It is by now widely accepted that all the fundamental interactions have an underlying gauge symmetry. But gauge symmetry implies that the gauge fields are massless, because an explicit gauge field mass term is not gauge invariant. On the other hand, short range forces like weak interactions have massive gauge field quanta. One way out of this dilemma is spontaneous breaking of the gauge symmetry. This is the mechanism used in the unified theory of electromagnetic and weak interactions, and is known as the Higgs mechanism. But an additional scalar field, the Higgs field, is there in the theory whose mass is indeterminate. Such an indeterminacy in the theory of fundamental interactions is rather disturbing and therefore several alternative mechanisms, like dynamical symmetry breaking, technicolour etc. were proposed but they met with very little success. In 2 + 1 dimensions the C-S term turns out to be an alternative to spontaneous symmetry breaking.

In this chapter we will discuss properties of the C-S term in brief [2.1]. Then we will review the charged vortex solutions of an abelian Higgs model with the C-S term [1.19]. In the end we will discuss the equivalence between the charged vortex solutions of Higgs models with the C-S term and neutral vortex solutions coupled to Fermions [2.2].

2.1 Properties of C-S term

Parity transformation connects a right handed coordinate system to a left handed one. In two space dimensions it amounts to reversing the direction of only one of the two axes. i.e. under parity transformation \((x, y) \rightarrow (-x, y)\). It can easily be seen that the C-S term is not invariant under parity transformation. It is also not invariant under time reversal transformation. The fermion mass term \(m \bar{\psi} \psi\) in 2 + 1 dimensions also possesses
the same \( P \) and \( T \) violating properties. Let us consider electrodynamics in the presence of the C-S term

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu}{4} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda
\]  

(2.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The equation of motion

\[
\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} = 0
\]

(2.2)

and the action \( \int \mathcal{L} d^3x \) are invariant under gauge transformation

\[
A_\mu \rightarrow A_\mu + \frac{1}{\epsilon} \partial_\mu \Lambda.
\]

(2.3)

The field equation can also be written as

\[
(g^{\mu\nu} + \frac{1}{\mu} \epsilon^{\mu\nu\alpha} \partial_\alpha) \ast F_\nu = 0
\]

(2.4)

where \( \ast F_\nu \) is dual field strength which is a vector in three dimensions. Multiplying on left by \( (g_{\beta\mu} - 1/\mu \epsilon_{\beta\mu\delta} \partial^\delta) \) we have

\[
(\partial_\mu \partial^\mu + \mu^2) \ast F_\beta = 0
\]

(2.5)

which shows that the gauge field excitations are massive. Since the theory is gauge invariant the massive C-S photon still has only one helicity. In the case of nonabelian gauge theories, the gauge field mass generated by the C-S term is quantized. Consider the nonabelian gauge field Lagrangian with the C-S term

\[
\mathcal{L} = \frac{1}{2g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{\mu}{2g^2} \epsilon^{\mu\nu\alpha} \text{Tr}(F_{\mu\nu} A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha)
\]

(2.6)

where \( F_{\mu\nu} \) and \( A_\mu \) are matrices. The field equations are

\[
D_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} = 0
\]

(2.7)
where $D_\mu = \partial_\mu + [A_\mu, ]$. Following the procedure of the abelian case one can show
that the gauge field has mass $\mu$. The action $I = \int \mathcal{L} d^3 x$, though invariant under in-
finitesimal gauge transformations, is not invariant under homotopically nontrivial gauge
transformations. For the gauge group $G$ if,

$$\pi_3(G) = \mathbb{Z}$$ (2.8)

then

$$I \longrightarrow I + \frac{8\pi^2 \mu}{g^2} m$$ (2.9)

where $m$ is an integer. From the path integral formulation we know that it is not the
action but $\exp(i I)$ that should be invariant. This implies that to have a sensible theory
we must write,

$$\frac{8\pi^2 \mu}{g^2} = 2n\pi \quad i.e. \quad \mu = \frac{g^2}{4\pi} n \quad n = 0, \pm 1, \pm 2, \ldots$$ (2.10)

Hence, the C-S mass for nonabelian gauge theories is quantized.

One of the important features of the C-S term is its relation with the parity anomaly
[2.3]. The C-S term is generated by radiative corrections due to the parity anomaly that
exists in the theory when an odd number of fermions are coupled to the gauge field. Let
us start with the action

$$I[A_\mu, \psi] = \int d^3 x \left[ \frac{1}{2g^2} \text{Tr} F_{\mu \nu} F^{\mu \nu} + i \bar{\psi} \slashed{D} \psi \right]$$ (2.11)

with an odd number of massless fermions coupled to SU(N) gauge fields in 3 dimensions.
This action is not only gauge invariant but also invariant under space as well as time
reversal. The effective action $I_{\text{eff}}[A_\mu]$ obtained by integrating out the fermionic degrees
of freedom is

$$I_{\text{eff}}[A_\mu] = \int d^3 x \left[ \frac{1}{2g^2} \text{Tr} F_{\mu \nu} F^{\mu \nu} - i \ln \mathcal{D} \right].$$ (2.12)

In regularising the logarithmic term in the effective action, either the gauge symmetry or
P and T symmetry is lost. For example, Pauli-Villars regularization introduces a large
fermion mass term as a regulator. But we already know that the fermion mass term breaks P and T symmetry. Thus the regularized effective action violates P and T (This anomaly does not exist in the case of an abelian gauge symmetry because there we can use a P and T conserving regulator.). In fact the regularized effective action contains the C-S term.

The C-S term can also be generated by spontaneous symmetry breaking [2.4]. Let us consider the generalized definition of a covariant derivative

\[ D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi + i\epsilon_{\mu\nu\alpha} \partial^{\nu}A^{\alpha} \frac{\phi}{|\phi|}. \]  

(2.13)

With this definition the Lagrangian

\[ \frac{1}{2}(D_\mu \phi)^*(D^\mu \phi) \]  

(2.14)

is gauge invariant. Substituting the generalized derivative we get, apart from the conventional covariant derivative, a term

\[ eg |\phi| \epsilon_{\mu\nu\alpha} (\partial^{\nu}A^{\alpha})A^\alpha \]  

(2.15)

so that if |\phi| acquires a nonzero vacuum expectation value then the abelian C-S term is generated.

2.2 Charged Vortices in Abelian Higgs model with the C-S term

The Landau-Ginzburg model as well as its relativistic generalisation - i.e. the abelian Higgs model admits neutral vortex solutions of finite energy. If the abelian Higgs model is coupled to the C-S term then it has charged vortex solutions of finite energy and angular momentum [1.19].

The abelian Higgs model in 2 + 1 dimensions with the C-S term is given by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(\partial_\mu - ieA_\mu)\phi^* (\partial^{\mu} + ieA^{\mu})\phi - c_4(|\phi|^2 - \frac{c_2}{2c_4})^2 + \frac{\mu}{4} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda \]  

(2.16)
where \( \mu \) is the topological mass of the gauge field. For the static \( n \)-vortex solution, the ansatz is

\[
\vec{A}(\vec{r}, t) = -\hat{e}_\theta c_0 A(r)/r , \quad A_0(\vec{r}, t) = c_0 A_0(r) , \quad \phi(\vec{r}, t) = c_0 \exp(i n \theta) f(r)
\]

(2.17)

Here we have rescaled the lengths and fields such that \( r, A(r), A_0(r) \) and \( f(r) \) are dimensionless variables.

\[
\rho = \frac{r}{e c_0} , \quad c_0 = \sqrt{\frac{c_2}{2c_4}} , \quad \lambda = \sqrt{\frac{8c_4}{e^2}}.
\]

(2.18)

With this ansatz the energy functional can be written as

\[
E_n = \pi c_0^2 \int_0^\infty r dr \left[ \frac{1}{r^2} \left( \frac{dA}{dr} \right)^2 + \left( \frac{df}{dr} \right)^2 + A_0^2 f^2 + \frac{(n + A)^2 f^2}{r^2} + \frac{(dA_0)^2}{r^2} + \frac{\lambda}{4} (1 - f^2)^2 \right].
\]

(2.19)

The boundary condition on the Higgs field configuration at infinity is related to the winding number of the map from the circle \( S^1 \) at spatial infinity to the group manifold. This in turn gives flux quantization,

\[
\Phi = \int d^2 x B = -\frac{2\pi}{e} \int_0^\infty r dr \frac{1}{r} \frac{dA}{dr} = \frac{2\pi}{e} n. \quad n = 0, \pm 1, \pm 2, ...
\]

(2.20)

Also on integrating the Gauss law equation one finds that these vortices are charged, i.e.

\[
Q = \int e^2 A_0 f^2 d^2 x = \mu \Phi = \frac{2\pi \mu}{e} n.
\]

(2.21)

For a finite energy solution the boundary conditions are,

\[
\lim_{r \to \infty} : f(r) = 1 , \quad A(r) = -n , \quad A_0(r) = 0 \quad (2.22a)
\]

\[
\lim_{r \to 0} : f(r) = 0 , \quad A(r) = 0 , \quad A_0(r) = \beta \quad (2.22b)
\]

where, \( \beta \) is arbitrary. The field equations are

\[
\frac{d^2 A}{dr^2} - \frac{1}{r} \frac{dA}{dr} - (n + A)^2 f^2 = \frac{\mu r}{ec_0} \frac{dA_0}{dr}
\]

(2.23a)
So far no analytic solution has been obtained to the field equations. However it is easily seen that for \( r \to \infty \), the asymptotic values of \( f \), \( A \) and \( A_0 \) are reached exponentially fast and are given by

\[ A(r) = -n + \alpha \sqrt{r} \exp(-\eta r) + \ldots \]  
\[ A_0(r) = -\frac{\alpha}{\sqrt{r}} \exp(-\eta r) + \ldots \]  
\[ f(r) = 1 + \beta_1 \exp(-\lambda r) + \ldots \]  

where \( \alpha \) and \( \beta_1 \) are dimensionless constants while

\[ \eta = (1 + \frac{\mu^2}{4e^2c_0^2})^{1/2} \pm \frac{\mu}{2ec_0}. \]  

From eq(2.25), it appears that there are two topologically nontrivial solutions, but it is not so. We will discuss more about it in chapter 3. One can also write down the behaviour as \( r \to 0 \) as

\[ A(r) = \alpha_1 r^2 + O(r^4) \]  
\[ A_0(r) = \beta + \alpha_1 \frac{\mu}{2ec_0} r^2 + O(r^4) \]  
\[ f(r) = \alpha_2 r \ln r + O(r \ln r + 2) \]  

where \( \alpha_1, \alpha_2 \) and \( \beta \) are constants. The qualitative features of the solution are as follows. The magnetic field \( B \) decreases exponentially fast from its maximum value at the origin to zero at infinity. The penetration depth is inversely proportional to the gauge boson mass, while the value of the Higgs field increases from zero at the origin to its vacuum expectation value at infinity. The coherence length is inversely proportional to the mass of the Higgs. The electric field vanishes both at the origin and at infinity reaching its
where $Q$ is the charge of the vortex defined earlier. The angular momentum is in general fractional but quantized. Thus these objects are anyons. They also possess an additional anomalous magnetic moment as compared to the usual neutral vortex magnetic moment

$$K_z = \int (\mathbf{r} \times \mathbf{j})_z d^2r = \frac{2\pi n}{e} + \frac{\mu}{e^2c_0} \int_0^\infty A_0(r)d^2r.$$  \hspace{1cm} (2.28)

Here we would like to note in passing another feature of the Higgs models with the C-S term. Since the C-S term gives a mass to the gauge fields we have massive gauge fields even in the absence of the Higgs mechanism. Therefore one can have vortex like configurations even without symmetry breaking. These vortices, however, do not have any topology coming from the Higgs field and hence can be called as nontopological vortices [2.5].

Vortices also exist in nonabelian gauge theories [2.6]. When $\pi_1(G/H)$ is nontrivial for a gauge group $G$ for which the vacuum is invariant under the subgroup $H$ we get charged vortex solutions to the Higgs models in the presence of the C-S term. For example in the SU(2) Yang-Mills theory, we can get $Z_2$ vortices by breaking the SU(2) symmetry down to $Z_2$ by using two Higgs fields. The two Higgs fields taken in the adjoint representation of SU(2) take vacuum expectation values in mutually orthogonal directions. The $Z_2$ vortices that we get possess flux, charge and angular momentum as follows

$$\Phi = 2\pi k/g, \quad Q = nk^2/2, \quad J = nk^2/4$$  \hspace{1cm} (2.29)

where $g$ is the gauge coupling constant and $'n'$ comes from the quantization of the C-S mass. It is worth noting that for $n = 1$ these vortices behave like half fermions.
2.3 Vortices in Higgs Models with and without the C-S term

Here we will show that the local charge and angular momentum induced on a neutral vortex by fermions is precisely the same as the local charge and angular momentum of a charged vortex in a C-S theory with the topological mass which is obtained by integrating out the fermions [2.2]. Let us consider an abelian Higgs model coupled to fermions

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu - ieA_\mu) \phi^* (\partial^\mu + ieA^\mu) \phi + V(\phi) + \bar{\psi}(i\partial + eA - m_f)\psi \]  (2.30)

The induced charge and spin on the vortex due to the fermions have been computed and found to be

\[ Q_{ind} = \frac{e}{2|m_f|} n, \quad J = \frac{n^2}{4} \]  (2.31)

Comparing this with the results of previous section shows that when

\[ \mu = \frac{e^2 m_f}{4\pi |m_f|} \]  (2.32)

the quantum numbers of the vortex excitations in both theories are the same. Thus the vortex excitations in the original theory with fermions and in the long wavelength limit of the theory obtained by integrating out fermions are the same. For a single quantum of flux the vortices are half-fermionic and pairs of them are bosonic.

Just as in the abelian case, let us consider a Lagrangian for nonabelian Higgs model coupled to fermions:

\[ \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} (D_\mu \phi^a)^* (D^\mu \phi^a) + \frac{1}{2} (D_\mu \chi^a)^* (D^\mu \chi^a) + V(\phi, \chi) + \bar{\psi}(iD - m_f)\psi \]  (2.33)

where \( \phi \) and \( \chi \) are two Higgs triplets that break SU(2) to \( Z_2 \) by taking vacuum expectation values in mutually orthogonal directions, while \( \psi \) is an SU(2) doublet of fermions. If we ignore fermions then this model has a neutral vortex solution of the form

\[ \phi^a = f(\rho)(\cos k\theta, \sin k\theta, 0), \quad \chi^a = \chi_0 \delta_{a3}, \quad A^a_\mu = \frac{A_\theta(\rho)}{\rho} \delta_{a3}. \]  (2.34)
This solution is such that it commutes with the diagonal generator $T_3$ of SU(2). Therefore we can compute the third component of the flux as

$$\Phi_3 = \frac{2\pi k}{g}, \quad (2.35)$$

and if coupled to fermions we can compute the charge $Q_3$ induced on the vortex as

$$Q_{3\text{ind}} = \frac{1}{4} g k \frac{m_f}{|m_f|}. \quad (2.36)$$

The extra factor of $1/2$ as compared to the abelian case occurs due to the group theory factor coming from $TrT_a T_b = \frac{1}{2} \delta_{ab}$. The statistics of the vortex may be computed using the arguments of Goldhaber et al. [2.7] that the phase factor of a charged vortex is given by $Q_{\text{ind}} \Phi/2$ instead of $Q\Phi$, when the charge is induced by a C-S term. The induced angular momentum can be determined from the statistics by demanding the consistency of the spin-statistics theorem and is given by

$$J_{\text{ind}} = \frac{k^2}{8}. \quad (2.37)$$

Comparing eq. (2.29), (2.36) and (2.37) we see that the quantum numbers of the vortex are in one-to-one correspondence for

$$\mu = \frac{g^2}{8\pi} \frac{m_f}{|m_f|} \quad \text{i. e.} \quad n = \frac{1}{2}. \quad (2.38)$$

Hence, the vortex excitations in both theories are identical for this particular value of parameter $\mu$.

Now consider a composite of two $Z_2$ vortices. It is topologically trivial since $k$ is defined modulo 2 and hence it carries no flux. However, it carries a charge $Q_3 = gn$ and a spin $J = J_1 + J_2 + (Q_1 \Phi_2 + Q_2 \Phi_1)/4\pi = n$. When $n$ is an integer as required by global gauge invariance, $Q_3$ is equivalent to zero, since the fundamental charge is $Q_3 = g/2$ and can only be measured modulo 2. Another way of saying it is that since we are measuring
charge with respect to a broken symmetry, any charge $g$ can be shielded by the massive Higgs and gauge fields inside the vortex which also have charges $\pm g$. However, when $n = 1/2$, the charge of the composite of two vortices (or a doubly charged vortex) is $g/2$ which cannot be shielded by the charge of gauge field or Higgs particles and its spin is $1/2$, i.e. the composite has precisely the quantum numbers of a fermion. However if the original fermion had other global quantum numbers those quantum numbers are not induced on the vortex. Hence, our composite of two vortices denotes a globally neutral fermion. Furthermore, when we consider a composite of four vortices, they can annihilate into the vacuum. Hence, we conclude that SU(2) theory with a single doublet of fermions integrated out has $Z_2$ solitons which are globally neutral and quarter fermionic and composites of two $Z_2$ solitons which are globally neutral and fermionic.

Unlike the U(1) case, however, there exists a subtlety in the SU(2) case, which in fact appears to be responsible for composite soliton excitations which are nontrivial. Firstly, introducing fermions in the theory with neutral $Z_2$ vortices breaks the $Z_2$ symmetry, thereby rendering the vortices unstable. We are ignoring this problem here and assuming that fermions can be introduced perturbatively. In the C-S theory, global gauge invariance demands that $n$ be an integer, whereas the induced $n$ turns out to be half integer. This subtlety is responsible for two vortices annihilating into a fermion, when there are an odd number of fermions in the theory. When there are an even number of fermions, the $Z_2$ symmetry can be restored in the SU(2) theory with fermions by a suitable redefinition of the $Z_2$ acting on fermions and in the C-S theory, $n$ is an integer. The vortex now has a charge $g/2$, $J = 1/4$ and pairs of them are trivial. So an SU(2) theory interacting with two doublets of fermions has the above mentioned vortex excitations which are globally neutral half fermions.

Now we will show why these vortices are relevant in condensed matter systems. Recently, Affleck et al. [2.8] showed that the $U \to \infty$ limit of the Hubbard model is the
Heisenberg model

\[ H = \frac{J}{4} \sum_{(ij)} (C^\dagger_i \bar{\sigma} \alpha \beta C_i \beta)(C^\dagger_j \bar{\sigma} \alpha \beta C_j \beta) \tag{2.39} \]

with the single occupancy constraint

\[ \sum C^\dagger_{i\alpha} C_{i\alpha} = 1 \tag{2.40} \]

is invariant under \((C_{i\dagger}, C^\dagger_{i\dagger})\) and \((-C_{i\dagger}, C^\dagger_{i\dagger})\) transforming as local SU(2) doublets. Furthermore, the single occupancy constraint forces the electric charge at every site to be \(-e\). Hence, in the half filled Hubbard model in the \(U \to \infty\) limit, electromagnetism is reduced to a global U(1) symmetry. The appropriate model to study is a local SU(2) theory supplemented by a global U(1) symmetry. Hence, the limit of the SU(2) Higgs model, where SU(2) breakdown to Z₂ is weak, is applicable here. Whether the appropriate theory has a single doublet of fermions or two appears controversial. If a single doublet of fermions is appropriate, then the long wavelength theory contains electric charge neutral \(J = 1/8\) solitons, pairs of which behave like electric charge neutral fermions. If, however, the model has two doublets of fermions, then long wavelength theory has electric charge neutral \(J = 1/4\) vortices pairs of which behave like bosons. Superconductivity is expected to occur via Bose condensation of charged bosons which occur when electrons are removed in the vicinity of the vortices.

In this chapter, we gave a brief review of the properties of the C-S term, which was followed by a review of the abelian Higgs model with the C-S term. Then we showed that vortex excitations in Higgs models interacting with fermions were in one-to-one correspondence with vortices in the Higgs model with a C-S term. In a nonabelian theory, we showed that a single fermion can be integrated out and then be realised in the effective C-S theory as a pair of Z₂ solitons. Finally, we commented on the connection between the quasi-particle excitations in this model and the quasi-particles appearing the \(U \to \infty\) limit of the half filled Hubbard model.
References


