Chapter 6

Summary and conclusion

This thesis deals with string theory in constant metric $G$ and constant antisymmetric field $B$ with point splitting regularization prescription, in particular noncommutative string theory. We present a summary of our works in this chapter.

In chapter 1, we have very briefly reviewed some of the salient features of string theory: its quantization and properties of the five known superstring theory and also the known concepts of Dp-brane and its charges. Its dynamics both for the single and coincident multiple branes.

In chapter 2, we have studied the origin of noncommutative theory through $O(d,d)$ transformation and such transformations have interesting consequences. One of our interesting results is that we started with a theory endowed with noncommutative string coordinates, $X^i$ i.e. satisfy the algebra

$$[X^i, X^j] = i \theta^{ij}, \quad (6.1)$$

where the $\theta^{ij}$ is defined as in eq. (3.2) and in this case the string is moving in the backgrounds of constant gravity $G_{ij}$ and constant second rank antisymmetric $B_{ij}$ fields. In this setting, we have intended to explore a scenario where one starts from a theory with noncommuting string coordinates and then go over to a dual set of coordinates and backgrounds and examine whether the dual theory is a noncommutative or commutative one.
We have also envisaged the situation when the two end points of the open string are attached to the same brane and we couple string to the resulting $U(1)$ gauge field with constant field strength, i.e. $F = dA$ is constant. We showed that the Hamiltonian obtained from the world sheet action can be cast in a form similar to the one derived for closed strings for constant backgrounds exhibiting $O(d, d)$ invariance and we have argued that there is an analogue of the $O(d, d)$ symmetry in this situation.

It is well known that if one has an open string where some of the coordinates obey the Dirichlet boundary conditions and the rest of them satisfy Neumann boundary conditions, then the usual T-duality operation interchanges these boundary conditions to one another. We have also studied how the boundary conditions are modified under the duality transformations mentioned above. We showed that there exist a set of dual coordinates and backgrounds such that these coordinates satisfy Dirichlet boundary conditions and these dual coordinates belong to a commutative theory. Thus there is an interesting interconnection between a theory which is endowed with the noncommutative and a commutative theories.

We have presented another example to show that the infinitesimal noncompact symmetry transformation $\Omega$ a $(2p+2) \times (2p+2)$ matrix, which is an arbitrary element of the the global $O(p+1, p+1)$ group, which is defined as

$$\Omega = \begin{pmatrix} 1 + \alpha & \beta \\ \lambda & 1 - \alpha^T \end{pmatrix} \tag{6.2}$$

where, $\alpha, \beta$ and $\lambda$ are infinitesimal parameters (actually $(p + 1) \times (p + 1)$ matrices). There is a constraint on $\beta, \lambda$ and is given by $\beta^T = -\beta$ and $\lambda^T = -\lambda$ follows from the $O(p + 1, p + 1)$ invariant condition i.e. $\Omega \eta \Omega^T = \eta$. Here

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.3}$$

is the metric of the $O(p + 1, p + 1)$ group, $\mathbf{1}$ being $(p + 1) \times (p + 1)$ unit matrix, generates mixed boundary condition for a special choice of the parameter i.e. taking the matrix $\alpha$ proportional to identity matrix and in this process we have generated a constant antisymmetric tensor field $B_{ij} = G_{ik} \lambda^{k} G_{ij}$.
We have also obtained the relation between the parameters of the duality transformation and the field strengths through the Seiberg-Witten map. Since, in our work, we are able to generate a constant B-field through duality transformations. Therefore, we can write down the Seiberg-Witten map explicitly in terms of the parameter of the noncompact symmetry transformation. The two field strengths are now related by the expression

\[ \hat{F} = G\lambda(G\lambda + FG^{-1})^{-1}F \]  

(6.4)

In chapter 3, we have studied the consequences of noncommutativity on the world sheet of the string i.e. the operator product expansion of various vertex operators. In particular in the Seiberg-Witten limit i.e.

\[ \alpha' \sim \epsilon^{\frac{1}{2}} \rightarrow 0 \]
\[ g_{ij} \sim \epsilon \rightarrow 0 \text{ for } i, j = 1, \ldots, r, \]  

(6.5)

while keeping B fixed. In this limit G and \( \theta \) becomes

\[ G_{ij} = \begin{cases} -2\pi\alpha' \left( Bg^{-1}B \right)_{ij} & \text{for } i, j = 1, \ldots, r \\ g_{ij} & \text{otherwise} \end{cases} \]

and

\[ \theta_{ij} = \begin{cases} \left( \frac{1}{B} \right)_{ij} & \text{for } i, j = 1, \ldots, r \\ 0 & \text{otherwise} \end{cases} \]

where \( r \) denotes the number of directions along which the brane is extended (this is done with the Euclidean signature). In this limit, it is quite easy to see that the two point correlation function takes the form

\[ \langle X^i(\tau)X^j(\tau') \rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau - \tau'), \]  

(6.6)

and hence it follows from the above correlation function that the operator product expansion of vertex operators on the world sheet of the string in the Seiberg-Witten limit is equal to the \( * \) product of the operators on the world sheet in the \( \tau \rightarrow 0 \) limit,
i.e. if we have inserted two operators one at \( r \) and the other at the origin then the operator product in the \( r \to 0^+ \) gives

\[
\lim_{r \to 0^+} : f(X(r)) \cdot g(X(0)) := f(X(r)) \star g(X(0)) ;
\]  

(6.7)

where the \( \star \) product is as defined

\[
f(x) \star g(x) = e^{\frac{i}{2} \theta^{ij} \partial_i \partial_j} f(y)g(z) \big|_{y=x}. 
\]  

(6.8)

Next, we have revisited the gauge invariance of the system and following Seiberg-Witten we have shown that the gauge field \( \hat{A}_i \), transforms as

\[
\delta \hat{A}_i = \partial_i \lambda + i[\lambda, \hat{A}_i]_\star, 
\]  

(6.9)

where the gauge fields, \( \hat{A}_i \), have been defined with point splitting regularization. Following Seiberg-Witten, we know that there exist a map which relate the commutative variables with the noncommutative variables and this is essentially given by the equations as written in (3.34). We have reviewed the the solutions to these differential equations for different choices of the parameters. For the following path

\[
\delta \theta^{kl} = \delta \alpha \theta^{kl}, 
\]  

(6.10)

we saw that the commutative and noncommutative variables are expressed in terms of a new kind of product, which is commutative but non associative in nature.

In the next section, we have reviewed the construction of gauge invariant operators in noncommutative gauge theory. The gauge invariant operators have been defined using open Wilson line \( W(x, C) \) as defined in eq. (3.52). The curve \( C \) is an open curve. As we have seen that, we can construct gauge invariant objects provided the end points are separated by \( l^i \) distances, where \( l^i = (k\theta)^i \). We have also seen that \( e^{ik \cdot x} \) acts as a translational operator in noncommutative theory i.e.

\[
e^{ik \cdot x} \star f(x) = f(x + k\theta) \star e^{ik \cdot x}. 
\]  

(6.11)

Using this property of \( e^{ik \cdot x} \), we have seen that any non local operator that transforms adjointly and is attached to Wilson line becomes a gauge invariant operator in noncommutative gauge theory i.e. the following non local operator is gauge invariant

\[
\tilde{O}(k) = \int [dx] O(x) \star W(x, C) \star e^{ik \cdot x}, 
\]  

(6.12)
with the curve $C$ is a straight path

$$\xi^i(\sigma) = (k\theta)^i \sigma \quad 0 \leq \sigma \leq 1.$$  (6.13)

We have seen that this way of introducing gauge invariant operators gives rise to generalized star product in short $*$ product. This product is symmetric, as written for $*$ in eq. (3.72). In the next section of this chapter, we have written down the action of the Dp-brane following Mukhi et al. [9]. Where the action has a background independent form.

In chapter 4, we have evaluated various scattering amplitudes in both noncommutative string theory and noncommutative gauge theory. We confirm that in the $\alpha' \to 0$ limit, the noncommutative string theoretic amplitude goes over to the noncommutative gauge theoretic amplitude, and the couplings are related as $g_{YM} = G_0 \sqrt{\frac{1}{2\alpha'}}$, where $G_0$ is related to $G_s$, the string coupling as $G_0 = 16(\pi^{7/2})G_s(\alpha'^2)$. This relation between the couplings of the gauge theory and the string theory is of the same form as that of the commutative theory. Furthermore we show that in this limit there will not be any correction to the gauge theoretic action because of absence of massive modes. In calculating the scattering amplitudes, we have shown that the form of the kinematic factor (K) in the case of the 4-point noncommutative amplitude involving gauge bosons to be of the same form as that in the corresponding commutative theory. We get sin/cos factors in the scattering amplitudes depending on the odd/even number of external photons. From the study of the noncommutative scattering amplitudes we conclude the phases that arises in each sector depends on the ordering of the vertex operators, also the phases of the amplitudes in each sector are not derivable by starting from any one of the amplitude, which we can say that the total amplitude in noncommutative string theory is not symmetric in the Mandelstam variables, even though the kinematic factor (K) is symmetric, but in the total noncommutative amplitude the presence of sin/cos factor will be determined by the coefficient of $e^{ik \cdot X}$ in the vertex operator; particularly the 3-point noncommutative amplitude of photons and tachyons leads to the appearances of phases in the total noncommutative amplitude as 'sin' and 'cos' times the kinematic factors respectively.
In chapter 5, we have shown that the higher order derivative $\alpha'$ corrections to the DBI and Chern-Simon action is derived from non-commutativity in the Seiberg-Witten limit, as conjectured by Das et al., (hep-th/0106024). In particular, we have evaluated the 4-derivative corrections to the $F^3$ term for DBI action and 4-form 4-derivative corrections at $F^4$, 6-form 4-derivative corrections at $F^4$, 8-form 8-derivative corrections at $F^4$ to the Chern-Simon action. The derivative corrections to both Dirac-Born-Infeld action and Chern-Simon has been shown to be in agreement with an independent calculation by Wyllard (hep-th/0008125) using boundary state technique. In calculating the corrections, we have expressed $\tilde{F}$ in terms of $F$, $\tilde{A}$ in terms of $A$ up to order $O(A^3)$, and made use of it.

As a final remark, it is possible that noncommutative field theory might have interesting relevance in particle physics due to the interconnection between UV and IR domain in noncommutative gauge theory. Furthermore, due to the noncommuting nature of the coordinates the description of spacetime in such theories is different from that of the commuting theories which might have several interesting consequences for physics at the plank scale.