} \hat{F}^{\hat{I}}_{\hat{\mu} \hat{\nu}} - \frac{1}{12} \hat{H}^{\hat{\mu} \hat{\nu} \hat{\lambda}} \hat{H}^{\hat{\mu} \hat{\nu} \hat{\lambda}} - 2\Lambda \right)$$  \hspace{1cm} (2.1)

where field strengths are defined as

$$\hat{H}^{\hat{\mu} \hat{\nu} \hat{\lambda}} = \partial_{\hat{\mu}} \hat{B}_{\hat{\nu} \hat{\lambda}} + \text{cyclic permutations},$$

$$\hat{F}^{\hat{I}}_{\hat{\mu} \hat{\nu}} = \partial_{\hat{\mu}} \hat{A}^{\hat{I}}_{\hat{\nu}} - \partial_{\hat{\nu}} \hat{A}^{\hat{I}}_{\hat{\mu}}$$

$$\hat{\mu}, \hat{\nu} = 1, \cdots, D; \ I = 1, \cdots, n.$$  \hspace{1cm} (2.2)
Here $\tilde{g}_{\mu\nu}$, $\Phi$, $\hat{A}_{\mu}^{I}$ and $\hat{B}_{\alpha\beta}$ denote the graviton, dilaton, $n$-component Abelian vector field and antisymmetric tensor fields, respectively. $R_g$ denotes $D$-dimensional scalar curvature and $\Lambda$ is the deficit in central charge (or nonperturbatively generated dilatonic potential term) which plays the role of cosmological constant. In dimensional reduction scheme, for backgrounds independent of $d$ coordinates (say, $x^\alpha$, $1 \leq \alpha \leq d$), with toroidal compactification on $T^d$ as in chapter-1, the action (2.1) can be rewritten as

$$S = \int d^{D-d}x \sqrt{-g} e^{-\Phi} \left[ R_g + g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} Tr \partial_\mu M^{-1} \partial^\mu M \right. \\
- \frac{1}{4} F_{\mu\nu} (M^{-1})_{ij} F^{ij\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - 2\Lambda \right]$$

(2.3)

where $\Phi = \tilde{\Phi} - \frac{1}{2} \ln \det G_{\alpha\beta}$ and the moduli matrix

$$M = \begin{pmatrix}
G^{-1} & -G^{-1} C & -G^{-1} A^T \\
-C^T G^{-1} & G + C^T G^{-1} C + A^T A & C^T G^{-1} A^T + A^T \\
-A G^{-1} & A G^{-1} C + A & 1 + AG^{-1} A^T
\end{pmatrix}$$

$$C_{\alpha\beta} = \frac{1}{2} A_{\alpha}^I A_{\beta}^I + B_{\alpha\beta}$$

(2.4)

with the space-time dependent background fields ($C_{\alpha\beta}$, $B_{\alpha\beta}$, $A_{\alpha}^I = \hat{A}_{\alpha}^I$) defining a generic point in moduli-space in the toroidal compactification of the heterotic string theory. The definitions of other two fields in (2.3) are

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2} A_{\mu}^I C_{i,j} F_{\nu\lambda}^i + \text{cyclic permutations}$$

$$F_{\mu\nu} = \partial_\mu A_{\nu}^i - \partial_\nu A_{\mu}^i, \quad i = 1, \ldots, 2d + n$$

(2.5)

where $B_{\mu\nu} = \tilde{B}_{\mu\nu} + \frac{1}{2} A_{\mu}^{(1)\alpha} A_{\nu\alpha}^{(2)} - A_{\mu}^{(1)\alpha} B_{\alpha\beta} A_{\nu}^{(1)\beta}$ and $A_{\mu}^i = (A_{\mu}^{(1)\alpha}, A_{\mu}^{(2)}, A_{\mu}^{(3)I})$ is a $(2d + n)$ component vector field with the following definition of its components,

$$A_{\mu\alpha}^{(1)} = \hat{G}_{\mu\alpha}$$

$$A_{\mu\alpha}^{(2)} = \hat{B}_{\mu\alpha} + \hat{B}_{\alpha\beta} A_{\mu}^{(1)\beta} + \frac{1}{2} \hat{A}_{\alpha}^I A_{\mu}^{(3)I}$$

$$A_{\mu}^{(3)I} = \hat{A}_{\mu}^I - \hat{\hat{A}}_{\alpha}^I A_{\mu}^{(1)\alpha}$$

(2.6)
Note that moduli $M$ satisfies the condition $M \mathcal{L} M^T = \mathcal{L}$, where $\mathcal{L}$ is the $O(d, d + n)$ metric,

$$
\mathcal{L} = \begin{pmatrix}
0 & I_d & 0 \\
I_d & 0 & 0 \\
0 & 0 & I_n
\end{pmatrix}
$$

(2.7)

where $I_d$ is $d$-dimensional identity matrix.

Now it is straightforward to check that the action (2.3) is manifestally invariant under global $O(d, d + n)$ transformations

$$
M \rightarrow \Omega M \Omega^T \\
\Phi \rightarrow \Phi, \ g_{\mu \nu} \rightarrow g_{\mu \nu}, \ B_{\mu \nu} \rightarrow B_{\mu \nu}, \\
A^i_\mu \rightarrow \Omega^a_j A^a_\mu
$$

(2.8)

where $\Omega$ is an $O(d, d + n)$ matrix such that $\Omega \mathcal{L} \Omega^T = \mathcal{L}$. Note that $A^i_\mu$ transforms as a vector multiplet under $O(d, d + n)$ transformations. We mention in passing that the transformation which gives $M \rightarrow M^{-1}$ is the analog of $R \rightarrow \frac{1}{R}$ duality for the generalised case. As noted in chapter-1, an $O(d) \times O(d + n)$ transformation, which is a subgroup of $O(d, d + n)$ generates new solutions which cannot be obtained from the old ones through general coordinate transformations or gauge deformations. Here we shall exploit these transformations to generate new solutions from some known solutions of the string equations of motion. There are specific examples where an $O(d) \times O(d)$ transformations can "boost away" curvature singularities and in fact can turn flat string backgrounds into nonflat ones and vice versa. It has been shown explicitly in ref.[49] that Nappi-Witten backgrounds in four-dimension can be obtained from $O(2, 2)$ boosting of the direct product of a pair of two-dimensional models.

### 2.1.1 Electro-vac backgrounds

We consider the following four-dimensional string effective action

$$
S = \int d^4 x \sqrt{-g} e^{-\phi} \left( R_g + \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{4} F^{I \mu \rho} F_{I \mu \rho} + 2 g_B^2 \right)
$$

(2.9)

The equations of motion of this action are satisfied by following background field configurations [62]

$$
\hat{ds}^2 = dx^2 + dy^2 + d\xi^2 - \cosh^2[\sqrt{2} g_B \xi] dt^2
$$
\[ \hat{A}_\mu = (0, 0, 0, \sqrt{2} \sinh(\sqrt{2} g_B \xi)) \]
\[ \hat{\Phi} = \text{constant}. \] (2.10)

The effective action (2.9) is similar to the \( N = 2 \) supergravity action considered by Freedman and Gibbons[44] with appropriate choice of backgrounds. The solution given by (2.10) are analog of the "electrovac" solutions of ref.[44]. Notice, however, that massless dilaton appears in the action (2.9). Now the dilaton equation of motion is required to be satisfied in addition to Einstein and Maxwell field equation. It is worthwhile to point out that this equation, even for constant dilaton background, imposes constraints on the background field equations. The topology of the spacetime is \( R^2 \times AdS_2 \) with the curvature scalar \( R = -4 g_B^2 \). The Maxwell field strength is covariantly constant, i.e. \( \nabla_\mu \hat{F}^{\mu\nu} = 0 \), and is of purely electric-type

\[ \hat{F}_{\xi 0} = 2 g_B \cosh(\sqrt{2} g_B \xi), \quad \hat{F}_{ij} = 0, \quad i, j = x, y, \xi, \]
and the field strength satisfies the constraint \( \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} = -8 g_B^2 \).

Note that backgrounds in (2.10) are independent of the coordinates \( x, y \) and \( t \). Therefore action (2.9) whose contents are graviton, dilaton and an Abelian gauge field is invariant under \( O(3,4) \) symmetry transformations [62]. Thus one can perform \( O(3,4) \) transformations on this background to obtain new classical solutions which can generate a nontrivial antisymmetric tensor field \( \hat{B}_{\mu\nu} \). For sake of definiteness, we shall choose a specific transformations restricted to \( t - y \) plane only [63]. This amounts to restricting ourselves to \( O(2,3) \) which is a subgroup of \( O(3,4) \). Let us first rewrite action (2.9) isolating \( y \)- and \( t \)-isometries [62]

\[ S = \int dy dt \int dx d\xi \sqrt{\det g_{\mu\nu}} e^{-\hat{\Phi}} \]
\[ \left( R_g + \partial_\mu \hat{\Phi} \partial^\mu \hat{\Phi} + \frac{1}{8} Tr(\partial_\mu M^{-1} \partial^\mu M) - \frac{1}{4} \mathcal{F}^{(i)(j)}(M^{-1})_{ij} \mathcal{F}^{(i)\mu\nu} + 2 g_B^2 \right) \] (2.11)

where

\[ g_{\mu\nu} = \delta_{\mu\nu}; \quad \mu, \nu = x, \xi \]
\[ \hat{\Phi} = \hat{\Phi} - \frac{1}{2} \ln \det G_{\alpha\beta} \]
\[ G_{\alpha\beta} = \begin{pmatrix} -\cosh^2(\sqrt{2} g_B \xi) & 0 \\ 0 & 1 \end{pmatrix}; \quad \alpha, \beta = t, y \]
\( A_\alpha = \left( \sqrt{2} \sinh[\sqrt{2} g_B \xi], 0 \right) \)
\( B_{\alpha\beta} = 0 \)
\( \mathcal{F}^{(i)}_{\mu\nu} = 0. \) \hfill (2.12)

The moduli matrix \( M \) can be constructed using the definition given in Eq.(2.4). To obtain new background we choose \( \Omega \) to be of the following form

\[ \Omega = \Omega_1 \times \Omega_2 \] \hfill (2.13)

where

\[ \begin{pmatrix}
    c+1 & -s & 1-c & -s & 0 \\
    -s & c+1 & s & c-1 & 0 \\
    1-c & s & 1+c & s & 0 \\
    -s & c-1 & s & c+1 & 0 \\
    0 & 0 & 0 & 0 & 2
\end{pmatrix} \]

\[ \begin{pmatrix}
    \hat{c}+1 & 0 & 1-\hat{c} & 0 & -\sqrt{2}\hat{s} \\
    0 & 1 & 0 & 0 & 0 \\
    1-\hat{c} & 0 & \hat{c}+1 & 0 & \sqrt{2}\hat{s} \\
    0 & 0 & 0 & 1 & 0 \\
    -\sqrt{2}\hat{s} & 0 & \sqrt{2}\hat{s} & 0 & 2\hat{c}
\end{pmatrix} \] \hfill (2.14)

and \( c = \cosh \theta, s = \sinh \theta, \hat{c} = \cosh \gamma, \) and \( \hat{s} = \sinh \gamma \) with \( 0 < \theta < \infty, 0 < \gamma < \infty \) being the boost parameters. Notice that our rotations involve the \( t-y \) plane and act on \( G_{\alpha\beta} \) and the gauge field \( A_\alpha \). These two noncompact rotations, generically, called “boosts”, appear because the \( t \)-coordinate is involved. The “boosts” \( \Omega_1 \) and \( \Omega_2 \) form the elements of the group \( O(1,1) \times O(1,2) \) which is a subgroup of \( O(2,3) \). The backgrounds thus generated satisfy the equations of motion and are not connected to the original background configuration by general coordinate transformations and/or gauge transformations.

Now we perform above \( O(1,1) \times O(1,2) \) transformations in a manner as summarized in eq.(2.8). After some straightforward calculation we get the new metric and
the gauge potential as following [62]

\[ ds^2 = dx^2 + d\xi^2 - \frac{1 + c^2 a(\xi)^2}{(1 + c \dot{s}a(\xi))^2} dt^2 + \frac{2s a(\xi)(ca(\xi) - \dot{s})}{(1 + c \dot{s}a(\xi))^2} dy^2, \]

\[ + \frac{1 + 2c \dot{s}a(\xi) + (c^2 - c^2)a(\xi)^2}{(1 + c \dot{s}a(\xi))^2} dy^2, \]

\[ \bar{A}_{\mu} = (0, 0, \frac{\sqrt{2c \dot{c}a(\xi)}}{1 + c \dot{s}a(\xi)}, -\frac{\sqrt{2s \dot{c}a(\xi)}}{1 + c \dot{s}a(\xi)}). \] (2.15)

where \( a(\xi) = \sinh[\sqrt{2}g_B \xi] \). Furthermore, the antisymmetric tensor field strength is nonzero after the \( O(1,1) \times O(1,2) \) transformations (although, the original background had vanishing \( \bar{H}_{\mu\nu\lambda} \))

\[ \bar{B}_{iy} = \frac{s \dot{a}(\xi)}{1 + c \dot{s}a(\xi)}. \] (2.16)

Moreover, we find that the transformed dilaton depends on coordinate \( \xi \) nontrivially and is given by

\[ \Phi = \bar{\Phi} - \ln(1 + c \dot{s}a(\xi)). \] (2.17)

The new Maxwell field strength, \( \bar{F}_{\mu\nu} \), has both electric and magnetic components and

\[ \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} = -8g_B^2 \frac{s^2}{(1 + c \dot{s}a(\xi))^2}. \]

The curvature scalar for new background in (2.15) is

\[ \bar{R} = -g_B^2 \frac{\mathcal{R}(\xi)}{(1 + c \dot{s}a(\xi))^5}, \] (2.18)

where the numerator is given by

\[ \mathcal{R}(\xi) = 4 + \dot{s}^2(1 + 7c^2) + (12 + 4\dot{s}^2(1 + 7c^2))c \dot{s} a(\xi) \]
\[ + (8 + 6\dot{s}^2(1 + 7c^2))c^2 \dot{s}^2 a(\xi)^2 - (8 - 4\dot{s}^2(1 + 7c^2))c^3 \dot{s}^3 a(\xi)^3 \]
\[ - (12 - \dot{s}^2(1 + 7c^2))c^4 \dot{s}^4 a(\xi)^4 - 4c^5 \dot{s}^5 a(\xi)^5. \]

Note that, the new background is singular whenever \( (1 + c \dot{s}a(\xi)) = 0 \) while the original background had a constant curvature \( -4g_B^2 \).

### 2.1.2 Euclidean backgrounds

Next we consider the string effective action in four Euclidean dimensions,
\[ S_E = \int d^4x \sqrt{g_E} e^{-\Phi} \left( R_{E}^{E} + \hat{g}_E^{\hat{\mu} \hat{\nu}} \partial_{\hat{\mu}} \hat{\Phi} \partial_{\hat{\nu}} \hat{\Phi} - \frac{1}{4} \hat{F}_{I \hat{\mu} \hat{\nu}} \hat{F}^{I \hat{\mu} \hat{\nu}} - 2\Lambda \right) \]  \hfill (2.19)

where \( \hat{\mu} = 1, \ldots, 4 \) and \( \hat{g}_E^{\hat{\mu} \hat{\nu}} \) is an Euclidean metric. As mentioned earlier, the presence of dilaton imposes constraints on the background. The background field configuration which satisfy the equations of motion for the Euclidean action was obtained in [62] and is given below,

\[ \hat{d}s^2 = \frac{dr^2}{f(r)^2} + \frac{r^2}{4f(r)^2} (d\psi + \cos \theta d\phi)^2 + \frac{r^2}{4f(r)} (d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ \hat{A}_{\hat{\mu}} = (0, 0, \sqrt{\frac{\Lambda}{2}} \frac{r^2 \cos \theta}{2f(r)}), \sqrt{\frac{\Lambda}{2}} \frac{r^2}{2f(r)} \]

\[ \hat{\Phi} = \text{constant} \]  \hfill (2.20)

where \( f(r) = 1 + \frac{\Lambda}{6} r^2 \). This is the Fubini-Study metric for \( CP^2 \) manifold and the solution has the interpretation of gravitational instanton [60]. The Weyl tensor for this background satisfies the anti-self-duality condition

\[ J_{\hat{m} \hat{n} \hat{p} \hat{q}} = \frac{1}{\frac{\epsilon}{2} g_{E}^{E_{M} N R S} C_{M N P Q}^{R S}} \]

which is as close as one can get to self-dual or anti-self-dual Ricci tensor. Strictly speaking one demands self-duality or anti-self-duality while looking for instanton like solutions. It is necessary to introduce a self-dual Abelian field strength in (2.19) in order to satisfy Einstein as well as matter field equations including dilaton equation.

Note, however, Gibbons-Pope instanton solution [60] was obtained for pure gravity in presence of cosmological constant, \( \Lambda \), whereas action (2.19) contains a gauge field and the dilaton in addition.

The solution (2.20) is a stringy background with vanishing antisymmetric tensor field strength. Also, all background fields are independent of two periodic coordinates, \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 4\pi \). Therefore, it is obvious that the action (2.19) is invariant under \( O(2, 3) \) transformations since the backgrounds are independent of \( \phi \) and \( \psi \). We can exploit this symmetry of the action to obtain new classical Euclidean backgrounds with nonvanishing antisymmetric tensor field and cosmological constant, by adopting a procedure similar to that discussed above. The first step is to construct
the corresponding moduli matrix $M$ for the problem at hand. The various moduli fields, which are used to construct moduli matrix $M$, can be read from eq.(2.20)

$$G_{\alpha\beta} = \begin{pmatrix} \frac{r^2}{4f(r)}(\cos^2 \theta + f(r) \sin^2 \theta) & \frac{r^2}{4f(r)} \cos \theta \\ \frac{r^2}{4f(r)} \cos \theta & \frac{r^2}{4f(r)} \end{pmatrix}$$

$$A_{\alpha} = \left( \frac{\Lambda r^2}{24f(r)} , \frac{\Lambda r^2}{24f(r)} \right)$$

$$B_{\alpha\beta} = 0,$$

(2.22)

where indices $\alpha, \beta$ run over coordinates $(\phi, \psi)$ in the given order. The next step is to choose an appropriate $O(2, 3)$ matrix $\Omega$. To generate inequivalent backgrounds from the one given in (2.20) we choose $\Omega$ to be a special element of $O(2) \times O(3)$, which is a subgroup of $O(2, 3)$, given by

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(2.23)

It is straightforward to perform the $O(d, d)$ transformations given in (2.8) using eqs.(2.23) and (2.22) above. We get the new field configuration

$$\tilde{G}_{\alpha\beta} = \begin{pmatrix} \frac{r^2 \sin^2 \theta}{4f(r)} & 0 \\ 0 & \frac{4f(r)}{r^2(1 + \frac{\Lambda}{4} r^2)} \end{pmatrix}$$

$$\tilde{A}_{\alpha} = \left( 0 , -\sqrt{2\Lambda} \frac{f(r)}{(1 + \frac{\Lambda}{4} r^2)} \right)$$

$$\tilde{B}_{\alpha\beta} = \begin{pmatrix} 0 & \cos \theta \\ -\cos \theta & 0 \end{pmatrix}$$

$$\tilde{\Phi}(r) = \hat{\Phi} - \ln \left( \frac{r^2(1 + \frac{\Lambda}{4} r^2)}{4(1 + \frac{\Lambda}{6} r^2)^2} \right),$$

(2.24)

while other fields remain unaffected. Interestingly, we have got a new field configuration in which the metric is diagonal. Also, dilaton and antisymmetric tensor fields are nontrivial. It should be noted that both Weyl tensor as well as vector field strength
have lost their respective self-duality properties after this transformation. Asymptotically (i.e., as \( r \to \infty \)) the new four-metric has the form

\[
\tilde{ds}^2 = \omega(r)^2(dr^2 + r^2d\Theta^2) + \frac{6}{4\Lambda}(d\Theta^2 + \sin^2 \theta d\phi^2)
\]  

(2.25)

where \( \omega(r) = \frac{6}{\Lambda r^2} \) and \( \Theta = \frac{2A}{9} \psi \). Let us for the sake of simplicity take \( \tilde{\Phi} = 0 \) and discuss the behavior of \( \tilde{\Phi} \) as a function of \( r \). Asymptotic value of the dilaton, \( \tilde{\Phi}_{\text{asym}} = -\ln \frac{3}{4\Lambda} \), depends on the cosmological constant \( \Lambda \). Notice that for \( r \sim 0 \), \( \tilde{\Phi}_{r\to0} \sim -\ln \frac{r^2}{4} \). We recall that \( e^{\tilde{\psi}} \) is the string coupling constant. It is interesting to note that string coupling has \( \frac{1}{r^2} \) behaviour near the origin while attains a constant value of \( \frac{4A}{9} \) at \( r \to \infty \). We observe that for large \( r \), \( e^{\tilde{\psi}} \) is a constant, if \( \Lambda \to 0 \) the string coupling tends to vanishing value. Thus for this simple model under consideration, there is a connection between string coupling constant and the cosmological constant.

### 2.2 Strong-weak duality and the cosmological constant

We have known from chapter-1 that the equations of motion derived from the four-dimensional heterotic string effective action are invariant under S-duality, although the action itself is not invariant. However, these results are derived in the absence of the cosmological constant term in the action. We shall learn here that in the presence of cosmological constant, \( \Lambda \), equations of motion associated with a four dimensional effective action, obtained through dimensional reduction, are not invariant under S-duality transformations. Nevertheless, the invariance of the equations of motion is recovered once we set \( \Lambda = 0 \). This leads us to conjecture that exact S-duality symmetry will force the cosmological constant to vanish. It is appropriate to recall here the hypothesis of 'naturalness' [64] expounded by 't Hooft which says that a parameter in any theory remains small, if the symmetry is enhanced by setting that parameter to zero. For example, in electrodynamics setting the electron mass \( m_e = 0 \) enhances the symmetry of the action and the chiral symmetry is restored. Therefore, it is guaranteed that \( m_e \) remains small and the corrections are proportional to \( m_e \) itself. We recall that according to 't Hooft, the vanishingly small value of cosmological constant is unnatural [64]; putting it equal to zero does not seem to increase the symmetry.
of Einstein-Hilbert action. However in the framework of string theory, $\Lambda$ obeys the naturalness criterion, i.e. putting $\Lambda = 0$ enhances the stringy symmetry. It should be borne in mind that S-duality is not a symmetry of the action and hence may not satisfy the strict criterion of naturalness.

The cosmological constant is a parameter measured very close to zero and it is a vintage theoretical problem to explain the smallness of $\Lambda$ which has eluded physicists for a long time [65]. Several attempts have already been made to explain the vanishing cosmological constant in the framework of string theory [66, 67, 10]. Recently, Witten [68] has argued that the vanishing of cosmological constant and the absence of massless dilaton might be explained by a duality between supersymmetric string vacuum in three dimensions and a non-supersymmetric string vacuum in four-dimensions. The issue has also been addressed in a more concrete model by Becker, Becker and Strominger [69].

2.2.1 Duality in presence of $\Lambda$

Consider the string effective action in $D$ spacetime dimensions with massless fields such as graviton, antisymmetric tensor, dilaton, and $n$ Abelian gauge fields. If we compactify coordinates on a $d = D-4$ dimensional torus and assume that the backgrounds are independent of these $d$ compact coordinates, the resulting four-dimensional reduced effective action takes the form,

$$S = \int d^4x \sqrt{-g} \, e^{-\Phi} \left[ R + g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} Tr \partial_\mu M^{-1} \partial^\mu M - \frac{1}{4} \mathcal{F}_{\mu\nu} (M^{-1})_{ij} \mathcal{F}^{i\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - 2\Lambda \right],$$ \hspace{1cm} (2.26)

where the definitions can be read from the last section. Above action has an explicit invariance with respect to $O(d, d + n; R)$ transformations. It is convenient for the implementation of S-duality transformations, discussed in chapter-1, to rescale the $\sigma$-model metric to Einstein metric, $g_{\mu\nu} \rightarrow e^{\Phi} g_{\mu\nu}$, and introduce the axion $\partial_\nu \chi = (\eta^2/6) \sqrt{-g} \epsilon_{\mu\nu\lambda\sigma} H^{\mu\nu\lambda}$ where $\eta = e^{-\Phi}$. Then (2.26) can be reexpressed as
\[ S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2\eta^2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + \frac{1}{8} \text{Tr}(\partial_\mu M^{-1} \partial^\mu M) \right. \]
\[ \left. - \frac{1}{4} \eta \mathcal{F}_{\mu\nu} M^{-1}_{ij} \mathcal{F}^{i\mu\nu} + \frac{1}{4} \chi \mathcal{F}^{i\mu\nu} \mathcal{L}_{ij} \mathcal{F}^{j\mu\nu} - \frac{2\Lambda}{\eta} \right], \quad (2.27) \]

where \( \Psi = \chi + i \eta \) is a complex scalar field and

\[ \mathcal{F}^{i\mu\nu} = \frac{1}{2} \sqrt{-g} e_{\mu\rho\sigma} \mathcal{F}^{i\rho\sigma}. \quad (2.28) \]

Now let us find out the equations of motion for various fields present in the theory. The equations of motion corresponding to \( \Psi, g_{\mu\nu} \) and \( A_\mu \) derived from the action (2.27) are

\[ \frac{\nabla_\mu \nabla^\mu \Psi}{\eta^2} + i \frac{\nabla_\mu \Psi \nabla^\mu \Psi}{\eta^2} - i \frac{1}{4} \mathcal{F} M^{-1} \mathcal{F} + \frac{1}{4} \mathcal{F} \mathcal{L} \mathcal{F} + i \frac{2\Lambda}{\eta^2} = 0, \quad (2.29) \]

\[ R_{\mu\nu} - \frac{\nabla_\mu \Psi \nabla_\nu \Psi}{2\eta^2} + \frac{1}{8} \text{Tr}(\partial_\mu M^{-1} \partial_\nu M) \]
\[ - \frac{\eta}{2} \mathcal{F}_{\mu\lambda} M^{-1} \mathcal{F}^\lambda + g_{\mu\nu} \left( \frac{\eta}{8} \mathcal{F} M^{-1} \mathcal{F} - \frac{\Lambda}{\eta} \right) = 0, \quad (2.30) \]

\[ \nabla_\mu \left( \eta (\mathcal{L}) \mathcal{F}^{\mu\nu} - \chi \mathcal{F}^{\mu\nu} \right) = 0, \quad (2.31) \]

and the Bianchi identity is

\[ \nabla_\mu \mathcal{F}^{i\mu\nu} = 0. \quad (2.32) \]

The S-duality transformations [20] correspond to

\[ \Psi \rightarrow \frac{a \Psi + b}{c \Psi + d}, \quad a d - b c = 1, \quad a, b, \ldots \in \mathbb{Z}, \]
\[ \mathcal{F}^{i\mu\nu} \rightarrow c \eta (\mathcal{L})_{ij} \mathcal{F}^{j\mu\nu} + (c \chi + d) \mathcal{F}^{i\mu\nu}, \quad (2.33) \]

while the metric \( g_{\mu\nu} \) and moduli \( M \) remain invariant.

Explicit calculations show that under S-duality the terms in eqs. (2.29) and (2.30) remain invariant when \( \Lambda = 0 \); however for nonvanishing \( \Lambda \) these equations are not S-duality invariant[70]. To analyze S-duality invariance of eqs. (2.29) and (2.30), let us consider a specific transformation \( (a = d = 0, b = -c = 1) \)

\[ \Psi \rightarrow -1/\Psi \text{ and } \mathcal{F}^{i\mu\nu} \rightarrow - \eta (\mathcal{L})_{ij} \mathcal{F}^{j\mu\nu} - \chi \mathcal{F}^{i\mu\nu}. \quad (2.34) \]
Now it is straightforward to find that first four terms on the left-hand-side of (2.29) are invariant under above transformation (2.34) while the last term with $\Lambda$ is not. Similarly, it can also be checked that except $\Lambda$-term all other terms in Eq. (2.30) make an invariant combination. Thus, in general, the presence of cosmological constant breaks the S-duality invariance of the string equations of motion. Furthermore, we might consider an effective action with nontrivial dilaton potential $V(\Phi)$ which might be generated due to nonperturbative effects. However, the corresponding equations of motion are S-duality invariant only if $V(\Phi) = 0$. We write the equations of motion involving $V(\Phi)$ (after rescaling to Einstein metric):

$$\frac{\nabla_\mu \nabla^\mu \Psi}{\eta^2} + \frac{i \nabla_\mu \nabla^\mu \bar{\Psi}}{\eta^3} - \frac{i}{4} \mathcal{F} M^{-1} \mathcal{F} + \frac{1}{8} \mathcal{F} \mathcal{L} \mathcal{F} + \frac{1}{4} \frac{2 \bar{V}(\eta)}{\eta^2} - \frac{2}{\eta} \frac{\partial \bar{V}(\eta)}{\partial \eta} = 0,$$

$$R_{\mu\nu} - \frac{\nabla_\mu \Psi \nabla_\nu \bar{\Psi}}{2 \eta^2} + \frac{1}{8} \text{Tr}(\partial_\mu M^{-1} \partial_\nu M) - \frac{\eta}{2} \mathcal{F}_{\mu \lambda} M^{-1} \mathcal{F}^{\lambda} - g_{\mu\nu} \left( \frac{\eta}{8} \mathcal{F} M^{-1} \mathcal{F} - \bar{V}(\eta) \right) = 0, \quad (2.35)$$

where $\bar{V}(\eta)$ is the dilaton potential reexpressed in terms of the new variable $\eta = e^{-\Phi}$. We note that the above equations of motion (2.35) are not invariant under the transformation (2.34) as long as the dilaton potential $V(\Phi)$ is nonzero.

### 2.2.2 WZW backgrounds with nonzero $\Lambda$

Having demonstrated that $\Lambda$-terms break S-duality invariance of the string equations of motion, now, let us consider an example\cite{71, 72} of six dimensional target space constructed by taking a tensor product of two WZW theories with the groups $SL(2, R)$ and $SU(2)$ respectively. Thus the underlying conformal field theory describing the above target space is exact. If we compactify one coordinate, say $\varphi$, of $SL(2, R)$ and another $\zeta$ of $SU(2)$ on tori then the resulting theory has a metric with $(---+)$ Minkowski signature. A pair of gauge fields $A^{(1)}_\mu (\alpha = \varphi, \zeta)$ appear from the metrics of the two groups and another pair of gauge fields $A^{(2)}_{\mu, \alpha}$ come from the antisymmetric tensor fields. The scalar multiplet consists of a pair from the moduli and the dilaton.
The exact conformal field theory backgrounds those satisfy equations of motion of four-dimensional string effective action with an appropriate cosmological constant are

\[ ds^2 = -\left(\frac{r^2}{K_{SL}} - M + \frac{J^2}{4r^2}\right)dt^2 + \left(\frac{r^2}{K_{SL}} - M + \frac{J^2}{4r^2}\right)^{-1}dr^2 + \frac{K_{SU}}{4}d\Omega_2^2, \]

\[ A^{(1)\mu}_\varphi = \left(-\frac{J^2}{2r^2}, 0, 0, 0\right), \quad A^{(1)\mu}_\zeta = (0, 0, 0, m \cos \theta), \]

\[ A^{(2)\mu}_\varphi = \left(-\frac{r^2}{r}, 0, 0, 0\right), \quad A^{(2)\mu}_\zeta = (0, 0, 0, \pm \frac{n}{4} \cos \theta), \quad A^{(3)\mu}_I = 0, \]

\[ G_{\alpha\beta} = \left(\begin{array}{cc} r^2 & 0 \\ 0 & \frac{n}{4m} \end{array}\right), \quad B_{\alpha\beta} = 0, \quad \Phi = -\ln r + \text{const.}, \quad H_{\mu
u\sigma} = 0, \quad (2.36) \]

where \( d\Omega_2^2 \) stands for the line element of a unit 2-sphere. The indices \( \alpha, \beta \) run over two compactified directions \( \varphi \) and \( \zeta \). Here \( G_{\varphi\varphi} = r^2 \) and \( G_{\zeta\zeta} = n/4m \) correspond to the moduli. The corresponding six-dimensional string effective action has graviton, antisymmetric tensor and dilaton only.

We mention in passing that the compactification of the direction \( \zeta \) gives rise to a \( U(1) \) gauge field with magnetic charge \( m \) and modular invariance imposes the constraint \( m n = K_{SU} \). Notice that large \( K_{SU} \) and \( K_{SL} \) limits corresponds to Bertotti-Robinson [73] space time in four dimensions. This solution describes the throat limit of extremal dilaton black holes with electric and magnetic charge investigated by Kallosh et. al.[74] and also describes the throat limit of the Reissner-Nordstrom black hole. We recall that large \( K_{SU} \) and \( K_{SL} \) can be envisaged as the classical limit since these constants play the role of \( 1/\hbar \) in the WZW theory. In this limit, the cosmological constant \( \Lambda \rightarrow 0 \).

Next, we present another way to obtain the four-dimensional black hole solutions. In this case, instead of compactifying the coordinate \( \varphi \) of the \( SL(2, R) \) alluded to above, we gauge the \( U(1) \) subgroup. The gauged \( SL(2, R) \) WZW action can be written in light cone coordinates \( (z, \bar{z}) \)

\[ S(U, A) = S(U) + \frac{K_{SL}}{2\pi} \int d^2z \left\{ \text{Tr} \left[ U^{-1} \partial_z U A_z + \partial_{\bar{z}} U U^{-1} A_{\bar{z}} + U^{-1} A_z U A_{\bar{z}} \right] + A_z A_{\bar{z}} \right\}, \quad (2.37) \]
where \( U(x) \in SL(2,\mathbb{R}) \) for \( x \) in the manifold \( M \). Integrating out the gauge fields in (2.37) by taking care of the Jacobian in the corresponding path-integral, one gets the two dimensional target space configuration[75]. As a result, the effective theory is a five-dimensional one with an appropriate cosmological constant term. The corresponding four dimensional effective action will arise from the compactification of this five-dimensional theory. In this prescription there are only two gauge fields, namely one coming from the metric and the other from the antisymmetric tensor field when \( \zeta \) coordinate of \( SU(2) \) is compactified. The background field configurations are:

\[
\begin{align*}
\sigma^2 &= -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{M}{r}\right)^{-1} \frac{K_{SL}}{8r^2} dr^2 + \frac{K_{SU}}{4} d\Omega_2^2,
\mathcal{A}^{(1)}_{\mu} &= (0, 0, 0, m \cos \theta), \mathcal{A}^{(2)}_{\mu, \zeta} = (0, 0, 0, \pm \frac{n}{4} \cos \theta), \mathcal{A}^{(3)}_{\mu} = 0,
G_{\zeta \zeta} &= \frac{n}{4m}, \quad \Phi = -\ln r + \text{const.}, \quad H_{\mu \nu \lambda} = 0.
\end{align*}
\]

(2.38)

This background configuration (2.38) describes a four dimensional magnetically charged static black hole solution. In the asymptotic limit, i.e. \( r \rightarrow \infty \), the topology of the spacetime is \( R^1 \times R^1 \times S^2 \).

These solutions (2.36) and (2.38) satisfy the background field equations of four dimensional heterotic string effective actions with nonvanishing cosmological constant terms, \( 2/K_{SU} - 2/K_{SL} \) and \( 2/K_{SU} - 4/K_{SL} \) respectively, which indeed break S-duality symmetry of the equations of motion [70]. The scalar curvatures corresponding to these background configurations are respectively given by

\[
\begin{align*}
\frac{8}{K_{SU}} - \frac{2}{K_{SL}} - \frac{3J^2}{2r^4} \quad \text{and} \quad \frac{8}{K_{SU}} + \frac{8M}{K_{SL}} \frac{1}{r}.
\end{align*}
\]

Similarly the backgrounds obtained in eqs. (2.15) and (2.24) do also break S-duality invariance of the string equations of motion [62, 70].

2.3 Summary and discussion

To summarize, in section-I, we have studied the solutions of four-dimensional heterotic string effective actions in presence of cosmological constant, \( \Lambda \). The solutions
of equations of motion corresponding to the Minkowskian string effective action are analogous to "electrovac" solution of Freedman and Gibbons [44]. This action has an invariance under $O(2,3)$ transformations. Using this property, we could generate new backgrounds with nontrivial $\tilde{B}_{\mu\nu}$. However, the curvature scalar corresponding to the new metric is singular while initial geometry had a constant curvature. Next, the Euclidean string action in four dimensions with $CP^2$ geometry and an Abelian self-dual gauge field strength admits "gravitational instanton" solution. This action also possesses an $O(2,3)$ invariance. We have generated new backgrounds such that the metric is diagonal and the dilaton as well as antisymmetric field acquire nontrivial coordinate dependences. Furthermore, we find that the Weyl tensor and $F_{\mu\nu}$ corresponding to new background, are no longer (anti)self-dual.

In section-II we have explored the consequences of S-duality transformations on the equations of motion with nonzero cosmological constant. First, we studied a four dimensional action in a general framework. The reduced action (2.26) could have been obtained from toroidal compactification of a heterotic string effective action in higher dimensions. Although these actions do not necessarily represent supersymmetric theories, S-duality invariance would have implied the absence of cosmological constant. We note that the cosmological constant term breaks S-duality for the exact conformal field theory backgrounds.

In this context, let us briefly discuss the presence of higher order terms in $\alpha'$ and the consequences of the S-duality transformations in the equations of motion. We write down the next higher order term in $\alpha'$ [32, 33] to the low energy string effective action (2.27) as

$$S' = \int d^4x \sqrt{-g} \eta \left( R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \right). \quad (2.39)$$

In presence of the higher order term the equation of motion (2.29) gets an additional contribution

$$\frac{i}{4} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$$

and similarly Eq.(2.30) is modified with the extra term

$$\eta \left( G_{\mu\nu} + g_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha} R \right),$$

where
\begin{equation}
G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R_{\alpha\beta\lambda\rho} R^{\alpha\beta\lambda\rho} + 2 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - 4 \nabla_{\alpha} \nabla^{\alpha} R_{\mu\nu} + 2 \nabla_{\mu} \nabla_{\nu} R - 4 R_{\mu\alpha} R_{\nu}^{\alpha} + 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta}.
\end{equation}

We have checked that under the S-duality transformation given in (2.34), eq.(2.29) with the additional term corresponding to $S'$ does not remain invariant. The graviton equation along with the higher order correction term as mentioned above is also not invariant under the S-duality. Thus, it can be argued that the presence of the higher order terms do not restore the S-duality invariance in the equations of motion. Notice that when we dimensionally reduce the terms involving quadratic in curvature, there will be additional terms in (2.39) involving moduli and gauge fields (arising from dimensional reduction). We have seen that the contribution of (2.39) to the equations of motion already breaks the S-duality. Therefore, even if we explicitly take into account the contribution coming from moduli and extra gauge fields in the corresponding equations of motion, the S-duality invariance will not be restored.