Chapter 2

Brief review of topological defects

- There are a large class of systems in which the unstable or metastable phase (with higher free energy) remains trapped in the form of points, lines or walls etc. after the phase transition in the background of the new phase. These localized regions of higher free energy phase are called topological defects. Topological defects arise in systems with non-trivial order parameter space or the vacuum manifold \( (M) \). A non-trivial vacuum manifold \( M \) either has disconnected pieces or there exist closed loops or surfaces in \( M \) which can not be smoothly shrunk to a point. Usually \( M \) is non-trivial for phase transitions associated with spontaneous breaking of some symmetry. For example, spontaneous breaking of \( U(1) \) local symmetry in superconductor leads to formation of flux-tubes. Here the order parameter space \( M \) is a circle and there are loops on this circle which can not be shrunk to a point. Monopoles, domain walls, cosmic strings etc. are various topological defects expected to be formed during phase transitions in the early Universe. Flux tubes in superconductor, vortices in super-fluid, disinclination lines and monopoles in liquid crystal are example of defects in condensed matter systems. There are important implications of studies of topological defects in cosmology. It is believed that topological defects may have played important role in the evolution of the Universe. There are models where network of cosmic strings formed in the early Universe phase transition leads to large scale structure formation in the Universe.

Conventionally there are two mechanisms responsible for the formation of topological...
defects. One is the thermal production in which defect-antidefect pairs can be produced via thermal fluctuation [2]. Other process of topological defect formation is the Kibble mechanism [3]. The Kibble mechanism is realized via formation of domains of uniform order parameter configuration in physical space after a spontaneous symmetry breaking phase transition. Defects form at the junction of these domains. There are two important predictions for defect formation via the Kibble mechanism. This mechanism predicts the number density of defects per domain once the vacuum manifold and the dimension of physical space is given. The second prediction is the existence of strong correlation between defects and antidefects produced via this mechanism. These predictions as such do not depend on the dynamics of phase transition, e.g. whether the phase transition is taking place in a non-relativistic condensed matter system or in a relativistic field theory system such as in the early Universe. Though there are theoretical studies of defect formation in early Universe phase transitions, experimental study of defect formation is possible only in condensed matter systems [4, 5]. For certain condensed matter systems both predictions of the Kibble mechanism have been experimentally verified [4].

In this chapter we will review some of the topics in topological defects. We will describe the connection between topology of the vacuum manifold $M$ and different types of topological defects in section 2.1. An example of topological defect will be given in detail in section 2.2. Classification of defects will be discussed in section 2.3. In section 2.4 we will explain the Kibble mechanism for formation of topological defects. Finally in section 2.5 we will briefly describe numerical simulation of first order phase transition demonstrating the formation of topological defects via the Kibble mechanism.
2.1 Topology of the Vacuum Manifold $M$ and Topological Defects

As we have already mentioned, topological defects form in phase transitions associated with spontaneous symmetry breaking (SSB). When there is SSB, the ground state may become degenerate, characterized by a set of vacua which constitute the vacuum manifold, $M$ (the order parameter space). For example, SSB of reflection symmetry ($\Phi \rightarrow -\Phi$, where $\Phi$ is a real scalar field) leads to a disconnected $M$, SSB of U(1) symmetry leads to $M = S^1$ (a circle), SSB of SO(3) rotational symmetry to SO(2) symmetry gives rise to $M = S^2$ (a 2-sphere), and $M = S^3$ for SSB of SU(2) etc. If the vacuum manifold $M$ has disconnected components, then domain walls form, if it is multiply connected (that is, if $M$ contains unshrinkable loops) strings will form, and if $M$ contains closed 2-surfaces which can not be shrunk to a point then monopoles will form in three dimensional physical space. All these defects can be characterized by non-trivial mappings from a $n$-dimensional sphere $S^n$ to the manifold $M$. By non-trivial mappings we mean maps which can not be smoothly deformed to the constant map. These mappings define a group called $n^{th}$ homotopy group $\pi_n(M, x)$, with a given element of the group consisting of all mappings from $S^n$ in to the manifold $M$ based at point $x \in M$ which can be smoothly deformed to each other. We will discuss in detail different homotopy groups in section 2.3 (most of the material in this section is based on Mermin's article [24]) where we describe classification of defects.

2.2 Example of a topological defect: string defect

String defects arise when the vacuum manifold $M$ contains loops which can not be smoothly shrunk to a point. Example of such a manifold is a circle $S^1$. We know that a loop winding $n$-times around the circle can not be smoothly deformed (without breaking the loop) to another loop of different winding on this circle. To see the existence of string defect we consider a theory for a complex scalar field $\Phi = \Phi_1 + i\Phi_2$ in 3+1 dimensions with spontaneous breaking
of $U(1)$ symmetry so that $M$ is a circle. The Lagrangian is given by,

$$L = (\partial_{\mu}\Phi^\dagger)(\partial^\mu\Phi) - \frac{\lambda}{4}\phi^2(\phi - \eta)^2.$$

(2.1)

$\phi$ is the magnitude of the complex field ($\Phi = \phi e^{i\theta}$) and $\theta$ is the phase. $\eta$ is positive real number. Here the effective potential is minimized by $\Phi \equiv \Phi_0 = \eta e^{i\theta}$ with $\theta$ taking any value in between 0 to $2\pi$. This implies that all points lying on the circle of radius $\eta$ centered at the origin in the complex $(\Phi_1, \Phi_2)$ plane are the possible ground states (vacua) of this theory. So the vacuum manifold is a circle, i.e $M \equiv S^1$.

For this model, though $\phi = \eta$ is minimum of the effective potential, magnitude $\phi$ of the field $\Phi$ can vanish on lines or loops in physical space which are known as string defects. To see this let us consider a specific field configuration $\Phi(\vec{r})$ such that $\phi = \eta$ on a hypothetical closed loop $l$ and $\theta$ changes by $\Delta\theta = 2\pi$ along the loop. [For simplicity we consider the loop to lie in a plane $P$ in 3-dimensional physical space.] Equivalently in the complex $(\Phi_1, \Phi_2)$ plane there will be a corresponding loop $l'$ winding once around $S^1$. Mathematically, $l'$ is the image of a map from the loop $l$ in physical space to the vacuum manifold $S^1$. If $\Delta\theta$ were $4\pi$ then $l'$ will wind around $S^1$ twice. As long as $\phi$ takes the value $\eta$ on loop $l$, any realization of the field $\Phi(\vec{r})$ will correspond to a closed loop in $S^1$ and winding of $l'$ on $S^1$ would depend on the net variation of $\theta$ on $l$ as explained above. One can think that $l'$ is the wrapping of $l$ on $S^1$. In Figure 2.1(a) we give a field configuration $\Phi(\vec{r})$ such that $\Delta\theta = 2\pi$ on $l$ and in Figure 2.1(b) we show the corresponding loop $l'$ on $S^1$. In the figure magnitude of the vectors represent $\phi$ and angle these vectors make with respect to x-axis represents the $\theta$.

Smooth deformation of $\theta$ distribution on $l$ will result in deformation of $l'$ in $S^1$ but the fact that $l'$ goes around $S^1$ once (i.e, the winding of $l'$ on $S^1$) does not change. This is because a closed loop winding once around $S^1$ can not be continuously shrunk to a point or to another loop of different winding.

Now possible values $\Delta\theta$ can take on any closed loop $l$ are $\pm 2\pi n$, $n$ being an integer. Equivalently $l'$ can loop around $S^1$ $n$ number of times. Here $n$ is called the winding number.
Figure 2.1: (a) A specific field configuration $\Phi(\tau)$ on the loop $l$ in physical space. (b) plot of $V(\Phi)$ vs. $\Phi$ showing the vacuum manifold

associated with loop $l$. In the case of above example $\Delta \theta = 2\pi$, hence $n$ is equal to 1. If we keep shrinking $l$ smoothly on the plane $P$, by successive infinitesimal deformations, and make sure that $\phi \neq 0$ on the loop $l$, then $\theta$ distribution on shrinking $l$ may change but $\Delta \theta = 2\pi$ will be maintained on $l$. Other possibility is that $\theta$ changes by a jump of $\pm 2\pi n$, on the shrinking loop $l$. It can only happen if the loop $l$ crosses a point where $\Phi = 0$ as $\theta$ is ill defined there. Otherwise it is not possible because infinitesimal changes in the loop can not lead to large changes in $\theta$ if $\theta$ is continuously varying. When $l$ becomes very small in size variation of $\theta$ by $2\pi$ on $l$ would lead to diverging gradient energy contribution, i.e $\phi^2 \partial_\mu \theta \partial^\mu \theta$, to the potential energy (Eqn.(2.1)) of the system. This contribution can only be made finite by making the magnitude of the complex field vanish as the loop shrinks to a point. So even though in the ground state magnitude of the complex field is non-zero ($\phi = \eta$) it has to vanish at least at a point inside the loop $l$ on which $\theta$ changes by $2\pi n$ for $n \neq 0$ for smooth $\Phi(\tau)$. For $n = 0$ it is not necessary that $\phi$ vanishes as $l$ shrinks to a point.
Now instead of shrinking on the plane one could lift the loop $l$ up from the plane $P$ while shrinking. Along the similar lines one can argue in this case also that $\Delta \theta = 2\pi$ will be maintained on the loop. Whenever the loop shrinks to a point, somewhere it must intersect a point where the magnitude of the complex field has to vanish because of the finite energy density condition. With this constraint in all different ways a hypothetical loop in three dimension can be shrunk to points of an infinite line or a closed loop on which $\phi = 0$. These objects in which $\phi = 0$ with $\theta$ changing by $2\pi n$ around them are called strings defects.

It is important to note that the existence of string defect is due to the fact that $l'$ corresponding to the field distribution on $l$ winds around $S^1$ once. In other words it is necessary for the existence of string defect that there are loops in $S^1$ which can not be shrunk to a point. For the theory we have considered here the fact that loop $l'$ can wind around $n$ times on $M$ relates to change of $\theta$ by $2\pi n$ on a $l$. One can extend these arguments to any arbitrary vacuum manifold $M$ containing unshrinkable loops. Unshrinkable $l'$ in $M$ would mean that a loop $l$ in physical space carries some non-zero topological quantity, like winding number $n$ in the above example, which will not change by smooth deformation of $l$.

2.3 Classification of Defects

In the previous example we concluded the existence of a string defect inside a hypothetical loop $l$ in physical space if the order parameter field configuration on $l$ corresponds to a unshrinkable loop $l'$ in the vacuum manifold $M$. Mathematically $\theta$ configuration on $l$ is a map from $l$ to $M$ and $l'$ is the image of $l$ under this map in $M$. Existence of different $l'$ in $M$ (which are images of maps from a circle $S^1$ to $M$ as loop $l$ is topologically equivalent to $S^1$) correspond to different string defects in physical space. Configurations of string defects, corresponding to different maps of $S^1$ to $M$ which can not be smoothly deformed to each other, are inequivalent in the sense the configuration of one string defect can not be smoothly deformed to other. This can be generalized to all kind of defects. All inequivalent non-trivial mappings from $S^n \to M$ corresponds to different class of defects of a particular type. This
is related to classifying defects by homotopy groups which we discuss in the following. We will first discuss the first homotopy group and classification of strings. Then we will discuss higher homotopy groups and end this section by defining the exact sequence for homotopy groups. For a review see Mermin's article ref. [24].

2.3.1 Classification of Strings and the first Homotopy Group

In order to classify strings one has to consider all possible mappings from the loop \( l \) in physical space (equivalent to a circle \( S^1 \)) to \( M \). Let us consider a simple example of \( M \) being \( \mathbb{R}^2 \) with a hole \( A \) as given in Figure 2.2. We consider all mappings from \( S^1 \) to \( M \) defined by a function \( f \) from the interval \( I=[0,1] \) into \( M \), such that \( f(t = 0) = f(t = 1) = x \) for some fixed \( x \in M \), here \( t \in [0,1] \). The constraints \( f(t = 0) = f(t = 1) \) makes the interval \([0,1]\) equivalent to a circle \( S^1 \). In Figure 2.2, as \( t \) varies from \( 0 \rightarrow 1 \) the loop \( f \) is traversed clockwise in \( M \).

![Figure 2.2: Loops in a manifold \( M \) based at \( x \). Only the loops \( f_1 \) and \( f_2 \) are homotopic to each other.](image)
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**Homotopic loops:** Two closed loops $f$ and $g$ in $M$ are called *based homotopic at* $x$, $x \in M$, if $f$ and $g$ are continuously deformable to each other being fixed at the point $x$ (called base point). (We will interchangeably use $f$ and $g$ for maps from $S^1$ to $M$ as well as for the images of $S^1$ under these maps in $M$.) In other words if there exists a map $h(s, t)$ from $I \times I$ to $M$ continuous in $s, t \in I \times I$ satisfying,

$$
\begin{align*}
  h(0, t) &= f(t) \\
  h(1, t) &= g(t) \\
  h(s, 0) &= h(s, 1) = x
\end{align*}
$$

then $f$ and $g$ are said to be based homotopic at $x$, denoted by $f \sim g$.

It is clear from Figure 2.2 that loop $f_1$ can be shrunk to a point, also can be smoothly deformed to any loop $f_2$ which is not winding around $A$. But loops winding around $A$ (for example $f$ in Figure 2.2) can not be smoothly shrunk to a point. So the two mappings $f_1$ and $f$ are not based homotopic to each other. All loops based homotopic to a loop $g$ at $x$ form the homotopy class of loops denoted by $[g]$, which is an element of the first homotopy group $\pi_1(M, x)$. Here the group multiplication operation is defined as,

$$
[f] * [g] = [f * g] \tag{2.3}
$$

where the product $f * g$ denotes the loop which traverses the loop $g$ first followed by traversing $f$. Mathematically $f * g$ is defined as,

$$
\begin{align*}
  f * g(t) &= g(r), \quad r = 2t, \quad 0 \leq t \leq \frac{1}{2} \\
  &= f(s), \quad s = 2t - 1, \quad \frac{1}{2} \leq t \leq 1.
\end{align*}
$$

It is easy to see that elements of $\pi_1(M, x)$ satisfy all group properties [24]. Now we have defined the first homotopy group $\pi_1(M, x)$ at the base point $x$ in $M$. One can show
that for a path connected $M$ these groups for different choice of the base point $x \in M$ are isomorphic. So one can refer to the first homotopy group for a path connected $M$ as $\pi_1(M)$, although the group operation is only defined for loops based at a particular point $x$. Each element in the first homotopy group would characterize a distinct string defect. But there are exceptions to this classification. For some manifolds, for example Figure 2.3, there are two loops $f$ and $g$ based at $x$ corresponding to same defect, i.e both winding around $B$.

$$\mathbb{R}^2$$

![Figure 2.3: The figure-eight space. The loops $f$ and $g$ are not based homotopic at $x$, but they are freely homotopic to each other.](image)

Here $f$ and $g$ can not be based homotopic at $x$, so do not belong to the same homotopic class of loops in $\pi_1(M, x)$. But these loops can be deformed to each other if freed from the base point $x$ which leads to the definition of freely homotopic loops. One can show that [24] two loops $f$ and $g$ based at $x$ are freely homotopic if there exists a based loop $c$ at $x$ such that,

$$f \sim c^{-1} * g * c. \quad (2.5)$$

In this case $f$ and $g$ are said to be in the same conjugacy class of loops as elements $f$ and $g$ of the group $\pi_1(M)$ are conjugate to each other. So string defects are classified by the equivalence classes of freely homotopic loops or conjugacy classes of elements of $\pi_1(M)$. Equation (2.5) describing conjugacy classes of loops in $\pi_1(M)$ defines action of $\pi_1(M)$ on


\( \pi_1(M) \). Only for abelian \( \pi_1(M) \) the set of different conjugacy classes in \( \pi_1(M) \) is \( \pi_1(M) \) itself, since \( c^{-1} * g * c = g \). Whether \( \pi_1(M) \) is abelian or non-abelian depends on the manifold \( M \).

For \( M \equiv \mathbb{R}^2 \) with one hole \( \pi_1(M) \) is abelian, while for \( M \equiv \mathbb{R}^2 \) with two holes it is non-abelian.

### 2.3.2 Coset Space representation of \( M \) and Computation of \( \pi_1(M) \)

For spontaneous symmetry breaking phase transitions, the vacuum manifold \( M \) can be expressed in terms of the symmetry groups involved in the symmetry breaking. Computation of \( \pi_1(M) \) is very much simplified for these cases. Suppose that in a spontaneous symmetry breaking a higher symmetry group \( G \) spontaneously breaks to a lower symmetry group \( H \) which is the subgroup of \( G \). If \( \phi_0 \) is a point in \( M \) characterizing one specific vacuum configuration then any other point \( \phi \in M \) can be obtained by transformation of the form \( \phi = D(g)\phi_0 \) for some element \( g \in G \) and \( g \notin H \). This is because \( H \) is the symmetry group of the order parameter after the symmetry breaking, so \( \phi = D(h)\phi \) for any \( h \in H \) and \( \phi \in M \).

Also \( D(g) = D(g*h) \) for any \( h \in H \). This implies for any \( \phi \) in \( M \) there is a corresponding left coset \( gH \). So the vacuum manifold \( M \) is equivalent to the set of left cosets of \( H \) in \( G \), \( M \equiv \frac{G}{H} \).

Once we represent \( M \) by the coset space \( \frac{G}{H} \) the first homotopy group can be computed using the first fundamental theorem which we state in the following.

**First fundamental theorem:** For simply-connected group manifold \( G \),

\[
\pi_1(M) \cong \pi_1(G/H) \cong \pi_0(H).
\] (2.6)

Here \( \cong \) denotes isomorphism of groups. For a simply connected space all loops are trivial and can be shrunk to a point. Here the elements of \( \pi_0(H) \) are the disconnected components of \( H \). Even if \( G \) is not simply connected one can consider a larger group \( \tilde{G} \) of which \( G \) is a subgroup. This larger group \( \tilde{G} \) covering \( G \) exists for all Lie groups \( G \). Since we have enlarged \( G \), \( H \) also gets enlarged to \( \tilde{H} \) and one can rigorously show that \( \frac{G}{H} = \frac{\tilde{G}}{\tilde{H}} \).
Application to U(1) strings

We know that when $G \equiv U(1)$ breaks completely, for example in superconducting phase transition, there arise string defects. The coset space $\frac{G}{H}$ is $U(1)$ itself, since the unbroken subgroup $H$ in this case is the identity element of $G$. This space is equivalent to a circle, which is multiply connected. To compute the first homotopy group we have to enlarge $U(1)$ group to a simply-connected group. For this let us consider the group of one dimensional translations ($R$ the real line) as $\tilde{G}$. Since any interval $[2\pi(n-1), 2\pi n)$ has a one-to-one map onto $U(1)$, $U(1) \simeq \frac{R}{Z}$, where $Z$ is the set of integers representing the set of translations by $2\pi n$, $n \in Z$, is the unbroken symmetry group $\tilde{H}$.

So $\pi_1(U(1)) \simeq \pi_1(R/Z) \simeq \pi_0(Z) \simeq Z$.

2.3.3 The second homotopy group $\pi_2(M)$

Analogous to the previous case where string defects arise due to non-trivial $\pi_1(M)$, nontrivial higher homotopy groups like $\pi_2(M)$ lead to point defects in three spatial dimensions. Classification of point defects is very similar to the case of strings which we discuss in the following.

The elements of the second homotopy group $\pi_2(M)$ are homotopically equivalent classes of closed surfaces (equivalent to $S^2$) in $M$. Two closed surfaces are called based homotopic if they can be smoothly deformed to each other being fixed at a base point $x$. As in the case of first homotopy group second homotopy group at any point $x$ is isomorphic to second homotopy group with any other base point $y$ if the space is path connected. Unlike the first homotopy group which can be non-abelian second homotopy group is always abelian. The second homotopy group can be computed using the second fundamental theorem for those manifolds for which $M \equiv \frac{G}{H}$ for simply connected $G$ (or $\tilde{G}$).

- **Second fundamental theorem:** For simply-path connected $G$, the second homotopy group is given by,
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$$\pi_2(G/H) \simeq \pi_1(H_0)$$  \hspace{1cm} (2.7)$$

where $H_0$ is the component of $H$ which contains the identity element. So the second homotopy group is isomorphic to the first homotopy group $\pi_1(H_0)$ of $H_0$.

Analogous to the previous case where strings are identified by conjugacy classes of loops in $\pi_1(M)$ here also point defects are classified by classes of closed surfaces in $M$ which are freely homotopic. One can find the classes of freely homotopic closed surfaces by considering non-trivial actions of $\pi_1(M)$ on $\pi_2(M)$. This amounts to moving the base point $x_0$ of a closed surface along a non-trivial loop in $M$. For $M \equiv S^2$, using the first and second fundamental theorems we can say that point defects are classified by action of $\pi_0(H)$ on $\pi_1(H_0)$. By action of $\pi_0(H)$ any loop in $H_0$, $h_0(t)$, is taken to another loop $h'_0(t) = h * h_0(t) * h^{-1}$, where $h \in H_i$ and $H_i$ is any disconnected component of $H$. Here $h_0(t)$ and $h'_0(t)$ correspond to the same point defect. If the subgroup $H$ is abelian then action of $\pi_0(H)$ is trivial, i.e $h_0(t) = h'_0(t)$.

Example:

As an example, we consider the symmetry breaking $SU(2) \rightarrow U(1) \otimes Z_2$ for a field $\Phi$, a traceless real symmetric $3 \times 3$ matrix. When a vacuum expectation value for $\Phi$ is chosen, for example $\Phi_0 = \eta \text{ diag}(1,1,-2)$, then the symmetries unbroken are the rotation around the $z$-axis corresponding to $U(1)$ and a discrete rotation of $180^\circ$ around the $x$-axis represented by $Z_2$. $\otimes$ in $U(1) \otimes Z_2$ is a semi-direct product as generators for both the symmetries do not commute. Applying the second fundamental theorem we get,

$$\pi_2(SU(2)/U(1) \otimes Z_2) \simeq \pi_1(U(1)) \simeq Z.$$  \hspace{1cm} (2.8)$$

Since the component $H_0$, connected to identity $e$ in $H \equiv U(1) \otimes Z_2$ is $U(1)$. Here the subgroup $H$ has two disconnected components, $H_0$ and $H_1$. Elements of $H_0$ are expressed as $h_0 = e^{i \alpha t \sigma}$ while elements of $H_1$ are represented by $h_1 = R_0 e^{i \alpha t \sigma}$, where $R_0 = \text{diag}(1,1,1)$ corresponding to rotation by $180^\circ$ around $x$-axis. Note that for any loop $h_0(t)$ in $H_0$,
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\( h_1^{-1} \ast h_0(t) \ast h_1 = h_0^{-1}(t) \) for any element \( h_1 \in H_1 \) where \( h_0^{-1}(t) \) has winding opposite to that of \( h_0(t) \). This implies that under the action of \( \pi_0(H) \) loops in \( H_0 \) change the sign of their winding. So loops in \( H_0 \) with winding \( m \) and \(-m\) correspond to the same point-defect of strength \( m \).

2.3.4 Relative homotopy groups \( \pi_n(G, H, e) \), Higher homotopy groups \( \pi_n(G/H) \) and exact homotopy sequence

For a given group \( G \) and its subgroup \( H \), \( n \)-th relative homotopy group \( \pi_n(G, H, e) \) is defined by all maps from a \( n \)-cube to \( G \) such that all the faces but one is mapped to \( e \) in \( G \), the face which is left (called base) is mapped anywhere in \( H \) with its boundary being mapped to \( e \). Here base is a \((n-1)\)-cube. In the coset space the same maps will look like a \( n \)-cube mapped to \( \frac{G}{H} \) with all the faces mapped to \( H \) which is a single element in \( \frac{G}{H} \). This map is equivalent to a map of \( S^n \) to \( \frac{G}{H} \) with base point as \( H \). This map will belong to one of elements in \( \pi_n(G/H) \). So there is a one-to-one correspondence between \( \pi_n(G, H, e) \) and \( \pi_n(G/H) \).

Now given this correspondence between \( \pi_n(G, H, e) \) and \( \pi_n(G/H) \) one can define three homomorphisms as follows. Each map in \( \pi_n(G, H, e) \) carries with it a map in \( \pi_{(n-1)}(H, e) \). This come from the fact that the base which is a \((n-1)\)-cube mapped to \( H \) with boundary being mapped to \( e \). This is equivalent to a based map from \( S^{(n-1)} \) to \( H \), so belongs to \( \pi_{(n-1)}(H, e) \). So there is a homomorphism \( \gamma_n \) from \( \pi_n(G, H, e) \) to \( \pi_{(n-1)}(H, e) \)

Each map in \( \pi_n(H, e) \) is also a map in \( \pi_n(G, e) \). This is because \( H \) is a subgroup of \( G \) so any \( S^n \) in \( H \) based at \( e \) is also a \( S^n \) in \( G \) at \( e \). So there is a homomorphism \( \alpha_n \) from \( \pi_n(H, e) \) to \( \pi_n(G, e) \). Apart from \( \gamma_n \) and \( \alpha_n \) there exists another homomorphism \( \beta_n \) from \( \pi_n(G, e) \) to \( \pi_n(G, H, e) \). This follows from the fact that for any map in \( \pi_n(G, e) \) which is equivalent to map from \( n \)-cube to \( G \) with all faces being mapped to \( e \) can be looked as a map in \( \pi_n(G, H, e) \) with all the faces including the base are mapped to \( e \). This is the case when the map of base to \( H \) with boundary of base being mapped to \( e \) is trivial.

Using these three homomorphisms \( \alpha_n, \beta_n \) and \( \gamma_n \) one can write down the following
sequence of homomorphisms.

\[ \cdots \to \pi_n(H) \to \pi_n(G) \to \pi_n(G, H, e) \to \pi_{n-1}(H) \to \pi_{n-1}(G) \to \cdots \quad (2.9) \]

Or

\[ \cdots \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \pi_{n-1}(G) \to \cdots \quad (2.10) \]

Where we have ignored the base point \( e \) and used the correspondence between \( \pi_n(G, H, e) \) and \( \pi_n(G/H, e) \). One can prove that the sequence is exact, that is the image of one homomorphism is the kernel of next homomorphism. One can show that the first and second fundamental theorems which we have already discussed are special cases of the exact sequence. To see this let us consider a small portion of the exact sequence,

\[ \cdots G_1 \to G_2 \to G_3 \to G_4 \cdots \quad (2.11) \]

If \( G_1 \) and \( G_4 \) are trivial, i.e., groups with only identity element then one can argue that \( G_2 \) and \( G_3 \) are isomorphic. Since the sequence is exact, image of the homomorphism from \( G_2 \) to \( G_3 \) is \( G_3 \) itself because it is the kernel of the next homomorphism from \( G_3 \) to \( G_4 \). So the homomorphism from \( G_2 \) to \( G_3 \) is onto. Since the image of the homomorphism from \( G_1 \) to \( G_2 \) is the identity in \( G_2 \) which is the kernel of the homomorphism from \( G_2 \) to \( G_3 \) so the homomorphism from \( G_2 \) to \( G_3 \) is one-to-one. So \( G_2 \) and \( G_3 \) are isomorphic, \( G_2 \equiv G_3 \).

Now in the exact sequence Eqn. (2.10) if \( \pi_n(G) \) and \( \pi_{n-1}(G) \) are trivial then one can easily get,

\[ \pi_n(G/H) \simeq \pi_{n-1}(H). \quad (2.12) \]
For \( n = 1 \) this equation corresponds to first fundamental theorem and for \( n = 2 \) corresponds to the second fundamental theorem. The choice of \( G \) for both the theorems, i.e. a connected and simply connected \( G \) implies both \( \pi_n(G) \) and \( \pi_{n-1}(G) \) are trivial, meeting the above criteria. With this brief introduction to classification of topological defects we now discuss the formation of topological defects via the Kibble mechanism in the next section.

### 2.4 Formation of defects via Kibble mechanism

The Kibble mechanism of formation of topological defects was first proposed by T. W. Kibble in the context of defect formation in the early Universe phase transitions [3]. There are two postulates underlying this mechanism, 1) after a spontaneous symmetry breaking phase transition physical space consists of regions, called domains, in which the configuration of the order parameter field is nearly uniform while varying randomly from one domain to another and 2) the order parameter field configuration in between the domains is such that the variation of the order parameter field is minimum on the vacuum manifold. This is known as Geodesic Rule. Application of these two postulates leads to a simple way to estimate the formation of topological defects at the junctions of these domains. As such these postulates do not assume any dynamics for the phase transition so this mechanism has applicability to all sorts of systems for formation of topological defects. In the following sections we will first discuss how domain formation is realized after a spontaneous symmetry breaking transition. Then we will explain the formation of defects using the geodesic rule.

For simplicity we consider spontaneous breaking of \( U(1) \) symmetry in a phase transition. Here the order parameter describing the transition is a complex number \( \Psi \) which is the value of the field \( \Phi \) minimizing the free energy or effective potential \( V_T(\Phi) \). We assume the free energy to be temperature dependent function of \( \Phi \) given by,

\[
V_T(\Phi) = \alpha_T \phi^2 - \beta \phi^3 + \gamma \phi^4, \tag{2.13}
\]

where \( \phi \) is the magnitude of the complex field \( \Phi \). Here we assume \( \beta \) and \( \gamma \) are temperature...
independent and $\alpha_T$ has the temperature dependence such that it is positive for temperatures larger than a critical value $T_c$ and negative below $T_c$. Non-zero value of the coefficient $\beta$ of the cubic term $\phi^3$ gives rise to first order phase transition. In Figure 2.4 we give the plots of $V_T(\Phi)$ for different values of $\alpha_T$ and $\beta$.

![Figure 2.4: (a) plot of free energy $V_T(\Phi)$ as a function of the field $\Phi$ at different temperatures for a case of second order phase transition. (b) shows similar plots for the case of first order phase transition.](image)

Now the free energy density $V_T(\Phi)$ is minimized by $\phi = \eta$ with $\eta$ being temperature dependent. For the above choice of $\alpha_T$ with $\beta = 0$, $\eta$ takes the value zero for $T > T_c$ and $\eta \neq 0$ for $T < T_c$ (Figure 2.4(a)). When $\beta > 0$ the free energy density has two minima for temperatures around $T_c$. $\phi = 0$ is the stable minimum and $\phi = \eta$ is local minimum of the free energy for temperatures slightly above $T_c$. This situation reverses as the temperature drops below the critical temperature so that $\phi = \eta$ is the stable minimum of $V_T(\Phi)$ (Figure 2.4(b)).

For the case of second order phase transition, i.e $\beta = 0$, the magnitude of the order parameter $\psi$ changes continuously from 0 to non-zero value as the temperature drops below $T_c$. While for first order phase transition, $\beta \neq 0$, $\psi$ changes discontinuously from 0 to a non-zero value. In both the cases below $T_c$ the stable minimum of the free energy is degenerate. For $T < T_c$, all the different values of the order parameter are characterized by
\[ \Psi = \eta e^{\theta}. \] Choice of a specific value of \( \theta \) corresponds to choosing a vacuum of the theory. Clearly this vacuum state is not invariant under \( U(1) \) transformation. This is the situation of spontaneous symmetry breaking. As mentioned before the vacuum manifold is a circle \( S^1 \) of radius \( \eta \).

2.4.1 Formation of domains or bubbles after a phase transition

At high temperatures \( T > T_c \), one would expect field configuration of the system be given by \( \Phi(\vec{r}) = 0 \) since \( \psi = 0 \). Since the system is at high temperature due to thermal fluctuations magnitude of the complex field \( \Phi(\vec{r}) \) can fluctuate to non-zero value temporarily in a given region in physical space. While fluctuating the field \( \Phi(\vec{r}) \) can take different values of \( \theta \) in regions separated by correlation length. This is because thermal fluctuations can be correlated only up to distances of the order of correlation length. So at a particular instant a typical field configuration in physical space would consist of domains in which \( \theta \) as well as the magnitude of \( \Phi(\vec{r}) \) is roughly uniform and \( \theta \) being randomly distributed in the domains [25]. In equilibrium, the size of these domains are given by the equilibrium correlation length of the theory for temperatures not too close to \( T_c \) [25]. These domains are unstable since any non-zero \( \phi \) will roll down to zero. But at any moment the typical configuration \( \Phi(\vec{r}) \) would look like as mentioned above.

This situation continues even below the critical temperature for the case of second order phase transition. Here the magnitude of the order parameter \( \psi \) changes continuously from zero to non-zero value as temperature \( T \) decreases below \( T_c \). For temperatures below \( T_c \) even if the magnitude of the order parameter is non-zero, field \( \Phi(\vec{r}) \) in a domain can fluctuate to zero value due to thermal fluctuations making the domains unstable. This continues down to a temperature called the Ginzburg temperature \( T_G (\leq T_c) \) [26]. Below \( T_G \) thermal fluctuations are suppressed and domains are stable against thermal fluctuations. Evolution of these domains is only via coalescing among themselves to minimize the gradient energy. For the case of non-equilibrium evolution domain size depends on the dynamics of the phase.
transition. For fast cooling rates, long-wavelength modes of $\Phi(\vec{r})$ grow exponentially during the transition period resulting in growth in the size of these domains. There are extensive studies of domain growth in literature both in particle and condensed matter physics [27].

In conventional studies of first order phase transition fluctuations on the scale of correlation length of the field $\Phi$ are neglected. As mentioned above, below $T_c$ there will be two phases, $\psi = 0$ is metastable phase and $\psi = \eta$ is the stable phase (true vacuum). Below $T_c$ bubbles of stable phase nucleate in the background of metastable phase $\psi = 0$. The configuration of bubbles of true phase are given by a uniform $\Phi(\vec{r})$ (near the bubble center in thin wall limit) with magnitude $\phi = \eta$ and $\theta$ taking some arbitrary value in between 0 and $2\pi$. After nucleation bubbles grow converting the $\psi = 0$ phase to the stable phase as this minimizes the free energy of the system. While growing, bubbles coalesce with other bubbles and complete the phase transition. Here one can take bubbles at the time of coalescing as the domains used in the Kibble mechanism. If the nucleation rate is small then only few bubbles will be nucleated which will grow to a large size before they coalesce. So at the end of phase transition there will be large domains of roughly uniform $\theta$ in physical space. On the other hand for high nucleation rate large number of bubble will be nucleated which will grow by a small amount before coalescing with other bubbles, hence there will be small domains after the phase transition. So for first order phase transition domain size (size of the bubbles at the time of coalescing) depends on the nucleation rate of the bubbles. At the end of this chapter we will briefly discuss numerical simulation of a first order phase transition.

2.4.2 Geodesic Rule

Since magnitude of the order parameter $\psi$ is equal to $\eta$, any typical field configuration $\Phi(\vec{r})$ will be such that magnitude of the field be fixed at $\eta$ except for the regions of defects. So we ignore $\phi$ and consider only $\theta$ configuration to simplify our discussion. As we have already said in both first and second order phase transition cases physical space consists of
domains with uniform $\theta$ configuration and $\theta$ takes random values in different domains. The field configuration in between the domains is obtained by applying the geodesic rule. To understand this let us consider domains $d_1$, $d_2$ and $d_3$ with $\theta$ taking values $\theta_1$, $\theta_2$ and $\theta_3$ respectively, see Figures 2.5(a) and 2.5(b) below. Here magnitude of the vectors represent magnitude of $\Phi(\mathbf{r})$ and the angle these vectors make with positive x-axis is the phase of $\Phi(\mathbf{r})$. Figure 2.5(b) shows the position of $\theta_1$, $\theta_2$ and $\theta_3$ on the vacuum manifold $S^1$.

Having prescribed the configuration of $\Phi$ in the domains we obtain the field configuration in between the domains as follows. Now between $\theta_1$ and $\theta_2$ in $S^1$ there are two paths $\theta_{12}$ shown inside the the circle $S^1$ and $\theta'_{12}$ shown outside $S^1$ in Figure 2.5(b). Here $\theta_{12}$ is the shortest path. So according to the geodesic rule the phase along $L_{12}$ from $d_1$ to $d_2$ will vary from $\theta_1$ to $\theta_2$ along the path $\theta_{12}$. This would imply $\theta$ in between the domains would take values from $\theta_{12}$. So we have put $\theta$ to be the average value in between $d_1$ and $d_2$ shown in Figure 2.5(a). This is how field in between domains is configured. Justification for the geodesic rule comes from the consideration of gradient energy. Shortest path on the vacuum manifold will lead to smallest possible gradient energy, and hence should be preferred in situations where dynamics of field is not dominated by fluctuations. We mention here that it is not important what value of $\theta$ on the arc $\theta_{12}$ is chosen for the region between $d_1$ and $d_2$. This is due to topological nature of arguments. Only important thing is whether $\theta_{12}$ is chosen or $\theta'_{12}$.

To see how the geodesic rule leads to defect formation consider a third domain $d_3$. Note that $\theta_3$ is lying on the shortest geodesic between $\theta'_2$ and $\theta'_1$ in $S^1$, where $\theta'_2$ and $\theta'_1$ are points opposite to $\theta_1$ and $\theta_2$ respectively. It is easy to see applying geodesic rule that field along $L_{23}$ from $d_2$ to $d_3$ will vary from $\theta_2$ to $\theta_3$ along $\theta_{23}$. Also along $L_{31}$ from $d_3$ to $d_1$ the field will vary from $\theta_3$ to $\theta_1$ along $\theta_{31}$. Now as the closed loop $L$ consisting of arcs $L_{12}$, $L_{23}$ and $L_{31}$ is traversed once the vacuum manifold $S^1$ is covered once. Hence $\Delta \theta = 2\pi$ on $L$. As we have argued in the section 2.2 because of this $2\pi$ variation of $\theta$ on $L$ there will be a vortex inside $L$ at the junction of domains $d_1$, $d_2$ and $d_3$.

Along Similar lines one can show that if $\theta_3$ does not lie on the arc between $\theta'_2$ and $\theta'_1$
Figure 2.5: (a) showing formation of a vortex at the junction of three domains $d_1, d_2$ and $d_3$. (b) shows the vacuum manifold $S^1$ with different arcs on this $S^1$ are mapping from paths connecting different domains to the $S^1$.

The net variation of the phase around $L$ will be zero leading to no defect inside the loop. Without using some prescription like the geodesic rule it is not possible to find the field configuration on the entire loop $L$ and hence one can not determine whether a defect is there inside $L$. Using the geodesic rule one can easily calculate the number density of defects per domain given the dimension of physical space and the vacuum manifold. Assuming the physical space to be 2 dimensional we compute the defect density per domain for $U(1)$ strings in the following.

### 2.4.3 Computing defect density per domain

We know that for any choice of $\theta_1$ and $\theta_2$ the average arc length between $\theta'_1$ and $\theta'_2$ is $\pi/2$, shortest arc length being 0 and longest being $\pi$. So the probability that $\theta_3$ falls on the arc $\theta'_{12}$ between $\theta'_1$ and $\theta'_2$ is $(\pi/2)/(2\pi) = \frac{1}{4}$ which is the probability of forming a defect at
the junction of three domains in 2-dimensions. Now a single domain can be part of three
junctions so the probability of forming a defect per domain is $\frac{1}{4}$. It is important to note
that we only considered three domains and used geodesic rule to determine $\theta$ in between the
domains. These consideration do not depend on the order of phase transition nor on the
dynamical details of the transition.

Number of defects per domains can be estimated for all kind of defects along the
above lines. It only depends on the geodesic rule and number of domains necessary to define
a defect. Number of domains necessary to define a defect in turn depends on the topology
of vacuum manifold and dimension of physical space. For detailed discussion of these issues
see ref. [1].

2.5 Formation of vortices via the Kibble mechanism:

Numerical simulation

In this section we briefly review numerical studies of the formation of vortices via the Kibble
mechanism for the case of first order phase transition which has been discussed in detail in
refs. [28, 29]. Here we present the simulation of refs. [28, 29] which we have repeated as a
preparation for setting up the simulations for our problem with explicit symmetry breaking
(which we will describe in the next chapter). We consider the following Lagrangian with
$U(1)$ global symmetry to study the formation of vortex defect in 2+1 dimensions,

$$L = \frac{1}{2} \partial_{\mu} \Phi^* \partial^{\mu} \Phi - \frac{1}{4} \phi^2 (\phi - 1)^2 + \epsilon \phi^3. \quad (2.14)$$

It is written in terms of dimensionless field $\Phi$ and length variables. $\phi$ and $\theta$ are the magnitude
and the phase of the complex scalar field $\Phi$ ($\Phi = \phi e^{i\theta}$). Eqn.(2.14) describes a theory where
- the $U(1)$ global symmetry is spontaneously broken so the vacuum manifold $M$ is a circle of
  radius $\eta = 1.25$, for $\epsilon = 0.05$, in the complex $(\Phi_1, \Phi_2)$ plane (with $\Phi = \Phi_1 + i\Phi_2$), where $\eta$ is
  the global minimum of the effective potential in Eqn.(2.14). This Lagrangian is equivalent
to Eqn.(2.1) except for the presence of the cubic term in $\phi$ which makes the phase $\phi = 0$ metastable and allows for first order phase transition.

At zero temperature bubbles of true vacuum with $\phi = \eta$ nucleate in the background of $\phi = 0$. To get the configuration of bubbles one first solves for $O(3)$-symmetric, least action, bounce solution of 3-dimensional Euclidean field equations [30]

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - V'(\phi) = 0$$

(2.15)

with boundary conditions $\phi(r = 0) = \eta$ and $\phi(r = \infty) = 0$. Here $V(\phi)$ is the effective potential in Eqn. (2.14) and $r$ is the Euclidean radial distance. The solution $\phi(r)$ for $\epsilon = 0.05$ is shown in the left figure in Figure 2.6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.6.png}
\caption{Figure on the left is the plot of magnitude profile of the bubble versus the radial distance $r$. Figure on the right is the plot of full field configuration $\Phi$ of a bubble at the time of nucleation. Here magnitude of $\Phi$ is proportional to the length of the vectors and phase of $\Phi$ is equal to the angle the vectors make with positive x-axis.}
\end{figure}

For magnitude part of $\Phi$ of the nucleated bubble in Mankowski space, $r$ in the bounce
solution is replaced by the radial coordinate in Mankowski space [30]. It follows from the boundary conditions of the bounce solutions that inside the bubble \( \phi = 1 \), the vacuum value in the thin wall limit (for the case when \( \epsilon << 1 \)), and zero outside. In between these regions \( \phi \) interpolates continuously from the vacuum value to zero which constitutes the bubble wall. \( \theta \) is taken to be uniform inside a bubble with arbitrary value in between 0 to \( \pi \). Having obtained a bubble configuration, nucleation of a bubble is achieved by replacing a region of false vacuum \( \Phi = 0 \) by the bubble configuration, see the right figure in Figure 2.6. In all the plots of \( \Phi \) magnitude of \( \Phi \) is proportional to length of the vectors while phase \( \theta \) is given by the angle these vectors make with positive x-axis. While nucleating a bubble is taken care that outer part of the bubble configuration matches smoothly with false vacuum, i.e \( \phi = 0 \).

This gives the initial configuration of a typical bubble in physical space as shown in the right figure in Figure 2.6. This initial field configuration is then evolved by the following field equations,

\[
\Box \Phi_i = -\frac{\partial V(\Phi)}{\partial \Phi_i}, \quad i = 1, 2
\]

(2.16)

where \( \Box \) is the d’Alembertian. Time derivatives of fields are set equal to zero at the initial time \( t = 0 \). Simulation is implemented by using a stabilized leapfrog algorithm of second order accuracy in both space and time. For these simulations, we choose a 800 \( \times \) 800 lattice with lattice spacing in spatial direction, \( \delta x \), equal to 0.16 and lattice spacing in the temporal direction, \( \delta t \), equal to \( \delta x/\sqrt{2} \) which satisfies the Courant condition for stability [28].

For checking the Geodesic rule we consider time evolution of a configuration of two bubbles with \( \theta_1 = 0^0 \) and \( \theta_2 = 120^0 \) as given in the left figure in Figure 2.7. In the circle \( S^1 \) in the complex \( (\Phi_1, \Phi_2) \) plane the geodesic between \( \theta_1 \) and \( \theta_2 \) is the arc connecting \( \theta_1 = 0^0 \) and \( \theta_2 = 120^0 \) which goes through \( \theta = 60^0 \). When these bubbles grow and coalesce \( \theta \) in the region between the bubbles interpolates along this arc as seen in the right figure in Figure 2.7.
Figure 2.7: Figure on the left is the plot of initial field configuration $\Phi$ for two bubbles. $\theta = 120^0$ for the bubble on the left and $\theta = 0^0$ for the bubble on the right. Figure on the right corresponds to $\Phi$ field at $t = 18.0$.

To see the formation of vortex we consider the initial configuration of three bubbles with $\theta_1 = 0$, $\theta_2 = 120^0$ and $\theta_3 = 270^0$ as shown in the left figure in Figure 2.8. Here $\theta_3$ lies on the geodesic between $\theta'_1$ and $\theta'_2$, where $\theta'_1$ and $\theta'_2$ are points opposite to $\theta_1$ and $\theta_2$ on $S^1$ respectively, which satisfies the criteria of forming a defect out of these three domains as discussed in the previous section. After these bubbles grow and coalesce, a vortex is formed in the junction of the three bubbles. We give the plot of $\Phi(\tau)$ at two different times. The left figure in Figure 2.8 is plot of $\Phi(\tau)$ at $t=0$ and the figure on the right is the plot of $\Phi$ at time $t = 15.84$. One can see formation of a defect in the right figure in Figure 2.8. On any loop around the junction of the three bubbles, the phase $\theta$ of the complex field $\Phi$ varies by $2\pi$ after the bubbles have coalesced. This demonstrates the formation of a vortex via the Kibble mechanism in a first order phase transition.
Figure 2.8: Figure on the left is the plot of initial field configuration $\Phi$ for three bubbles. $\theta = 0^0$ for the bubble on the right, $\theta = 120^0$ for the bubble on the left with larger y coordinate and $\theta = 240^0$ in the third bubble. Figure on the right is the plot of $\Phi$ field at $t = 15.84$ where the bubbles have coalesced. Formation of a defect can be easily seen at the junction of the three bubbles.