Chapter 4

On the Compactification of Type IIA String Theory

This chapter will be devoted to study some of the aspects of type IIA theory where we shall consider compactification of the ten-dimensional type IIA action in the presence of a cosmological constant term. We have mentioned, in chapter-1, that there have been very important developments in understanding the unity of all the five string theories and the importance of extended objects (generically called p-branes) have been recognised. First, we recall some of the salient features of the p-branes before proceeding to derive the compactified action.

Recently, considerable progress has been made in our understanding of the non-perturbative features of superstring theories [22, 23, 26]. It is now realised that the five consistent superstring theories might be envisioned as various phases of a single unique theory [24]. Dualities play a cardinal role in revealing the intimate connections between different string theories in diverse spacetime dimensions and provide deeper insight into string theory dynamics. We recall that the predictions of T-duality are subject to tests in the perturbative frame work; whereas, the predictions and tests of S-duality are beyond the realms of perturbation theory. The p-branes, which appear as classical solutions of the string effective action, have been instrumental in our understanding of various duality conjectures in string theory [25]. The RR p-branes are interpreted as D-p-branes of type II theories [82]. The type IIA string admits even D-branes, $p = 0, 2, 4, 6$ and type IIB theory, on the other hand, has the odd ones,
i.e. $p = 1, 3, 5$ with the identification that $-1$-brane is the instanton of the theory. Furthermore, for ten-dimensional spacetime, dual of a $p$-brane is the $(6 - p)$ brane and consequently, those $p$-branes with $p \leq 6$, have duals with $p \geq 0$. Thus, for $D = 10$, the 8-brane and 7-brane appearing in type IIA and type IIB string theories respectively have special roles different from the other branes alluded to above.

A $p$-brane couples to $(p + 1)$-form potential; therefore, the 8-brane will couple to the potential $A_9$ whose corresponding field strength is the ten form $F_{10}$. In standard type IIA supergravity, the presence of the potential $A_9$ is rather obscure. From the perspective of type IIA string theory, we know that the theory admits 8-D-brane [82, 83]. Notice that the equations of motion arising from the kinetic energy term $F_{10}^2$ only give rise to a conservation law and the presence of this term does not introduce any new dynamical degree of freedom. However, the effect of this additional term amounts to introduction of cosmological constant, when we introduce the Poincare dual of ten form field strength instead. In this context it is worthwhile to mention that it had been realised several years ago that the introduction of a four-form field strength in four spacetime dimensions amounts to having a cosmological constant term in that supergravity theory [84]. Romans [85], subsequently, constructed the massive ten dimensional type IIA supergravity theory and a complete construction was given in ref.[86].

The study of type IIA superstring effective action in the presence of $F_{10}$, or alternatively the theory with cosmological constant has drawn attention of several authors [87, 88, 89, 90] in the recent past and it has been argued that the cosmological constant takes only quantized values. It is well known, for the massless theory, that type IIA compactified on $S^1$ with radius $R$ is T-dual to type IIB compactified on another circle with reciprocal radius [91]. Thus the issue of compactification of massive type IIA theory to $D = 9$ has been addressed in the context of its T-duality to type IIB theory[89]. It has been argued that the ten-dimensional type IIB theory, when compactified according to the ‘generalised’ Scherk-Schwarz [39] prescription, yield a massive theory in nine dimensions and then one can explore the T-duality. Moreover, there have been attempts to obtain various brane solutions in type IIA, IIB and M-theory [92, 93, 94, 95, 89] and relate these solutions in lower dimensions by adopting sequential steps in dimensional reductions.

Now we proceed to obtain dimensionally reduced string effective action for massive
type IIA superstring action and the compact space is a $d$-dimensional torus. We investigate the symmetry properties of the reduced effective action [96]. In particular, we show that the six-dimensional effective action for the case of the massive theory does not respect the $SO(4,4)$ symmetry of the corresponding six-dimensional massless theory [97]. Furthermore, we find new black hole solutions in five and four spacetime dimensions from the reduced effective action in the presence of cosmological constant and the dilatonic potential term. We also present four-brane solutions in six dimensions.

4.1 Review: massive supergravity

The bosonic part of massive type IIA supergravity action, in ten dimensions, is of interest to us. The action was introduced by Romans [85] and we write an action in the string frame metric

$$S_m = \int d^{10}x \sqrt{-g} \left[ e^{-2\Phi} \left( R_g + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2 \cdot 3!} H_{\mu \nu \lambda} H^{\mu \nu \lambda} \right) - \frac{1}{2 \cdot 2!} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2 \cdot 4!} G_{\mu \nu \lambda \rho} G^{\mu \nu \lambda \rho} - \frac{1}{2} m^2 \right], \quad (4.1)$$

where $\Phi$ is the dilaton field, $g_{\mu \nu}$ is the string $\sigma$-model metric and $m$ is the mass parameter. This action can be identified as the low energy limit of the type IIA string theory with $m^2$ playing the role of cosmological constant. The NS-NS and R-R field strengths are defined as follows:

$$F_{\mu \nu} = \partial_{[\mu} A_{\nu]} + mf_{\mu \nu},$$
$$H_{\mu \nu \lambda} = \partial_{[\mu} B_{\nu \lambda]},$$
$$G_{\mu \nu \lambda \rho} = \partial_{[\mu} C_{\nu \lambda \rho]} + 2A_{[\mu} H_{\nu \lambda \rho]} + 2m g_{\mu \nu \lambda \rho}, \quad (4.2)$$

where coefficients of the mass parameter terms are $f_{\mu \nu} = B_{\mu \nu}$ and $g_{\mu \nu \lambda \rho} = B_{[\mu \nu} B_{\lambda \rho]}$. The notation $[\mu \nu \cdots]$ implies the antisymmetrization of the indices. Note that the field strengths have mass dependent terms and are the generalisations of their massless counterparts. The advantage of writing massive type IIA action as in (4.1) is that the action for the massless theory can be obtained by taking the limit $m \to 0$ [89]. The
action has the invariance under massive ‘Stückelberg’ gauge transformations

\[ \delta A_\mu = -mA_\mu \]
\[ \delta B_{\mu\nu} = \partial_{[\mu}A_{\nu]} \]
\[ \delta C_{\mu\nu\lambda} = -2mA_{[\mu}B_{\nu\lambda]} . \]  

The above action has \( N = 2 \) supersymmetry even though it involves mass terms. The constant mass term in the R-R sector of the theory which has the interpretation of the cosmological constant can also be envisaged as the dual of 10-form filed strength alluded to earlier. Therefore, in ten dimensions, the appearance of \( m^2 \) terms provides a clue for the presence of an 8-brane in type IIA theory with the hindsight.

### 4.2 Toroidal compactification and SO(4,4) duality

Let us consider compactification of the ten dimensional effective action in (4.1), in presence of the cosmological constant term, on a \( d \)-dimensional torus. We adopt the prescription of Schwarz and one of the authors (JM) [11]. The coordinates of D-dimensional spacetime are denoted by \( x^\mu \), whereas the rest which make the internal dimensions, the \( d \)-dimensional torus, are denoted as \( x^\alpha \). In our notational conventions, we denote ten-dimensional fields with hats over the fields as well as over the tensor indices (\( \hat{F}, \hat{g}_{\mu\nu}, \text{etc.} \)), while reserve the quantities without hats for D-dimensional ones. Furthermore, we assume that the fields are independent of the “internal” coordinates, \( x^\alpha \). The ten dimensional vielbein can be expressed in the following form

\[ \hat{e}_\mu^a = \begin{pmatrix} e_\mu^a & A_\mu^{(1)\beta} E_\beta^a \\ 0 & E_\alpha^a \end{pmatrix} \]

and “spacetime” metric \( g_{\mu\nu} = e_\mu^\alpha \eta_{\alpha\beta} e_\nu^\beta \), “internal” metric \( G_{\alpha\beta} = E_\alpha^a \delta_{ab} E_\beta^b \).

Thus, the ten-dimensional metric components will be expressed in terms of the D-dimensional metric, Kaluza-Klein gauge fields and scalars \( G_{\alpha\beta} \)

\[ \hat{g}_{\mu\nu} = g_{\mu\nu} + A_\mu^{(1)\alpha} A_\nu^{(1)\beta} G_{\alpha\beta}, \quad A_{\mu\alpha} = \hat{g}_{\mu\alpha}, \quad G_{\alpha\beta} = \hat{g}_{\alpha\beta}. \]  

Similarly for the antisymmetric tensor field, coming from the NS-NS sector, the decompositions are

\[ A_{\mu\alpha}^{(2)} = \hat{B}_{\mu\alpha} - A_\mu^{(1)\beta} b_{\alpha\beta}, \quad b_{\alpha\beta} = \hat{B}_{\alpha\beta}, \]
\[ B_{\mu\nu}^{(1)} = \hat{B}_{\mu\nu} - A^{(1)\alpha}_{[\mu} A^{(2)\alpha}_{\nu]} - A^{(1)\alpha}_{\mu} A^{(1)\beta}_{\nu} b_{\alpha\beta}. \]  

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and the R-R fields can be decomposed as follows:

\[
c_{\alpha\beta\gamma} = \hat{C}_{\alpha\beta\gamma}, \quad a_{\alpha} = \hat{A}_{\alpha}
\]

\[
A^{(3)}_{\mu\alpha\beta} = \hat{C}_{\mu\alpha\beta} - A_{\mu}^{(1)\delta} c_{\alpha\beta\delta},
\]

\[
B^{(2)}_{\mu\nu\alpha} = \hat{C}_{\mu\nu\alpha} + A^{(1)\delta}_{\mu} \hat{C}_{\nu}\delta_{\alpha} + A^{(1)\delta}_{\nu} A^{(1)\epsilon}_{\mu} c_{\delta\epsilon\beta\alpha},
\]

\[
C_{\mu\nu\lambda} = \hat{C}_{\mu\nu\lambda} - \left( A_{\mu}^{(1)\delta} C_{\delta\nu\lambda} + \text{cyclic perms. in } \mu, \nu, \lambda \right),
\]

\[
+ \left( A^{(1)\delta}_{\mu} A^{(1)\epsilon}_{\nu} \hat{C}_{\delta\epsilon\beta\lambda} + \text{cyclic perms in } \mu, \nu, \lambda \right),
\]

\[
- A_{\mu}^{(1)\delta} A^{(1)\epsilon}_{\nu} A^{(1)\zeta}_{\lambda} c_{\delta\epsilon\beta\lambda},
\]

\[
A^{(4)}_{\mu\alpha} = \hat{A}_{\mu} - A_{\mu}^{(1)\delta} a_{\delta}.
\] (4.6)

Recall that the scalars are constructed in the ten dimensional theory by contracting the hat indices of various tensors. In order to obtain tensors with unhatted indices, i.e. tensors in D-dimensions we adopt the following prescription:

\[
\mathcal{H}_{\mu\nu...\alpha\beta..} = \mathcal{O}_{\mu}^{\hat{\mu}} \mathcal{O}_{\nu}^{\hat{\nu}} ... \hat{H}_{\hat{\mu}\hat{\nu}...\alpha\beta..}
\] (4.7)

where, \( \mathcal{O}^{\hat{\mu}}_{\mu} = \epsilon_{\mu}^{\hat{\mu}} \) and \( \hat{H}_{\hat{\mu}\hat{\nu}...\alpha\beta..} \) is a tensor in ten dimensions. Thus scalars constructed out of contraction of ten dimensional indices, as is the case with kinetic energy terms in the action, can be expressed in the following form in terms of scalars constructed out of various tensors in D-dimensions, obtained through the dimensional reduction procedure,

\[
\hat{H}_{\hat{\mu}\hat{\nu}...} \hat{H}_{\hat{\nu}\hat{\mu}...} = \mathcal{H}_{\mu\nu...} \mathcal{H}^{\mu\nu...} + n\mathcal{H}_{\mu\nu...} \mathcal{H}^{\mu\nu...} + \frac{n(n-1)}{2!} \mathcal{H}_{\mu\nu...} \mathcal{H}^{\mu\nu...} \\
+ \cdots + \mathcal{H}_{\alpha\beta...} \mathcal{H}^{\alpha\beta...}
\] (4.8)

for the n–form field strength. Following (4.7) NS-NS field strengths are obtained as below,

\[
H^{(1)}_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} - F^{(1)\delta}_{[\mu\nu} A_{\rho]\delta],
\]

\[
H^{(2)}_{\mu\nu\alpha} = F^{(2)}_{\mu\nu\alpha} - F^{(1)\delta}_{\mu\nu} b_{\alpha}\delta,
\]

\[
H^{(4)}_{\mu\alpha\beta} = \partial_{\mu} b_{\alpha\beta}.
\] (4.9)

where \( F^{(1)}_{\mu\nu} = \partial_{[\mu} A^{(1)}_{\nu]} \). The Chern-Simon (CS) term in eq.(4.2), \( \hat{A} \wedge \hat{H} \), will give

\[
[CS]_{\mu\alpha\beta\gamma} = - \left( a_{\alpha} \partial_{\mu} b_{\beta\gamma} + \text{cyclic perms. of } \alpha, \beta, \gamma \right),
\]

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\[ [CS]_{\mu\nu\alpha\beta} = A^{(4)}_{[\mu} \partial_{\nu]} b_{\alpha\beta} + \left\{ a_\alpha (F^{(2)}_{\mu\nu\beta} - b_{\beta\delta} F^{(1)}_{\mu\nu\delta}) - (\alpha \leftrightarrow \beta) \right\} \]
\[ [CS]_{\mu\nu\rho\sigma} = A^{(4)}_{[\mu} \left( F^{(2)}_{\nu\rho]} - F^{(1)}_{\nu\rho]} b_{\alpha\delta} \right) - a_\alpha H^{(1)}_{\mu\nu\rho} \]
\[ [CS]_{\mu\nu\rho\sigma} = A^{(4)}_{[\mu} H^{(1)}_{\nu\rho\sigma}] . \] (4.10)

Then RR field strengths reduce as given below,

\[ G_{\alpha\beta\gamma\delta} = 2m b_{[\alpha\beta\gamma\delta]} \]
\[ G_{\mu\nu\alpha\beta} = \partial_\mu c_{\alpha\beta\gamma} + [2[CS]_{\mu\nu\alpha\beta} + 2m A^{(2)}_{\mu\nu\alpha\beta}] \]
\[ G_{\mu\nu\alpha\beta} = \partial_\mu A^{(3)}_{\nu\rho\alpha\beta} + F^{(1)}_{\mu\nu\beta} c_{\delta\alpha\beta} + 2[CS]_{\mu\nu\alpha\beta} \]
\[ + 2m \left( B^{(1)}_{\mu\nu\alpha\beta} - (A^{(1)}_{\mu\nu\alpha\beta} - (\alpha \leftrightarrow \beta)) \right) \]
\[ G_{\mu\nu\rho\sigma} = \partial_\mu B^{(2)}_{\nu\rho\sigma\alpha} + F^{(1)}_{\mu\nu\rho\sigma\beta} + 2[CS]_{\mu\nu\rho\sigma} + 2m B^{(1)}_{\mu\nu\rho\sigma\alpha} \]
\[ G_{\mu\nu\rho\sigma} = \partial_\mu C_{\nu\rho\sigma\alpha} + F^{(1)}_{\mu\nu\rho\sigma\beta} + 2[CS]_{\mu\nu\rho\sigma} + 2m B^{(1)}_{\mu\nu\rho\sigma} \]
\[ F_{\alpha\beta} = m b_{\alpha\beta} \]
\[ F_{\mu\alpha} = \partial_\mu a_\alpha + m A^{(2)}_{\mu\alpha} \]
\[ F_{\mu\nu} = F^{(4)}_{\mu\nu} + F^{(1)}_{\mu\nu\beta} a_\delta + m B^{(1)}_{\mu\nu} . \] (4.11)

Now, to consider a specific case, let us look at the reduced effective action in six spacetime dimensions. We utilise the identity (4.8) and use various definitions from eqs.(4.9) and (4.11) to write down the six-dimensional massive IIA action,

\[
S_m = \int d^6 x \sqrt{-g} \left[ e^{-2\phi} \left[ R + 4 \partial_\mu \partial^\mu \phi - \frac{1}{12} H^{(1)}_{\mu\nu\lambda} H^{(1)\mu\nu\lambda} + \frac{1}{8} \text{Tr} \partial_\mu M^{-1} \partial^\mu M - \frac{1}{4} F^{(i)}_{\mu\nu} M^{-1} F^{(j)}_{\mu\nu} \right] \right. \\
\left. - \sqrt{G} \left\{ \frac{1}{2 \cdot 2!} \left[ (F^{(4)}_{\mu\nu} + F^{(1)}_{\mu\nu\beta} a_\delta + m B^{(1)}_{\mu\nu})^2 + 2(\partial_\mu a_\alpha + m A^{(2)}_{\mu\alpha})^2 + (m b_{\alpha\beta})^2 \right) \right. \right. \\
+ \frac{1}{2 \cdot 4!} \left[ (\partial_\mu C_{\nu\rho\sigma\alpha} + F^{(1)}_{\mu\nu\rho\sigma\beta} a_\delta + 2 A^{(4)}_{\mu\nu\rho\sigma} + 2 m B^{(1)}_{\mu\nu\rho\sigma})^2 \right. \\
+ 4(\partial_\mu B^{(2)}_{\nu\rho\sigma\alpha} + F^{(1)}_{\mu\nu\rho\sigma\beta} a_\delta + 2 A^{(3)}_{\mu\nu\rho\sigma} + 2 A^{(4)}_{\mu\nu\rho\sigma} + 2 m B^{(1)}_{\mu\nu\rho\sigma})^2 \right. \\
+ 6(\partial_\mu A^{(3)}_{\nu\rho\sigma\alpha} + F^{(1)}_{\mu\nu\rho\sigma\beta} a_\delta + 2 A^{(4)}_{\mu\nu\rho\sigma} + 2 \left\{ a_\alpha (F^{(2)}_{\mu\nu\beta} - b_{\beta\delta} F^{(1)}_{\mu\nu\delta}) - (\alpha \leftrightarrow \beta) \right\} \\
+ 2 m \left[ B^{(1)}_{\mu\nu\alpha\beta} + A^{(2)}_{\mu\nu\alpha\beta} \right] + 2 m b_{[\alpha\beta\gamma\delta]} \right] + \frac{1}{2} m^2 \left. \right\} \right]. \] (4.12)
where $\phi = \hat{\Phi} - \frac{1}{2} \ln G$ is shifted dilaton field and the scalars coming from $G_{\alpha\beta}$ and $b_{\alpha\beta}$ have been combined to form the symmetric $8 \times 8$ matrix

$$M = \eta M^{-1} \eta = \begin{pmatrix} G^{-1} & -G^{-1}b \\ bG^{-1} & G - bG^{-1}b \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \tag{4.13}$$

where $\eta$ is $O(4,4)$ metric and $I_4$ is 4-dimensional identity matrix. We mention in passing that, if we had chosen to consider compactification on the d-dimensional torus, $T^d$, then the corresponding $2d \times 2d$ symmetric M-matrix will appear, defined in terms of scalars coming from the NS-NS sector and the metric $\eta$ for $O(d,d)$ group with off diagonal identity matrix $I_d$ has to be introduced. Let us recall how various fields appear in the six dimensional action (4.12), after the compactification. In the NS-NS sector we have dilaton field, $\phi$, graviton, $g_{\mu\nu}$, tensor field, $B_{\mu}^{(1)}$, eight vector fields, coming from ten dimensional metric and two index antisymmetric tensor fields after compactification and sixteen scalar fields, $\hat{g}_{\alpha\beta}$ and $\hat{B}_{\alpha\beta}$, appearing in matrix $M$ which parameterize the coset $O(4,4)/O(4) \times O(4)$. On the other hand in the R-R sector there are eight scalars from $\hat{A}_\alpha$ and $\hat{C}_{\alpha\beta\gamma}$, seven vectors from $\hat{A}_\mu$ and $\hat{C}_{\mu\alpha\beta}$, four 2-rank potentials from $\hat{C}_{\mu\nu\alpha}$ and one 3-rank potential $C_{\mu\nu\lambda}$.

Let us recapitulate the symmetry of the six dimensional effective action for the case when $m = 0$ following the works of Sen and Vafa [97]. It was shown in ref.[97] that the action is invariant under $SO(4,4)$ symmetry after the transformation properties of scalar, vector, and tensor fields were defined. In fact the equations of motion are invariant under a larger noncompact symmetry group $SO(5,5)$. On this occasion, the massless case, one can combine dual of 3-rank tensor field $C_{\mu\nu\lambda}$ with seven other RR vector fields to form 8-dimensional spinorial representation, $\psi_a^{(1)}(1 \leq a \leq 8)$, of $SO(4,4)$. Similarly, 3-form field strengths $G_{\mu\nu\lambda\alpha}$ can be taken to be (anti)self-dual to form another 8-dimensional spinorial representation, $\psi_a^{(3)}$. Note that eight RR-scalars do also transform under one of these spinor representation. The afore mentioned symmetry of massless six-dimensional theory was exploited in [97] to generate type II dual pairs. By using this symmetry one can generate type II strings with both NS-NS and R-R charges starting from a background with either NS-NS or R-R charges. However, one can explicitly check that in the case of dimensionally reduced massive theory, the action is no longer invariant under the above noncompact symmetry group.

Now, we investigate the gauge invariance properties of the dimensionally reduced
massive type II action (4.12). This action is invariant under the following set of Stückelberg transformations (although it is tedious calculation),

\[
\begin{align*}
\delta A^{(4)}_\mu &= -m \Lambda_\mu, \\
\delta a_\alpha &= -m \lambda_\alpha, \\
\delta b_{\alpha\beta} &= 0, \\
\delta A^{(1)}_{\mu\alpha} &= 0, \quad \delta A^{(2)}_{\mu\alpha} = \partial_\mu \lambda_\alpha, \\
\delta B^{(1)}_{\mu\nu} &= \partial_{[\mu} \Lambda_{\nu]} + F^{(1)}_{\mu\nu} \lambda_\delta, \\
\delta c_{\alpha\beta\gamma} &= -2m \lambda_{[\alpha b_{\beta\gamma]}, \\
\delta A^{(3)}_{\mu\alpha\beta} &= -2m (\Lambda_\mu b_{\alpha\beta} + A^{(2)}_{\mu[\alpha} \lambda_{\beta]}), \\
\delta B^{(2)}_{\mu\nu\alpha} &= -2m (A^{(2)}_{[\mu} A^{(2)}_{\nu]} + \lambda_\alpha B^{(1)}_{\mu\nu}), \\
\delta C_{\mu\nu\rho} &= -2m A^{(1)}_{[\mu} B^{(1)}_{\nu\rho].}
\end{align*}
\]  

(4.14)

Here, \(A_\mu\) and \(\lambda_\alpha\) are vector and scalar gauge functions respectively. Note that in (4.14) RR-scalars do also transform under Stückelberg transformations in lower dimensions.

4.3 Maximally symmetric black holes in massive IIA theory

Next, we present black hole solutions in five and four dimensions. While looking for black hole solutions, we keep only dilaton and the two form field strengths in the action (4.12) and set the other scalar and tensor fields to zero. First we consider, the following D-dimensional action (in Einstein frame)

\[
S_m = \int d^D x \sqrt{-g} \left[ \left( R_g - \frac{4}{D-2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2 \cdot 2!} \frac{1}{2} e^{\frac{D-4}{2}} F_{\mu\nu} F^{\mu\nu} - 2 \lambda e^{\frac{4-D}{2}} \Phi \right) \\
- \frac{1}{2 \cdot 2!} e^{\frac{2D-4}{2}} F_{R_{\mu\nu}} F^{R_{\mu\nu}} - \frac{1}{2} m^2 e^{2D-4} \Phi \right],
\]  

(4.15)

We have added the term, \(\lambda e^{-\frac{4-D}{2}} \Phi\), to the action and the presence of this term can be interpreted as a dilatonic potential which owes its origin from the NS-NS sector and might appear due to some nonperturbative effects. The \(m^2\) piece comes from the
massive ten dimensional action (4.1) after compactification. In eq. (4.15) the gauge field strengths $F_{\mu \nu}$ and $F_{R \mu \nu}$ come from the NS-NS and RR sectors respectively.

The equations of motion for various fields in (4.15) are

\[
\nabla_\mu \nabla^\mu \phi + \frac{1}{8} e^{-\frac{4}{D-2} \phi} F^2 - \frac{D-4}{16} e^{-\frac{2(D-4)}{D-2} \phi} F_R^2 - \lambda e^{\frac{4}{D-2} \phi} - \frac{m^2 D}{8} e^{\frac{4}{D-2} \phi} = 0, \\
(R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) - \frac{4}{D-2} (\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2) - \frac{e^{-\frac{4}{D-2} \phi}}{4} (2 F_{\mu \lambda} F^\lambda_{\nu} - \frac{1}{2} g_{\mu \nu} F^2) \\
\frac{1}{4} e^{-\frac{2(D-4)}{D-2} \phi} (2 F_{R \mu \lambda} F^\lambda_{R \nu} - \frac{1}{2} g_{\mu \nu} F_R^2) + \frac{1}{2} g_{\mu \nu} (2 \lambda e^{\frac{4}{D-2} \phi} + \frac{1}{2} m^2 e^{\frac{4}{D-2} \phi}) = 0, \\
\partial_\mu e^{\frac{4}{D-2} \phi} F^{\mu \nu} = 0, \\
\partial_\mu e^{\frac{2(D-4)}{D-2} \phi} F_R^{\mu \nu} = 0. 
\]

We seek maximally symmetric black holes solutions [98] and we choose constant dilaton backgrounds $\phi = \phi_0$ with the metric ansatz

\[
ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_{D-2}^2, 
\]

where $e_{D-2}$ and $d\Omega_{D-2}^2$ are the volume element and metric on unit $S^{D-2}$, respectively.

- Our first example is a black hole with the following choice of backgrounds: $F = 0$, $F_R = 0$, with $\lambda < 0$; the solution (4.17) is Schwarzschild-Anti-deSitter(SAdS) space with

\[
e^{2\phi_0} = \frac{8}{5} \frac{\vert \lambda \vert}{m^2}, \\
f(r) = 1 - \frac{2M}{r^2} + \frac{\vert \lambda \vert}{10} \frac{8 \vert \lambda \vert}{m^2} \left[ \frac{1}{5} \right] r^2. 
\]

Note that the black hole solution is asymptotically an AdS space with effective cosmological constant $\Lambda = \frac{2\vert \lambda \vert}{5} \left[ \frac{8\vert \lambda \vert}{5m^2} \right]^{\frac{1}{3}}$.

- The second example corresponds to the backgrounds: $F \neq 0$, $F_R \neq 0$ and the following condition

\[
Q_R^2 m^2 = \frac{8}{5} Q^2 \vert \lambda \vert. 
\]

is satisfied. The charges are defined as

\[
Q = \frac{1}{2\pi^2} \int_{S^3} * e^{-\frac{3}{2} \phi} F, \\
Q_R = \frac{1}{2\pi^2} \int_{S^3} * e^{\frac{3}{2} \phi} F_R. 
\]
The solution (4.17) is the Reissner-Nordstrom-AntideSitter (R-N-AdS) black hole in 5-dimensions

\[ e^{2\phi_h} = \frac{1}{2} \frac{Q_R^2}{Q^2} \]

\[ f(r) = [1 - (r^+)^2][1 - (r^-)^2] + \frac{|\lambda|}{10} \left[ \frac{8|\lambda|}{5m^2} \right] r^2 \]

\[ e^{-\frac{3}{2} \phi} F = Q \epsilon_3, \quad e^{3/2} F_R = Q_R \epsilon_3, \]

\[ r^2_\pm = M \pm (M^2 - \frac{e^2}{2})^{\frac{1}{2}}, \tag{4.21} \]

where \( M \) is a parameter, analog of mass (notice that the space is not asymptotically flat) and \( e = \frac{1}{2} \left[ \frac{Q_R^2 Q^2}{2} \right]^{\frac{1}{2}} \) is related to the product of the two charges \( Q_R \) and \( Q \) defined through (4.20). It follows from eqs. (4.19) and (4.21) the string coupling at the black hole horizon is given by the ratio of the two charges \( Q_R \) and \( Q \), and thus can be adjusted to be small through the judicious choice of the ratio of the two charges. Note that the spacetime in (4.21) is not asymptotically flat but has the curvature equal to \( 5A \). We see from (4.21) that near extremal black hole solution can be obtained in the limit when \( \lambda \) goes to zero and the two horizons come very close to each other, i.e., \( r_+ \sim r_- \). Moreover, for \( \lambda = 0, m = 0 \) above solution in (4.21) reduces to the Strominger-Vafa’s five-dimensional extremal black hole solution [27] as expected.

- We also find a black hole solution in four dimensions for the case when only the 2-form RR-field strength is nonzero and as is well known for \( D = 4 \), the gauge field couples to gravity conformally. The black hole solution of the type (4.17) in four dimensions exists when \( F = 0, F_R \neq 0, \lambda < 0 \). The solution is R-N-AdS with

\[ e^{2\phi_h} = \frac{2|\lambda|}{m^2}, \]

\[ f(r) = (1 - \frac{2M}{r} + \frac{Q_R^2}{r^2}) + \left[ \frac{\lambda^2}{3m^2} \right] r^2, \]

\[ *F_R = Q_R \epsilon_2, \tag{4.22} \]

where \( Q_R = \frac{1}{4\pi} \int_{S^2} *F_R. \)
4.4 Four-brane solution

Next we turn our attentions to obtain 4-brane solutions in six-dimensional model with cosmological constant term in the RR sector, i.e. we set $\lambda = 0$ in this case. We choose the background field configurations in (4.12) so that fields except dilaton and moduli matrix $M$ are nonvanishing. The resulting action takes the following form

$$S_m = \int d^6 x \; \sqrt{-g} \left[ e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi + \frac{1}{8} \text{Tr} \partial_\mu M^{-1} \partial^\mu M \right) - \sqrt{G} \left( \frac{1}{2 \cdot 2!} (m b_{\alpha\beta})^2 + \frac{1}{2 \cdot 3!} \left( m b_{(\alpha\beta)\gamma} \right)^2 + \frac{1}{2} m^2 \right) \right],$$

(4.23)

Note that with the introduction of mass term, $m$, in the ten dimensional effective action (4.1), the reduced action, with the specific choice of the backgrounds, gets a piece which amounts to introducing a potential term involving the moduli $G_{\alpha\beta}$ and $b_{\alpha\beta}$, $\alpha, \beta = 6, 7, 8, 9$. We seek for a four-brane solution around $b_{\alpha\beta} = 0$ and, in the Einstein frame, $(g^E_{\mu\nu} = e^{-\phi} g_{\mu\nu})$ the 4-brane solution is

$$ds^2_E = U^4 \left[ -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \right] + U^\frac{5}{2} dy^2,$$

$$e^{-\frac{5}{2}\phi} = U^\frac{1}{2}, \quad U = \pm m |y - y_0|,$$

$$G_{\alpha\beta} = \delta_{\alpha\beta} e^{\frac{3}{2}\phi}, \quad b_{\alpha\beta} = 0,$$

(4.24)

these background configurations satisfy all the equations of motion derived from (4.23). This is a domain wall solution with a kink singularity (delta-function) at $y = y_0$. The solution is not asymptotically flat, however, for the choice, $U = |y - y_0|$ at large distances, $e^\phi$ vanishes. This solution can be oxidised to obtain 8-brane solution in ten dimensions.

4.5 Summary and discussion

We considered ten-dimensional effective action of type IIA theory in the presence of cosmological constant term which arises as the dual of the ten dimensional field strength coming from the RR sector. The action is dimensionally reduced on a d-dimensional torus with the assumption that the fields do not depend on internal coordinates. The gauge invariance properties of the reduced action is investigated and the transformation properties of the fields in the NS-NS and RR sectors are derived in
the presence of the cosmological constant term. One of the interesting result is noticed in the six dimensional theory. It is found, that in the case of the massive theory, in the presence of this cosmological constant term, the $SO(4,4)$ invariance is lost; whereas the massless theory respects this symmetry. Thus, the cosmological constant coming from the RR sector, in this case, breaks the T-duality symmetry: $SO(4,4)$. Moreover, it is quite evident that, for the six dimensional massive theory, the equations of motion do not respect the $SO(5,5)$ symmetry unlike the massless case [97]. We recall that when one considers a four dimensional heterotic string theory and introduces a cosmological constant (in this case assumed to come from NS-NS sector as central charge deficit), the equations of motion do not respect the S-duality invariance, as found in chapter-2 [70]. We have also presented classical solutions of the effective action. In five dimensions we find black hole solutions in the presence of cosmological constants. It is possible to get near extremal solutions for the choice of small values of cosmological constant parameter, $\lambda$. In this context, we would like to point out that our black holes are anti-de Sitter type and these solutions do not correspond to asymptotically flat case. Therefore, one has to define the Hawking temperature with some care. There have been attempts to understand thermodynamic properties of black holes with (negative) cosmological constant term [99]. Brown, Creighton and Mann [100] identify the thermodynamic internal energy of such a black hole and equate the entropy to $\frac{1}{4}$ of the area of the black hole event horizon. The temperature on the boundary can be defined through thermodynamic relation between these two quantities, such that the black hole temperature, $T_H$, is $\frac{\kappa_H}{2\pi}$ times the redshift factor [101] for temperature in stationary gravitational field. The desired relation is

$$2\pi T(R) = \frac{\kappa_H}{N(R)}$$

(4.25)

where $\kappa_H$ is the surface gravity at the horizon of the black hole and $N(R) = \sqrt{-g_{tt}}$, is the lapse function. The temperature, accordingly, depends on the location of the boundary. We have mentioned earlier that a massive type IIB effective action can be obtained in nine dimensions from the ten dimensional type IIB theory through generalised dimensional reduction due to Scherk and Schwarz. One can adopt the toroidal compactification for that nine dimensional theory to obtain reduced effective action in a way similar to the one presented recently [102] and explore various symmetries in the massive theory.