Chapter 4

Noncommutative String theory and its low energy limit

4.1 Introduction

Recently there have been interesting developments which exhibit intimate connections between the noncommutative string theory and the noncommutative gauge theories. It is well known that in the zero slope limit, $\alpha' \to 0$, the scattering amplitudes computed from the noncommutative string theory reproduces the corresponding noncommutative field theoretic results. For example, in the case of open bosonic string theory which contains massless spin one particle in its spectrum, when one computes the tree level scattering amplitude involving the gauge bosons in noncommutative string theory and takes $\alpha' \to 0$ limit then the resulting amplitude coincides with the tree level noncommutative field theoretic amplitude. Moreover, generalizations to noncommutative non-Abelian gauge theories can be accomplished using the well known prescription of introducing Chan-Paton factors.

It has been shown that the end points of open strings attached to D-branes do not commute if there is a constant B-field along the brane directions\cite{1, 2, 3}. Therefore, it is natural to explore the connections between string theories (with noncommuting coordinates) and the noncommutative field theories. Indeed, Seiberg and Witten in
their seminal paper, have investigated relations between noncommutative gauge theories and the noncommutative string theory.

In this chapter we carry out some explicit calculations of scattering amplitudes of open string states in noncommutative string theory and then take $\alpha' \to 0$ limit. Then we compare these results with that of the noncommutative field theoretic ones. We may recall that the non-commutativity parameter, $\theta$, defined below is related to the $B$-field and therefore setting $B = 0$ amounts to going over to commutative theory.

The non-commutativity is defined as $[X^\mu, X^\nu] = i\theta^{\mu\nu}$. One can derive a stringy uncertainty relation for the noncommutative coordinates, starting from the noncommutative algebra, which is

$$\Delta X^\mu \Delta X^\nu \geq \frac{|\theta^{\mu\nu}|}{2} \quad (4.1)$$

From this equation it is very easy to see that the small change along one direction in $X^\mu$ is related to the large change in another direction in $X^\mu$. But we know that $\Delta X^\mu \to 0$ corresponds to the UV divergence whereas $\Delta X^\mu \to \infty$ corresponds to the presence of IR divergence. It therefore implies that this kind of commutation relation gives rise to the UV/IR mixing, but to see the appearance explicitly one needs to go to loop level.

The presence of $B_{\mu\nu}$ field makes the open string to behave differently in two distinct ways from that of a theory without this field. (1) The scattering amplitude depends on the ordering of vertex operators, (2) NCOS theory: theories where open strings decouple from closed strings [4, 5, 7] (and there by makes a large class of massive open string modes stable). Apart from this UV/IR mixing [11], as mentioned above, the field theory associated to the string theory, in the low energy limit, is nonlocal, because the fields in the action are multiplied by a (deformed) $\ast$ product. The deformed product is defined as

$$A \ast B(x) = \exp(i\frac{1}{2}\theta^{\mu\nu}\partial^\mu\partial^\nu)A(y)B(z)\big|_{y=z=x} \quad (4.2)$$

This implies the presence of (infinitely many) derivatives in the action, hence the theory becomes non-local. We must mention that $\ast$ product has no effect on the
kinetic energy term under integral. We can classify the noncommutative theories in 3 ways and the classification will be characterized through the non-commutativity parameter $\theta^{\mu\nu}$ i.e. depending on the nonvanishing components along different directions. (1) Space/Space noncommutative theories: Which is defined as $\theta^{\mu\nu} \neq 0$ and the other components of it are 0, one gets these type of theories by suspending the D-brane in a background magnetic field. Moreover, one can show that this theory has low energy limit i.e. $\alpha' \rightarrow 0$ limit and as has been suggested in [9] that unitary field theories can be derived from string theory in a suitable limit. Since we shall show that space/space noncommutative field theories are derived from string theory it implies, according to the above argument, that these theories are unitary. In passing we mention that in this background there exist various dualities that has been mentioned in [1, 6]. (2) Space/time noncommutative theories: This class of theories has nonvanishing components only along the time and spatial directions i.e. $\theta^{0\mu} \neq 0$, one gets these kind of theories by suspending open strings in the background electric fields. As is shown in [8] that the field theory associated with this type of string theory will become non-unitary which means that this type of string theory does not have a well-defined low energy limit. (3) Light-like theories: Which is defined as $\theta^{0\mu} = -\theta^{1\mu}$, these theories are the results of embedding open strings in both electric and magnetic fields. These theories have a low energy or field theoretic limit which implies that the low energy theory is unitary[9]. We have summarized these properties in a table below.

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1 Which can be seen in momentum space easily.

2 Throughout in our calculations we shall take $\mu, \nu...$ as spatial directions as we shall be dealing with space/space non-commutativity only.
In \( \alpha' \to 0 \) limit of noncommutative string theory, as we shall see that at tree level all the massive open string states decouple and the relevant degrees of freedom are the massless open string states. As has been shown in [1] there exists a decoupling limit in which the string dynamics can be decoupled from the gauge theory degrees of freedom, where the gauge field theory lives on a noncommutative space [10].

The plan of this chapter is as follows: In section 4.2 we shall review the four point gauge boson amplitude in noncommutative string theory, which is of the same form as that of Type I superstring four point amplitude in commutative theory, except the phase factors that is multiplied with the kinematic factors in each sectors. We shall derive the 3-point amplitude for the scattering of open strings in noncommutative string theory, and the \( \text{SL}(2,\mathbb{R}) \) invariance can be fixed by inserting the vertex operators at 0, 1, \( \infty \), and shall show that in the \( \alpha' \to 0 \) limit the noncommutative string theory goes over to the corresponding noncommutative field theory, through explicit calculations. In section 4.3, we derive four point amplitudes for the case of one photon, two photon, three photon with three tachyon, two tachyon and one tachyon respectively in noncommutative bosonic string theory.

In this study we have demonstrated explicitly that in the \( \alpha' \to 0 \) limit noncommutative string theory reduces to the noncommutative gauge theory and in the said limit no corrections to the noncommutative gauge theoretic action is observed due to the absence of massive string modes. In particular we have shown the form of the kinematic factor \( (K) \) in the case of the 4-point noncommutative amplitude involving
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gauge bosons to be of the same form as that in the corresponding commutative theory with the desired properties. We also get sin/cos factors in the noncommutative scattering amplitudes depending on the number of external photons. From the study of the noncommutative scattering amplitudes we conclude that the phases arises depending on the ordering of the vertex operators but in the total noncommutative amplitude the presence of sin/cos factor will be determined by the coefficient of $e^{ik \cdot X}$ in the vertex operator, particularly the 3-point amplitude of photon and tachyon leads to the appearances of phases in the total amplitude as 'sin' and 'cos' times the kinematic factors respectively. The couplings of noncommutative string theory $G_s$ and the noncommutative gauge theory $g_{YM}$ are related as $g_{YM} = \frac{G_0}{\sqrt{2a'}}$, where $G_0$ is related to $G_s$, the string coupling. This relation between the couplings of the gauge theory and the string theory is of the same form as that of the commutative theory.

4.2 Tree level amplitudes

We shall show explicitly that the noncommutative field theoretic amplitudes can be reproduced by taking the $\alpha' \to 0$ limit of the noncommutative string theory amplitudes. Also, we shall relate the couplings of both the theories by demanding the equality of the tree level amplitudes in this limit.

4.2.1 4-point amplitude of massless gauge bosons at tree level in noncommutative superstring theory

In order to evaluate the scattering amplitude of four gauge bosons in noncommutative string theory, we shall follow the procedure of [12]. For the sake of completeness we shall demonstrate it. The scattering amplitudes are computed by evaluating the correlation function of the vertex operators corresponding to the asymptotic states in the scattering process. The form of the amplitude for the processes that we are interested in is

$$A(e_1, p_1; e_2, p_2; e_3, p_3; e_4, p_4) = \int \frac{dx_1 dx_2 dx_3 dx_4}{V_{CKG}} (: e_1 V_0(p_1, x_1) : e_2 V_0(p_2, x_2) :$$
Figure 4.1: The vertex operators are inserted at $x_1, x_2, x_3, x_4$ on the disk and we are fixing $\text{SL}(2,\mathbb{R})$ by inserting three of them at $0, 1, \infty$. The position of the vertex operator increases along the direction of arrow.

\begin{equation}
: e_3.V_{-1}(p_3, x_3) :: e_4.V_{-1}(p_4, x_4) : \right),
\end{equation}

where $V_{\text{CKG}}$ is the volume of the conformal killing group which remains as a residual gauge symmetry even after choosing the conformal gauge and $e_i$'s are the polarization vectors of the massless gauge bosons. The form of the vertex operators are

\begin{equation}
V_0^\mu(p, z) = G_0(2\alpha')^{-\frac{3}{2}}(\partial X^\mu(z) + 2i\alpha' p.\psi\psi^\mu)e^{ip.X}
\end{equation}

\begin{equation}
V_{-1}^\nu(p, z) = G_0e^{-\phi(z)}\psi^\nu(z)e^{ip.X},
\end{equation}

where $G_0$ is defined as $G_0 = 16(\pi^{7/2})G_s(\alpha')^2$, $G_s$ is the open string coupling constant: $G_s = g_s(\frac{\det(g + 2\pi\alpha' B)}{\det g})^{\frac{3}{2}}$, $g$ and $B$ are the closed string background fields. The subscript 0 and -1 in the vertex operator denote the super-ghost charge carried by the vertex operators. The total ghost charge on a disk must be -2. This is a consequence of
superdiffeomorphism invariance on the string world sheet[14]. The following operator product expansion (OPE) will be useful for computation of scattering amplitudes

\begin{equation}
(X^\mu(x_1)X^\nu(x_2)) = -\alpha' G^{\mu\nu} \log(x_1 - x_2)^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon(x_1 - x_2) \tag{4.6}
\end{equation}

\begin{equation}
\langle \psi^\mu(x_1)\psi^\nu(x_2) \rangle = -\frac{G^{\mu\nu}}{x_1 - x_2} \tag{4.7}
\end{equation}

where $G^{\mu\nu}$ and $\theta^{\mu\nu}$ are as defined in [1], which is $G^{\mu\nu} = (\frac{1}{g+2\pi\alpha'B}g^{\mu\nu})$ and $\theta^{\mu\nu} = -(2\pi\alpha')^2(\frac{1}{g+2\pi\alpha'B}B_{\mu\nu})$. We shall map the disk on to the upper half plane and follow, what is called, the traditional method for canceling the conformal Killing volume is to set the vertex operators on the real axis, as shown in the figure 1, but there are 6 inequivalent ordering of the vertex operator on the disk. For the first 3 diagrams the integral of $x_1$ ranges from $-\infty < x_1 < 0$, $0 < x_1 < 1$, $1 < x_1 < \infty$ and the integration ranges for the last 3 diagrams can be obtained by interchanging the vertex operator inserted at position $x_3$ with $x_4$, i.e. $x_3 \leftrightarrow x_4$. For the evaluation of the amplitude, we shall use transversality and masslessness properties $e.p = 0$ and $p^2 = 0$ respectively for the gauge bosons in our calculations.

In order to write the amplitudes in each sector in a manifestly Lorentz invariant manner, we shall introduce the well known Mandelstam variables for the gauge bosons, which are defined as $s = -2p_1.p_2$, $t = -2p_1.p_3$, and $u = -2p_1.p_4$, since $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ for the gauge bosons. The amplitudes for the first 3 diagrams are

\begin{equation}
A(s - t) = 8 \frac{G^2_0}{2\alpha' st} \frac{\Gamma(1 - \alpha's)\Gamma(1 - \alpha't)}{\Gamma(1 + \alpha'u)} \times e^{-(p_1^\mu p_2^\nu - p_3^\mu p_4^\nu)} \tag{4.8}
\end{equation}

\begin{equation}
A(t - u) = 8 \frac{G^2_0}{2\alpha' ut} \frac{\Gamma(1 - \alpha'u)\Gamma(1 - \alpha't)}{\Gamma(1 + \alpha's)} \times e^{(p_1^\mu p_4^\nu + p_2^\mu p_3^\nu)} \tag{4.9}
\end{equation}

\begin{equation}
A(s - u) = 8 \frac{G^2_0}{2\alpha' us} \frac{\Gamma(1 - \alpha'u)\Gamma(1 - \alpha's)}{\Gamma(1 + \alpha't)} \times e^{(p_1^\mu p_2^\nu + p_3^\mu p_4^\nu)} \tag{4.10}
\end{equation}

3\(\Gamma(m)\) and $B(m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$ are the well-known Gamma and Beta functions respectively.
It is obvious to see that the phases of the amplitudes in each sector are not derivable by starting from any one of the amplitude, thus we conclude that the total amplitude in noncommutative string theory is not symmetric in the Mandelstam variables. The indices are contracted as, \( p\theta k = p_\mu \theta^{\mu \nu} k_\nu \), where \( K \) is

\[
K = (2\pi)^{10} \delta^{10}(\sum_i p_i) \times [-\frac{1}{4}(ute_1.e_2.e_3.e_4 + su_1.e_3.e_2.e_4 + su_1.e_4.e_2.e_3)
\]

\[
\]

\[
+ \frac{t}{2}(e_1.e_4.e_3.p_4.e_2.p_1 + e_4.p_3.e_1.p_2.e_3 + e_2.p_3.e_1.p_4.e_3.e_4 + e_1.e_2.e_3.p_4.p_1) +
\]

\[
+ \frac{u}{2}(e_1.e_2.e_3.p_1.e_4.p_2 + e_3.p_4.e_1.e_2.e_4 + e_4.p_3.e_2.p_1.e_3 + e_2.p_4.e_3.p_3.e_4)] \tag{4.11}
\]

To evaluate the rest of the 3 diagrams interchange \( x_3 \) with \( x_4 \) and \( p_3 \) with \( p_4 \). Thus the total amplitude for the scattering of 4 massless gauge bosons at tree level of the noncommutative string theory in the \( \alpha' \to 0 \) limit is

\[
A_{Total}^{Abelian} = \frac{16 G_6^2}{2\alpha'} \left[ \frac{K}{st} \cos(\frac{p_1 \theta p_2 - p_3 \theta p_4}{2}) \right] + \frac{K}{su} \cos(\frac{p_1 \theta p_2 + p_3 \theta p_4}{2}) + \frac{K}{ut} \cos(\frac{p_1 \theta p_4 + p_2 \theta p_3}{2}) \tag{4.12}
\]

It is easy to derive the non-Abelian noncommutative string theory amplitude by multiplying the trace of the product of the Chan-Paton factors to the amplitudes in different sectors in accordance with the insertion of the vertex operators. Finally the full (non-Abelian noncommutative) 4-point amplitude is

\[
[A(s-t)tr(\lambda^{a_2}\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}) + A(u-t)tr(\lambda^{a_2}\lambda^{a_3}\lambda^{a_1}\lambda^{a_4}) + A(s-u)tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}) + 3 \leftrightarrow 4]
\]

\[
\tag{4.13}
\]

In the \( \theta \to 0 \) limit this amplitude matches with the four point amplitude of Type I superstring theory[13]. The scattering amplitudes of massless scalars can be derived by choosing the polarizations to lie in the transverse direction, \( e_1.p_m = 0 \). For example, the four massless scalar amplitude in noncommutative string theory in the \( \alpha' \to 0 \) limit is

\[
A_{scalar}^{\alpha'} = \frac{16 G_6^2}{2\alpha'} \left[ \frac{M}{st} \cos(\frac{p_1 \theta p_2 - p_3 \theta p_4}{2}) \right] + \frac{M}{su} \cos(\frac{p_1 \theta p_2 + p_3 \theta p_4}{2}) + \frac{M}{ut} \cos(\frac{p_1 \theta p_4 + p_2 \theta p_3}{2})
\]

\[
M = (2\pi)^{10} \delta^{10}(\sum_i p_i) \times [-\frac{1}{4}(ute_1.e_2.e_3.e_4 + su_1.e_3.e_2.e_4 + su_1.e_4.e_2.e_3)] \tag{4.14}
\]
From the expression (4.12) one can derive the scattering amplitude for two gauge bosons with two scalars, three gauge bosons with one scalar and one gauge boson with three scalars using $\epsilon_i \cdot p_m = 0$ in the kinematic factor $K$.

The intermediate particles that propagates in the s-t channel of the $U(1)$ noncommutative string theoretic amplitude are $\alpha m^2 = 0, 1, 2, 3, \ldots$, but going to its low energy limit, one sees from (4.12) that only the massless particles propagates at the intermediate stages.

**Noncommutative gauge theoretic amplitude at tree level**

The action for the noncommutative gauge theory that we are interested in is [1]

$$S = -\frac{1}{4} \int d^{10}x \hat{F}_{\mu\nu} \times \hat{F}^{\mu\nu} \quad (4.15)$$

Where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - igYM[\hat{A}_\mu, \hat{A}_\nu]$, the fields are multiplied through $\times$ product and it is defined as $A \times B(x) = \exp(i \frac{1}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu) A(y) B(z) |_{y = \lambda = x}$. The Feynman rules for this action are

$$V(p_1; p_2; p_3) = 2igYM(2\pi)^4 \delta^{10}(p_1 + p_2 + p_3)[G^{\rho\sigma}(p_3 - p_1)\nu + G^{\rho\sigma}(p_2 - p_3)\mu + G^{\rho\mu}(p_1 - p_2)\rho] \sin\left(\frac{p_1 \cdot p_2}{2}\right)$$

$$V(p_1; p_2; p_3; p_4) = 4gYM^2(2\pi)^4 \delta^{10}(p_1 + p_2 + p_3 + p_4)[\sin\left(\frac{p_1 \cdot p_2}{2}\right) \sin\left(\frac{p_3 \cdot p_4}{2}\right) \sin\left(\frac{p_1 \cdot p_4}{2}\right) \sin\left(\frac{p_2 \cdot p_3}{2}\right)]$$

$$\left[ (G^{\mu\rho} G^{\nu\sigma} - G^{\mu\sigma} G^{\rho\nu}) + \sin\left(\frac{p_1 \cdot p_3}{2}\right) \sin\left(\frac{p_2 \cdot p_4}{2}\right) \left( G^{\mu\nu} G^{\rho\sigma} - G^{\mu\sigma} G^{\rho\nu} \right) + \sin\left(\frac{p_1 \cdot p_4}{2}\right) \sin\left(\frac{p_2 \cdot p_3}{2}\right) \left( G^{\mu\nu} G^{\rho\sigma} - G^{\mu\sigma} G^{\rho\nu} \right) \right] \quad (4.17)$$

We shall use the propagator as $-\frac{G_{\delta}^2}{q^2}$ in the evaluation of the noncommutative amplitude. The corresponding diagrams are shown in figure 2. $V(p_1; p_2; p_3)$ and $V(p_1; p_2; p_3; p_4)$ are the 3-point and the 4-point vertex functions respectively. So at this level i.e. to order $gYM^2$ there are 4 diagrams, without loops, one is the contact interaction term and the other 3 are s, t and u channel diagrams. The total noncommutative amplitude to order $gYM^2$ is
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\[ A_{\text{Total}} = 16g_Y^2 \left[ \frac{K}{st} \cos \left( \frac{p_1 \theta p_2 - p_3 \theta p_4}{2} \right) + \frac{K}{su} \cos \left( \frac{p_1 \theta p_2 + p_3 \theta p_4}{2} \right) + \frac{K}{ut} \cos \left( \frac{p_1 \theta p_4 + p_2 \theta p_3}{2} \right) \right] \] (4.18)

The form of this noncommutative gauge theoretic amplitude is same as that of the noncommutative string theoretic amplitude (4.12) apart from over all constants. This implies that there will not be any correction to the noncommutative gauge theoretic action due to the absence of other modes. Now comparing the amplitudes of noncommutative string theory with noncommutative gauge theory, we get a relation between the string parameters with the Yang-Mills coupling namely, \( g_Y^2 (2\alpha') = G_0^2 \)

### 4.2.2 3-point amplitude in noncommutative open string theory

The 3-point noncommutative scattering amplitude of open string massless modes on the disk is

\[ A(e_1, p_1; e_2, p_2; e_3, p_3) = \frac{1}{\alpha' G_0^2} \left[ cV_{-1}(p_1, x_1) :: cV_{-1}(p_2, x_2) :: cV_0(p_3, x_3) : \right] + 1 \leftrightarrow 2 \] (4.19)

The vertex operators are written as earlier and here \( c \) stands for are the ghost fields moreover we need super ghost charge on the disk to be -2 as mentioned earlier. Let us map the disk to the upper half plane and fix the vertex operators at \( x_3 = 0, x_2 = 1, x_1 = \infty \) to cancel the residual SL(2,R) invariance that is left over even after working in the conformal gauge. We shall evaluate the correlation function using the OPE's as written earlier and next, going over to \( \alpha' \to 0 \) limit we find that the noncommutative amplitude derived agrees with the one obtained from the corresponding \( U(1) \) noncommutative gauge theory apart from over all constants. The 3-point amplitude in \( U(1) \) noncommutative gauge theory is given by

\[ A(e_1, p_1; e_2, p_2; e_3, p_3) = 2ig_Y(2\pi)^{10} \delta^{10}(\sum p_i) \times \]

\[ [e_1 \cdot e_3(p_3 - p_1) \cdot e_2 + e_2 \cdot e_3(p_2 - p_3) \cdot e_1 + e_1 \cdot e_2(p_1 - p_2)] \sin \left( \frac{p_1 \theta p_2}{2} \right) \] (4.20)
On looking at the kinematic factors we see that in the $\alpha' \to 0$ limit there are not any poles, which implies that the noncommutative gauge theoretic action will remain unchanged, at least in tree level, as suggested in [1]. The coupling in both the noncommutative theories are related as $g_{YM} = G_0 \sqrt{\frac{1}{2\alpha'}}$. This relation between the couplings are in the same form as found in the commutative theory. The corresponding non-Abelian noncommutative 3-point amplitude in the $\alpha' \to 0$ limit is same as that of the noncommutative gauge theoretic one and it is

$$A \sim (2\pi)^{10} \delta^{10}(\sum_i p_i)[e_1.e_3(p_3-p_1).e_2 + e_2.e_3(p_2-p_3).e_1 + e_1.e_2(p_1-p_2)] \times$$

$$[e^{-i\frac{\epsilon_2\epsilon_3}{2}} \text{tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) - e^{i\frac{\epsilon_1\epsilon_3}{2}} \text{tr}(\lambda^{a_2} \lambda^{a_1} \lambda^{a_3})]$$

(4.21)

### 4.3 4-point amplitudes in noncommutative open bosonic string theory

In this section we shall evaluate 4-point amplitudes at tree level in noncommutative bosonic string theory, and consider cases where the external particles are gauge bosons and tachyons. We shall show that if we are taking odd number of photons then the amplitude will have a 'sin' factor and for even number of photons it will have a cosine factor. Before evaluating the scattering amplitude of photons with the tachyons, we shall evaluate the 3-point and 4-point tachyonic amplitude which will demonstrate that the 'sin' factor arises due to the presence of odd number of photons (even though we are checking it by taking tachyons only).

#### 3-point tachyonic amplitude:

The noncommutative amplitude is

$$A(p_1; p_2; p_3) = \frac{1}{\alpha' G_0^2} x_{12} x_{13} x_{23} \langle V(p_1) V(p_2) V(p_3) \rangle + 2 \leftrightarrow 3$$

(4.22)

where $V(p) = G_0 e^{ip.X}$, we shall evaluate it by fixing the vertex operators at $x_1 = 1, x_2 = 0, x_3 = \infty$ which will take care of the $\text{SL}(2,\mathbb{R})$ invariance and using the mass
shell condition as $p^2 = \frac{1}{\alpha'}$, we get

$$A(p_1; p_2; p_3) = 2 \frac{G_0}{\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i p_i \right) \cos \left( \frac{p_1 \theta p_2}{2} \right)$$

(4.23)

Scattering of four tachyons:

As has been mentioned in the evaluation of 4 massless gauge bosons, there are 6 diagrams that one has to evaluate in order to derive the total amplitude, and there are 2 diagrams in each channel. The amplitude in the s-u channel can be derived by fixing the vertex operators at $x_2 = 0, x_3 = 1, x_4 = \infty$ plus $3 \leftrightarrow 4$, in order to cancel the $SL(2,\mathbb{R})$ invariance and for these 2 diagrams the integration ranges of $x_i$ are from $-\infty$ to 0 and to 1 respectively. The form of the bosonic photon vertex that we are taking is $V(e : p) = \frac{G_0}{\sqrt{\alpha'}} e \partial X e^{ip \cdot X}$. The noncommutative amplitude in this sector is

$$A_{su}(p_1; p_2; p_3; p_4) = -2 \frac{G_0}{\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i p_i \right) \cos \left( \frac{p_1 \theta p_2 + p_3 \theta p_4}{2} \right) \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' u)}{\Gamma(2 + \alpha' t)}$$

(4.24)

The amplitude in the s-t sector can be derived by fixing the vertex operators at $x_2 = 0, x_3 = \infty, x_4 = 1$ and integrating $x_1$ from 0 to 1, and for the 2nd diagram in this sector we fix the vertex operators at $x_2 = 0, x_3 = \infty, x_4 = 1$ and integrating $x_1$ from $-\infty$ to 0. We get the noncommutative amplitude in this sector as

$$A_{st}(p_1; p_2; p_3; p_4) = -2 \frac{G_0}{\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i p_i \right) \cos \left( \frac{p_1 \theta p_2 - p_3 \theta p_4}{2} \right) \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' t)}{\Gamma(2 + \alpha' u)}$$

(4.25)

For the amplitude in the u-t channel, we fix the vertex operators at $x_2 = 0, x_3 = 1, x_4 = \infty$ and $x_2 = 0, x_3 = \infty, x_4 = 1$ and integrating $x_1$ from 1 to $\infty$ for these two diagrams, one gets the noncommutative amplitude in this sector as

$$A_{ut}(p_1; p_2; p_3; p_4) = -2 \frac{G_0}{\alpha'} (2\pi)^{26} \delta^{26} \left( \sum_i p_i \right) \cos \left( \frac{p_1 \theta p_4 + p_2 \theta p_3}{2} \right) \frac{\Gamma(-1 - \alpha' u) \Gamma(-1 - \alpha' t)}{\Gamma(2 + \alpha' s)}$$

(4.26)

The total amplitude is the sum of the amplitudes in each sector.

Scattering of one photon with 3 tachyons:

Here the fixing of the vertex operators shall be the same as in the previous subsection
and photon has been inserted at $x_2$. The total noncommutative amplitude is the sum of amplitudes in each channel, $A_{st} + A_{su} + A_{ut}$, which is

$$A(p_1; e_2, p_2; p_3, p_4) = -\frac{4G^2_e}{\sqrt{\alpha'}} \times \left[ \sin\left(\frac{p_1 p_2 + p_3 p_4}{2}\right) (e_2.p_3(1 + \alpha' t) - e_2.p_4(1 + \alpha' u)) \right]$$

$$A(p_1; e_2, p_2; p_3, p_4) = \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' u)}{\Gamma(2 + \alpha' t)} - \sin\left(\frac{p_1 p_2 - p_3 p_4}{2}\right) (e_2.p_3(1 + \alpha' t) - e_2.p_4(1 + \alpha' u))$$

$$A(p_1; e_2, p_2; p_3, p_4) = \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' t)}{\Gamma(2 + \alpha' u)} - \sin\left(\frac{p_1 p_3 + p_2 p_4}{2}\right) (e_2.p_3(1 + \alpha' t) - e_2.p_4(1 + \alpha' u))$$

$$A(p_1; e_2, p_2; p_3, p_4) = \frac{\Gamma(-1 - \alpha' t) \Gamma(-1 - \alpha' u)}{\Gamma(2 + \alpha' s)} \left[ \right]$$

(4.27)

Scattering of two photons with two tachyons:

Here the fixing of the vertex operators is same but the photons are inserted at $x_1, x_2$ and the tachyons are at $x_3$ and $x_4$. Total noncommutative amplitude in this case is

$$A(e_1, p_1; e_2, p_2; p_3, p_4) = \frac{4G^2_e}{\alpha'} \cos\left(\frac{p_1 p_2 + p_3 p_4}{2}\right) ((e_1.e_2 - 2\alpha'(e_1.p_3 e_2.p_3 + e_1.p_4 e_2.p_4))$$

$$(1 + \alpha' t)(1 + \alpha' u) + 2\alpha'(e_2.p_4 e_1.p_3 \alpha' u(1 + \alpha' u) + e_2.p_3 e_1.p_4 \alpha' t(1 + \alpha' t)))$$

$$\frac{\Gamma(-1 - \alpha' u) \Gamma(-1 - \alpha' t)}{\Gamma(2 + \alpha' s)} + \cos\left(\frac{p_1 p_2 - p_3 p_4}{2}\right) ((e_1.e_2 - 2\alpha'(e_1.p_3 e_2.p_3 + e_1.p_4 e_2.p_4))$$

$$(1 + \alpha' t)(1 + \alpha' u) - 2\alpha'(e_2.p_4 e_1.p_3 \alpha' u(1 + \alpha' u) + e_2.p_3 e_1.p_4 \alpha' t(1 + \alpha' t)))$$

$$\frac{\Gamma(-1 - \alpha' t) \Gamma(-1 - \alpha' u)}{\Gamma(2 + \alpha' u)} + \cos\left(\frac{p_1 p_2 - p_3 p_4}{2}\right) ((e_1.e_2 - 2\alpha'(e_1.p_3 e_2.p_3 + e_1.p_4 e_2.p_4))$$

$$(1 + \alpha' t)(1 + \alpha' u) - 2\alpha'(e_2.p_4 e_1.p_3 \alpha' u(1 + \alpha' u) + e_2.p_3 e_1.p_4 \alpha' t(1 + \alpha' t)))$$

$$\frac{\Gamma(-1 - \alpha' u) \Gamma(-1 - \alpha' t)}{\Gamma(2 + \alpha' s)}$$

(4.28)

Instead of inserting the photons at $x_1, x_2$ and the tachyons at $x_3, x_4$, if we shall insert the tachyons at $x_2, x_4$ and photons at $x_1, x_3$ then the total noncommutative amplitude is

$$A(e_1, p_1; e_3, p_3; p_4) = \frac{4G^2_e}{\alpha'} \cos\left(\frac{p_1 p_2 + p_3 p_4}{2}\right) ((e_1.e_3(1 + \alpha' u)(1 + \alpha' s) +$$

$$2\alpha'(e_1.p_3 e_3.p_1 \alpha' u(1 + \alpha' u) + e_1.p_4 e_3.p_2 \alpha' t(1 + \alpha' t) - (e_1.p_3 e_3.p_2 + e_1.p_4 e_3.p_1)$$

$$(1 + \alpha' u)(1 + \alpha' t))) \frac{\Gamma(-1 - \alpha' s) \Gamma(-1 - \alpha' u)}{\Gamma(2 + \alpha' t)} + \cos\left(\frac{p_1 p_2 - p_3 p_4}{2}\right)$$
\[ (-e_1.e_3(1+\alpha' u)(1+\alpha s)+2\alpha'(-e_1.p_3.e_3.p_1\alpha' u(1+\alpha' u) - e_1.p_4.e_3.p_2\alpha' t(1+\alpha' t) + (e_1.p_3.e_3.p_2 + e_1.p_4.e_3.p_1)(1+\alpha' u)(1+\alpha' t))\frac{\Gamma(-1-\alpha s)\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' u)} + \\
\cos\left(\frac{p_1.\theta p_2 + p_3.\theta p_4}{2}\right)(-e_1.e_3(1+\alpha' u)(1+\alpha s)+2\alpha'(-e_1.p_3.e_3.p_1\alpha' u(1+\alpha' u) - e_1.p_4.e_3.p_2\alpha' t(1+\alpha' t) + (e_1.p_3.e_3.p_2 + e_1.p_4.e_3.p_1)(1+\alpha' u)(1+\alpha' t))\frac{\Gamma(-1-\alpha' t)\Gamma(-1-\alpha' u)}{\Gamma(2+\alpha' s)} \right) \quad (4.29) \]

Scattering of 3-photons with one tachyons:

Here vertex operators are fixed as in earlier cases, but the tachyon is at \( x_1 \) and photons are located at rest positions. Total noncommutative amplitude is

\[
A(p_1; e_2, p_2; e_3, p_3; e_4, p_4) = \frac{8G^2}{\sqrt{\alpha'}}(\sin(\frac{p_1.\theta p_2 + p_3.\theta p_4}{2})|e_2.e_3.e_4.p_1 B(-1-\alpha u, -\alpha' s) + \\
(e_2.e_3.e_4.p_2 - e_2.e_4.e_3.p_2)B(-\alpha' s, -\alpha' u) - e_2.e_4.e_3.p_1 B(-1-\alpha' u, -\alpha' s) + \\
2\alpha' e_4.p_2.e_3.p_1.e_2.p_4 B(-1-\alpha' s, 2-\alpha' u))\frac{\sin(\frac{p_1.\theta p_2 + p_3.\theta p_4}{2})}{2} \\
\left[ -e_2.e_3.e_4.p_1 B(1-\alpha' t, -\alpha' s) + (-e_2.e_3.e_4.p_2 + e_2.e_4.e_3.p_2)B(-\alpha' s, -\alpha' t) + \\
e_2.e_4.e_3.p_1 B(-1-\alpha' t, -\alpha' s) + (e_3.e_4.e_3.p_2 - 2\alpha'(e_4.p_1.e_3.p_1.e_2.p_3 + e_4.p_2.e_3.p_2.e_2.p_3 + \\
e_4.p_1.e_3.p_2.e_4.p_4)) B(-\alpha' s, 1-\alpha' t) + (e_3.e_4.e_4.p_4 - 2\alpha'(e_4.p_1.e_3.p_1.e_2.p_4 + \\
e_4.p_2.e_3.p_2.p_4 + e_4.p_3.e_3.p_1.e_2.p_3)) B(1-\alpha' t, -1-\alpha' s) - 2\alpha' e_4.p_1.e_3.p_2.e_2.p_3 \\
B(-1-\alpha' s, 2-\alpha' t) - 2\alpha' e_4.p_2.e_3.p_1.e_2.p_4 B(-1-\alpha' s, -1-\alpha' t)\right] \\
(\sin(\frac{p_1.\theta p_4 + p_2.\theta p_3}{2})|e_2.e_3.e_4.p_1 B(-1-\alpha' u, 1-\alpha' t) + \\
(e_2.e_3.e_4.p_2 - e_2.e_4.e_3.p_3)B(-\alpha' t, -\alpha' u) + e_2.e_4.e_3.p_1 B(1-\alpha' u, -1-\alpha' t) + \\
e_4.p_2.e_3.p_1.e_2.p_3)) B(1-\alpha' u, -\alpha' t) + 2\alpha' e_4.p_1.e_3.p_2.e_2.p_3 B(2-\alpha' t, -1-\alpha' u) - \\
2\alpha' e_4.p_2.e_3.p_1.e_2.p_4 B(-1-\alpha' t, 2-\alpha' u)) \right) \quad (4.3c)\]
By going through different noncommutative amplitudes in this section we conclude that 'sin' factor arises only when we have odd number of photons, which implies that in the corresponding commutative theory the amplitude vanishes.

4.4 Conclusions

In this study we have demonstrated explicitly that in the $\alpha' \to 0$ limit noncommutative string theory reduces to the noncommutative gauge theory and in the said limit no corrections to the noncommutative gauge theoretic action is observed (due to the absence of massive string modes), and this is in agreement with conclusions of [1]. In particular we have shown that the form of the kinematic factor (K) in the case of the 4-point noncommutative amplitude involving gauge bosons is found to be of the same form as that in the corresponding commutative theory [13] with the desired properties, (1) Absence of tachyons, and this is in agreement with the fact that we are dealing with a super-symmetric theory. (2) Satisfy the on-shell gauge invariance. (3) It is cyclic in the external photons. We also gets sin/cos factors in the noncommutative scattering amplitudes depending on the number of external photons. From the study of the noncommutative scattering amplitudes we conclude that the phases arising in each sector depends on the ordering of the vertex operators, also the phases of the amplitudes in each sector are not derivable by starting from any one of the amplitude, which we can say that the total amplitude in noncommutative string theory is not symmetric in the Mandelstam variables, even though the kinematic factor is symmetric, but in the total noncommutative amplitude the presence of sin/cos factor will be determined by the coefficient of $e^{ik \cdot X}$ in the vertex operator; particularly the 3-point noncommutative amplitude of photons and tachyons leads to the appearances of phases in the total noncommutative amplitude as 'sin' and 'cos' times the kinematic factors respectively, and the case of 4-point noncommutative amplitude, we get the 'sin' or 'cos' factors times the kinematic factor in the noncommutative amplitude depending on the presence of odd or even number external photons respectively. Moreover, we have shown that the couplings of noncommutative string theory $G_s$ and the noncommutative gauge theory $g_{YM}$ are related as $g_{YM} = \frac{G_0}{\sqrt{2\alpha'}}$, $G_0$ is related to
as mentioned earlier, and the form of the couplings matches with [1]. This is of the same form as that of the commutative theory.

References


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Figure 4.2: Feynman diagrams for the noncommutative gauge theory for contact, s,t and u channels.