Chapter Six

WKB ANALYSIS OF THE STRING EQUATION

Recent developments in the study of random surfaces and two-dimensional gravity [6.1] have brought forth a number of interesting questions of both a mathematical and a physical nature. One of the remarkable achievements has been a non-perturbative formulation of strings in \( d < 1 \) [1.5-1.7]. The string susceptibility is described as the solution of a non-linear differential equation, often referred to as the string equation of motion. The study of the correlation functions of these theories reveals the close connection of this theory to the KdV hierarchy and its generalizations. The string equations of motion are described by commutation relations between the differential operators of the KdV hierarchy.

One of the interesting questions that arises however, is the nature of the boundary conditions on the string equations of motion. While some of these are fixed by the requirement of matching perturbation theory, the rest can be fixed by the location of the movable poles of the solutions of the string equations of motion. It was subsequently shown that the presence of poles on the real axis for the string susceptibility was inconsistent with the Schwinger -Dyson equations of the theory [6.2]. It was also shown that the boundary conditions can be fixed by specifying the behaviour at \( x \to \infty \) for the \( k = \text{odd} \) one-matrix models. In these cases numerical work suggests that the string susceptibility has no poles on the real axis [6.3].

Here, we shall study a different method of understanding the boundary conditions starting from the commutation relation description of the string equations of motion. We show a straight-forward connection between the differential operators described by Douglas [6.4] and the theory of monodromy-preserving deformations of ordinary differential equations [6.5,6.6]. It turns out that the boundary conditions may be equivalently
described by the behaviour of the theory for large eigenvalues in the scaling limit. There is also a neat relation with the KdV structure in the theory. Though the method has implications for the study of renormalization group flows between the different one-matrix models, most of our work is confined to the study of the $k = 2$ matrix model. After completing this work, we became aware of a preprint by G. Moore [1.8] where, among other things he also addressed the question of uniqueness of the solution of the string equation using the framework of monodromy preserving deformation. The ideas presented in this chapter overlap to a great extent with his work [1.8] and for ease of presentation we have appropriated some of his ideas.

6.1 Zero Curvature Method and Isomonodromy Deformation

Our starting point is the observation by Douglas [6.4] that the string equation of motion is given by the relation

$$[P, Q] = 1$$

where $P$ and $Q$ are operators corresponding to $d/d\lambda$ and $\lambda$ respectively. In the double scaling limit they are differential operators. For pure gravity the operators are

$$Q \equiv D^2 - u$$

$$P \equiv \frac{1}{2} D^3 - \frac{3}{4} u D - \frac{3}{8} u'$$

where $D \equiv \frac{\partial}{\partial x}$. If $\psi$ is the wave-function of the Schrödinger operator $Q$ we can write

$$Q \psi = \lambda \psi$$

$$P \psi = \frac{d\psi}{d\lambda}$$

These equations may now be converted to matrix form by defining $\Psi \equiv \left( \begin{array}{c} \frac{\partial \psi}{\partial x} \\ \psi \end{array} \right)$ ; this gives us the following system of matrix equations

$$\frac{d\Psi}{d\lambda} = A(x, \lambda) \Psi$$
where
\[ A(x, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda^2/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \lambda/2 \\ 0 & -\lambda/4 \end{pmatrix} + \begin{pmatrix} -\frac{y}{8} & \frac{y^2}{8} + x \\ \frac{y}{8} & -\frac{y^2}{8} \end{pmatrix} \] (6.6)

and
\[ \frac{\partial \psi}{\partial x} = B(x, \lambda) \psi \] (6.7)

where
\[ B(x, \lambda) = \begin{pmatrix} 0 & \lambda + u \\ 1 & 0 \end{pmatrix} \] (6.8)

Note that equation (6.6) may be obtained from eq.(6.2) by using the Schrodinger equation. The string equation follows naturally as a zero-curvature condition
\[ \frac{\partial A}{\partial x} - \frac{\partial B}{\partial \lambda} + [A, B] = 0. \] (6.9)

For purposes of further analysis, it is necessary to 'shear' eq.(6.6). The shearing transformation is such that it diagonalises the highest power of \( \lambda \) in the matrix \( A(x, \lambda) \), i.e. for \( \lambda = \xi^2 \)
\[ \Psi(x, \lambda) = \sqrt{\xi} \begin{pmatrix} 1 & \xi \\ \xi & -1 \end{pmatrix} Y(x, \lambda) \] (6.10)

so that
\[ \frac{dY}{d\xi} = A(x, \xi) Y \] (6.11)

where
\[ A(x, \xi) = \left[ (\xi^4 + \frac{u^2}{8} + x)\sigma_3 - (\frac{u}{2}\xi^2 + \frac{u^2}{8} + x)i\sigma_2 - (\frac{u'}{4}\xi + \frac{1}{2\xi})\sigma_1 \right] \] (6.12)

Equation (6.7) may also be similarly transformed. This method can obviously be extended to all string equations with arbitrary \( k \). Written in this system form, these equations may be recognised to precisely correspond to the form of monodromy-preserving deformations of ordinary differential equations with an irregular singular point at infinity. In particular, these equations are similar to those written down by Jimbo and Miwa.
The basic idea is as follows. The fundamental equation is the equation
\[ \frac{d\Psi}{d\lambda} = A(x, \lambda)\Psi \] (6.13)
referred to as the 'equation in \( \lambda \). However, in practice we will use the equation in \( \xi \) as
the equation in \( \lambda \). The solutions of this equation namely \( Y \) provide a set of monodromy
data. The second equation (6.7) deforms the solution, by varying the parameter \( x \) while
keeping the monodromy data fixed.

To define the monodromy data we first write down the asymptotic solution as \( \xi \to \infty \),
\[ Y \approx \xi^{-\infty} \left( I + \frac{\bar{Y}_1}{\xi} + \frac{\bar{Y}_2}{\xi^2} + \cdots \right) \exp T(\lambda) \] (6.14)
where
\[ T(\lambda) = \frac{4}{5} \left( \begin{array}{cc} 1 & \xi^5 \\ -1 & \end{array} \right) - \frac{1}{2} \ln \xi \] (6.15)
and
\[ \bar{Y}_1 = \left( \begin{array}{c} -H_I \\ H_I \end{array} \right) \quad \text{and} \quad \bar{Y}_2 = \left( \begin{array}{c} u \\ v \end{array} \right) \] (6.16)
with \( H_I = \frac{1}{2} u^2 - (2u^3 + xu) \).

This asymptotic solution is not valid in all angular sectors at \( \infty \). The growing and
decaying solutions interchange roles in neighbouring sectors. In general, the decaying
solution in one angular sector can be analytically continued into the neighbouring sector
to a growing solution. The growing solution needs, however, an addition of a decaying
part in order to reproduce asymptotically a decaying solution [6.5].

We may define angular sectors at \( \infty \), separated by lines where \( \text{Re} \xi^5 = 0 \) and hence
the decaying and growing solutions interchange their roles. These lines (anti-Stokes lines)
separate angular sectors \( A_{j+1} = \frac{(2j-1)\pi}{10} \theta < \frac{(2j+1)\pi}{10} \quad j = 1 \text{ to } 10 \). The solution in each
sector is labelled as \( Y_j \) and
\[ Y_{j+1} = Y_j S_j \quad \text{where} \ S_j \text{ is} \]
\[
\begin{pmatrix}
1 & 0 \\
S_j & 1
\end{pmatrix}
\] for \(j\) odd and
\[
\begin{pmatrix}
1 & S_j \\
0 & 1
\end{pmatrix}
\] for \(j\) even. (6.17)

This is referred to as the Stokes phenomenon and the matrices the Stokes matrices. There is also a fixed square-root branch in the solution at \(\infty\) and a square-root branch at zero. The solutions at zero and \(\infty\) in each angular sector may be related by a connection matrix \(C\)

\[Y = \Phi C\]

with \(C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) and \(\alpha \delta - \beta \gamma = 1\) (6.18)

where \(\Phi\) is the asymptotic solution at zero. The collection of monodromy data which is to be held fixed under deformation is given by the set

\[s_j, j = 1, \ldots, 10; \alpha, \beta, \gamma, \delta\] with \(\alpha \delta - \beta \gamma = 1\) (6.19)

This large collection of data is not independent and the symmetries of the equation reduce the total number. We can check that if \(Y(x, \xi)\) is a solution so is \(MY(x, -\xi)\) where \(M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\). Thus \(s_j = s_{j+5}\) and the number of Stokes parameters is reduced by half. We can also compute the monodromy matrix at 0 and \(\infty\).

\[\Phi(x, \xi e^{2\pi i}) = \Phi(x, \xi)M_0 \quad \text{and} \quad Y_1(\xi) = Y_{10}(\xi e^{2\pi i})S_{10}M_\infty\] (6.20)

Therefore, using the connection matrix, we can write the following constraint

\[\prod_j S_j M_\infty = C^{-1} M_0^{-1} C.\] (6.21)

However this constraint is not always true. A sharper constraint is obtained by the following considerations. Under the \(\xi \to \xi e^{\pi i}\) we can see by direct inspection of the asymptotic solution that

\[Y_0^{(1)}(\xi, x) \sim MY_1^{(2)}(\xi e^{\pi i}, x)\]

where \(M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\) (6.22)

The asymptotic solution at 0 gives us

\[\Phi(\xi e^{\pi i}) = \Phi(\xi)J.\] (6.23)

65
Now,
\[ Y(\xi e^{\pi i}) = \Phi(\xi e^{\pi i})A = \Phi JA \] \hspace{1cm} (6.24)

\[ Y_1(\xi)S_5S_4S_3S_2S_1 = Y_6(\xi) \] \hspace{1cm} (6.25)

But
\[ Y_6(\xi) = (Y_1^{(2)}(\xi e^{\pi i}), Y_1^{(1)}(\xi e^{\pi i})). \] \hspace{1cm} (6.26)

Putting all these together we get
\[ S_5S_4S_3S_2S_1 = MA^{-1}JA. \] \hspace{1cm} (6.27)

Thus with these relations we see that only two of the Stokes parameters are independent. Of course, a proper proof is needed that the second equation indeed generates deformations that do not change the monodromy data. We assume that such a proof can be provided following the lines indicated by Jimbo and Miwa [6.5]. Since the two independent Stokes parameters are invariants of the deformation flow, they may be considered as providing the boundary conditions on the Painleve-type equations.

6.2 The Inverse and Direct Problem of Monodromy Theory

The inverse problem consists in determining what \( u(x) \) is obtained by a specification of the monodromy data. This means that we are solving the initial value problem of the string equations. However, in general this problem is quite complicated and the number of attempts to solve for similar equations like the Painleve-II have not led to much progress. The most recent attempt has been to reduce the problem to that of solving a Riemann-Hilbert problem. However, in the case of Painleve-II even this method is useful only in special cases and that too in conjunction with the techniques of the direct problem. The inverse problem for the Painleve-I and its generalizations appears to be open in the mathematical literature. Hence, we will discuss here only the direct problem in any substantial detail.
The direct problem consists of determining the Stokes data that characterises different solutions of the string equations of motion. The method of choice here is the WKB method. From the string equations we can obtain appropriate asymptotics for different solutions at \( x \to \pm \infty \). Since the equation in \( \lambda \) becomes an equation with \( x \) as a large parameter, it is natural to apply the WKB method. In general, for the string equations we know that perturbation theory fixes only the boundary conditions at \( x \to \infty \). Thus the Stokes parameters would not be fully determined. However, in the odd \( k \) one-matrix models we could use the suggestion of BMP for the boundary conditions at \( \infty \) to fix the Stokes parameters completely \([6.7]\). It is much harder however, to relate this result to the location or indeed the non-existence of the poles of \( u(x) \) on the real axis.

### 6.2.1 The WKB method

It is easy to show that for \( x \to -\infty \) to leading order \( u(x) \approx \sqrt{\frac{-8x}{3}} \). Using the rescaled variable \( \xi = \eta \left( \frac{-8x}{3} \right)^{1/4} \), and substituting this in eq.(6.12) we obtain

\[
\frac{dY}{d\eta} = \tau \left[ (\eta^4 - \frac{1}{4})\sigma_3 - (\frac{\eta^2}{2} - \frac{1}{4})i\sigma_2 - \frac{1}{2\eta\tau} \left( \frac{2\eta^2}{3} + 1 \right)\sigma_3 + O(\eta^{-2}) \right]
\]

(6.28)

We must first transform the equation such that the \( O(1) \) terms on the right-hand side are diagonal, giving an equation of the form

\[
\frac{d\tilde{Y}}{d\eta} = \tau \mu \sigma_3 \tilde{Y} + \tau F \tilde{Y}
\]

(6.29)

where \( \mu \) is diagonal and order 0 in \( \tau \) while \( F \) is of order \( 1/\tau \). The turning points are then given by the zeroes of \( \mu \). Thus, we compute the leading WKB expression

\[
Y_{WKB} \approx T(\eta) \exp \left( \int_{\eta_c}^{\eta} \mu d\eta \right) - \text{diag}(T^{-1} dT/\eta).
\]

(6.30)

In our case the corresponding calculations give the following:

\[
\mu = \sqrt{(\eta^4 - \frac{1}{4})^2 - \frac{1}{4}(\eta^2 - \frac{1}{2})^2 + \frac{1}{4\eta^2\tau^2} \left( \frac{2\eta^2}{3} + 1 \right)^2}
\]

(6.31)
and

\[ T(\eta) = \begin{pmatrix} 1 & -\frac{1}{2\eta^2}\left(\frac{2n^3}{3}+1+\frac{n^2}{3} \right) \\ \frac{\mu-n\left(\frac{2n^3}{3}+1\right)}{\mu_n^2+\frac{n^2}{3}} & 1 \end{pmatrix} \]  

(6.32)

The next step is to find the domains of \( \text{Re} \int_{\eta}^{\eta'} \mu = +ve \) and \(-ve\). We therefore plot the lines of \( \text{Re} \int_{\eta}^{\eta'} \mu = 0 \). These are given by the curves in \( u \) and \( v \) which are the real and imaginary parts of \( \eta \). We note, that asymptotically, the lines tend to the anti-Stokes lines at \( \infty \) as already described.

In the neighbourhood of the real root of \( \mu \), we expand \( \mu \) in powers of \( \eta - \eta_0 \). Defining \( \zeta_0 = \tau^{1/2}(\eta - \eta_0) \), we get the equation

\[ \frac{dY_0}{d\zeta_0} = [(A\sigma_3 + B\sigma_2)\zeta_0 + O(\tau^{-1/2})]Y_0. \]  

(6.33)

This equation can be diagonalised and its solution is asymptotic to

\[ Y_0 = T' \exp\left[\sqrt{A^2 + B^2}\sigma_3 \frac{\sigma_2^2}{2}\right] \]  

(6.34)

where, \( T' \) diagonalises the matrix \( A\sigma_3 + B\sigma_2 \). Now, using the formal asymptotic solution, the WKB solution and the solution near the turning point we determine the connection matrices relating the WKB solution to the other two solutions. The connection matrices are,

\[ C = \lim_{\eta \to \infty} \exp[\tau \int_{\eta}^{\eta'} \mu(\eta')d\eta' - \frac{\eta^5}{5} - x\eta] \]  

(6.35)

and

\[ N = \exp[\tau \int_{\zeta}^{\zeta'} \mu(\zeta')d\zeta' - \sqrt{A^2 + B^2}\zeta^2]. \]  

(6.36)

Following the procedure of ref.[6.8] we can write down the Stokes matrices as,

\[ S_k = C_{k+1}^{-1}N_{k+1}^{-1}N_k C_k. \]  

(6.37)

Determination of all the Stokes matrices using this method and then imposing the constraints derived earlier is a long and tedious calculation [6.9]. Here, instead we will follow
the approach of Moore [1.8]. The theorem that he proved is as follows. Suppose that at a turning point four Stokes lines join to form three open regions, then the Stokes matrix for the middle region is trivial. Using this theorem and the fact that the reality condition on \( u \) relates the Stokes matrices symmetric about the \( \text{Im} \eta \) axis with each other, we determine the Stokes parameters for the case at hand. From fig. 1 and using the theorem it is easy to see that the Stokes parameter \( s_5 \) vanishes. Using this information in eq.(6.21) and (6.27) we can eliminate one of the two independent Stokes parameters, which implies that there is a one parameter family of solutions consistent with the asymptotic behaviour, -i.e., a one parameter family of solutions consistent with perturbation theory.
References


VORTICES IN HIGGS MODELS WITH AND WITHOUT CHERN–SIMONS TERMS

Dileep P. JATKAR and Sumathi RAO
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, India

Received 23 May 1989

We note that neutral vortices in a fermionic background acquire the same local charge and spin quantum numbers as charged vortices in a Chern–Simons theory, provided the Chern–Simons mass is obtained by integrating out the fermions. We also point out that in an SU(2) theory involving (globally) charged fermions, (globally) neutral fermions appear as pairs of $Z_2$ solitons and comment on their relevance to condensed matter systems.

Vortices that arise in abelian and non-abelian Higgs models [1] have been well studied for a while, especially due to their applicability in theories of superconductivity [2]. However, more recently the charged vortex solutions [3,4] which occur when a Chern–Simons mass term [5] is added to the theory have gained in importance, ever since it has been realised [6] that such a mass term for the gauge field can be induced by fermions.

In this letter, we note that the local charge [7] and angular momentum [8] induced on a neutral vortex by fermions is precisely the same as the local charge and angular momentum of a charged vortex in a Chern–Simons theory with that value of the topological mass which is obtained by integrating out the fermions. For an abelian Higgs model, either interacting with a single fermion or with an appropriate Chern–Simons mass, the vortices are $J = \frac{1}{2}, Q = \frac{1}{2}g$ fermions. For a non-abelian Higgs model, the vortices are $J = \frac{1}{2}, Q = \frac{1}{2}g$, global $U(1)$ charged bosonic-fermions, pairs of which are globally neutral fermions [9]. However, the SU(2) theory with two doublets of fermions, which is $Z_2$ invariant and globally gauge invariant, has $J = \frac{1}{2}, Q = \frac{1}{2}g$ charged bosonic-fermions, pairs of which are deformable to the vacuum. Finally, we note that these globally neutral vortices may be identified with the quasiparticle excitations of the RVB theory [10].

The lagrangian for an abelian Higgs model with a Chern–Simons term [1,11] is

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_{\mu} - ieA_{\mu})\phi^*(\partial^{\mu} + ieA^{\mu})\phi + V(\phi) + \frac{1}{4}\mu_{\mu\nu\rho\sigma}A^{\mu}F_{\nu\rho}A^{\sigma},$$

(1)

where $\mu$ is the topological mass for the gauge field and $V(\phi)$ is the Higgs potential for the complex scalar field $\phi$. A $k$-vortex solution of this model is given by [3]

$$\phi(\rho) = \exp(i k \theta) f(\rho),$$

$$A_1(\rho) = -\frac{\partial_0 A(\rho)}{\rho}, \quad A_0(\rho) = A_0(\rho),$$

(2)

where $\rho$ and $\theta$ are cylindrical coordinates and the boundary conditions on the radial functions are

$$\lim_{\rho \to 0} f(\rho) = c,$n

$$\lim_{\rho \to 0} A(\rho) = -\frac{k}{e}, \quad \lim_{\rho \to 0} A_0(\rho) = 0,$$

(3)

where $c$ minimises $V(\phi)$ and

$$\lim_{\rho \to 0} A_0(\rho) = \lim_{\rho \to 0} A(\rho) = \lim_{\rho \to 0} f(\rho) = 0.$$n

(4)

This $k$-vortex solution describes an excitation with

$$\Phi = \frac{2\pi k}{e}, \quad Q = \mu \Phi \Rightarrow J = \frac{Qk}{2e}.$$n

(5)

Since $Q$ and $J$ are zero when $\mu$ is zero, without a topological mass term, the theory has charge neutral, bosonic vortex excitations [1]. However, consider the case when $\mu$ is zero, but the gauge fields are coupled to fermions – i.e. in eq. (1), set $\mu = 0$ and add to it.
The induced charge and spin on the vortex due to the interaction with fermions have been computed and found to be [7,8]

\[ Q_{\text{ind}} = \frac{e}{2 |m_t|} k, \quad J = \frac{1}{2} k^2 \]  

By comparing eqs. (5) and (7) we see that when

\[ \mu = \frac{e^2 m_t}{4\pi |m_t|}, \]  

the quantum numbers of the vortex excitations in both theories are the same. This, perhaps, is not surprising, because this is precisely the value of \( \mu \) we get by integrating out the fermion. Hence, the vortex excitations in the original theory with fermions and in the long wavelength theory obtained by integrating out fermions are the same. For a single quantum of flux, \((k=1)\) the excitations are half fermionic and pairs of them (or \(k=2\) vortices) are bosonic.

The non-abelian Higgs model is described by [1,4]

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{1}{2} \mu_\alpha \phi^\alpha \left(F^{a \mu \nu} A^a_{\mu \nu} + \frac{1}{4} g e^{abc} A^a_{\mu \nu} A^b \phi^c \right) + \frac{1}{2} \left(D_\mu \phi^a \right)^* \left(D^\mu \phi^a \right) + \frac{1}{2} \left(D_\mu \chi^a \right)^* \left(D^\mu \chi^a \right) + \frac{1}{2} m_1^2 |\phi|^2 + \frac{1}{2} m_2^2 \chi^2 - \frac{1}{4} \lambda_1 |\phi|^4 - \frac{1}{2} \lambda_2 |\chi|^4 - \frac{1}{2} \beta (|\phi| \chi)^2 - \frac{1}{2} \gamma |\phi|^2 |\chi|^2, \]  

(9)

where \( \chi \) and \( \phi \) are two Higgs triplets that break SU(2) down to \( Z_2 \) when \( m_1^2 > 0 \), \( D_\mu \phi^a = \partial_\mu \phi^a + g e^{abc} A^b \phi^c \) and \( \mu \) is the topological mass which is quantised as [5]

\[ 4\pi \mu g^2 = n = \text{integer}, \]  

(10)

due to global gauge invariance. This model has a vortex solution of the form

\[ \phi^a = f(\rho) \left( \cos k\theta, \sin k\theta, 0 \right), \]  

\[ \chi^a = \chi_0 \delta_{a3}, \]  

\[ A_\mu^a = \frac{\psi_0}{\rho} A_\mu A_0(\rho) \delta_{a3}, \quad A_\mu^a = A_0(\rho) \delta_{a3}, \]  

(11)

where \( k \) is the winding number of the solution. The boundary conditions on the solution are given by

\[ \lim_{\rho \to 0} f(\rho) = \phi_0, \]  

\[ \lim_{\rho \to 0} A_\mu(\rho) = -\frac{k}{g} \rho, \quad \lim_{\rho \to 0} A_0(\rho) = 0, \]  

(12)

where \( \phi_0 \) and \( \chi_0 \) in eqs. (11) and (12) minimise the potential and

\[ \lim_{\rho \to 0} f(\rho) = \lim_{\rho \to 0} A_\mu(\rho) = \lim_{\rho \to 0} A_0(\rho) = 0. \]  

(13)

The vortex excitation has

\[ \Phi_3 = \frac{2\pi k}{g}, \quad Q_3 = \mu \Phi_3 = \frac{1}{2} ngk, \quad J = \frac{1}{2} nk^2, \]  

(14)

where \( n \) is defined in eq. (10). A non-trivial solution only exists for \( k = 1 \). Since the vortices are \( Z_2 \), a \( k = 2 \) vortex is equivalent to the vacuum. Hence, the charge and angular momentum are also defined only for \( k = 1 \).

Just as in the abelian case, let us consider what happens when \( \mu \to 0 \) in eq. (9) and the lagrangian is supplemented by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{1}{2} \mu \phi^a \left(F^{a \mu \nu} A^a_{\mu \nu} + \frac{1}{4} g e^{abc} A^b \phi^c \right) + \frac{1}{2} \left(D_\mu \phi^a \right)^* \left(D^\mu \phi^a \right) + \frac{1}{2} \left(D_\mu \chi^a \right)^* \left(D^\mu \chi^a \right) + \frac{1}{2} m_1^2 |\phi|^2 + \frac{1}{2} m_2^2 \chi^2 - \frac{1}{4} \lambda_1 |\phi|^4 - \frac{1}{2} \lambda_2 |\chi|^4 - \frac{1}{2} \beta (|\phi| \chi)^2 - \frac{1}{2} \gamma |\phi|^2 |\chi|^2, \]  

(15)

where \( \psi \) is an SU(2) doublet of fermions. Since the vortex solution commutes with the \( T_3 \) generator of SU(2), we may compute [7] the \( Q_3 \) charge induced on the vortex as

\[ Q_{3 \text{ind}} = \frac{1}{4} g k m_t, \]  

(16)

The extra factor of \( \frac{1}{2} \) as compared to the abelian case occurs due to the group theory factor coming from \( \text{Tr} T_a T_b = \frac{1}{2} \delta_{ab} \). The induced angular momentum on the vortex has not been computed directly for the non-abelian model. However, the statistics of the vortex may be computed using the argument in ref. [12] that the phase factor for a charged vortex given by \( \frac{1}{2} Q_{\text{ind}} \Phi \) instead of \( Q \Phi \), when the charge is induced by a Chern–Simons term. \( \frac{1}{2} Q_{\text{ind}} \Phi = \frac{1}{2} g k^2 \) leads to a spin of

\[ J_{\text{ind}} = \frac{1}{2} k^2, \]  

(17)

in order to be consistent with the spin-statistics theorem. Once again comparing eqs. (14), (16) and (17), we see that the quantum numbers of the vortex are in one-to-one correspondence for
\[ \mu = \frac{g^2 m_t}{8\pi |m_t|} \quad \text{i.e. } n = \frac{1}{2}. \]  

Hence, the vortex excitations in both theories are identical for this particular value of the parameter \( \mu \).

Now consider a composite of two \( Z_2 \) vortices \([9]\). It is topologically trivial since \( k \) is only defined modulo 2 and hence it carries no flux. However, it carries a charge \( Q_2 = gn \) and a spin \( J = J_1 + J_2 + \frac{1}{2} (Q_1 \Phi_1 / 2\pi + Q_2 \Phi_2 / 2\pi) = n \). When \( n \) is an integer as required by global gauge invariance, \( Q_2 \) is equivalent to zero, since the fundamental charge is \( Q_2 = \frac{1}{2} g \) and can only be measured modulo 2. Another way of saying it is that since we are measuring charge with respect to a broken symmetry, any charge \( g \) can be shielded by the massive Higgs and gauge fields inside the vortex which also have charges \( \pm g \). However, when \( n = \frac{1}{2} \), the charge of the composite of two vortices (or a doubly charged vortex) is \( \frac{1}{2} g \) which cannot be shielded by the charge of gauge field of Higgs particles and its spin is \( \frac{1}{2} \) i.e. the composite has precisely the quantum numbers of a fermion. However, if the original fermion had other global quantum numbers those quantum numbers are not induced on the vortex.

Hence, our composite of 2 vortices denotes a globally neutral fermion. Furthermore, when we consider a composite of four vortices, they can annihilate into the vacuum. Hence, we conclude that the SU(2) theory with a single doublet of fermions integrated out has \( Z_2 \) solitons which are globally neutral and quarter fermionic and composites of two \( Z_2 \) solitons which are globally neutral and fermionic.

Unlike the \( U(1) \) case, however, there exists a subtlety in the SU(2) case, which in fact appears to be responsible for composite soliton excitations which are non-trivial. Firstly, introducing fermions in a theory with neutral \( Z_2 \) vortices breaks the \( Z_2 \) symmetry, thereby rendering the vortices unstable. We are ignoring this problem here and assuming that fermions can be introduced perturbatively. In the Chern–Simons theory, global gauge invariance demands that \( n \) be an integer, whereas the induced \( n \) turns out to be half integer. This subtlety is responsible for two vortices annihilating into a fermion, when there are an odd number of fermions in the theory. When there are an even number of fermions, \( Z_2 \) symmetry can be restored in the SU(2) theory with fermions by a suitable definition of \( Z_2 \) acting on the fermions and in the Chern–Simons theory, \( n \) is an integer. The vortex now has a charge \( \frac{1}{2} g \), \( J = \frac{1}{2} \) and pairs of them are trivial. So an SU(2) theory interacting with two doublets of fermions has vortex excitations that have local charge \( \frac{1}{2} g \), \( J = \frac{1}{2} \) and are globally neutral half-fermions.

Finally, why are these vortices relevant in condensed matter systems? Recently, Affleck et al. \([13]\) showed that the \( U \to \infty \) limit of the Hubbard model i.e. the Heisenberg model

\[ H = \frac{J}{4} \sum_{\langle \alpha \beta \rangle} (C^\dagger_{\alpha \sigma} \sigma_{\alpha \beta} C_{\beta \beta})(C^\dagger_{\beta \sigma} \sigma_{\alpha \beta} C_{\alpha \alpha}) \]

with the single occupancy constraint

\[ \sum C^\dagger_{\alpha \sigma} C_{\alpha \sigma} = 1 \]

is invariant under \( (C_{\alpha \sigma}, C_{\sigma \alpha}) \) and \( (C_{\alpha \sigma} - C_{\sigma \alpha}) \) transforming as local SU(2) doublets. Furthermore, the single occupancy constraint forces the electric charge at every site to be \(-e\) (i.e. the charge of a single electron). Hence, in the half-filled Hubbard model in the \( U \to \infty \) limit, electromagnetism is reduced to a global \( U(1) \) symmetry. The appropriate model to study is a local SU(2) theory supplemented by a global U(1) symmetry. Hence, the limit of the SU(2) Higgs model, where SU(2) breakdown to \( Z_2 \) is weak is applicable here. Whether the appropriate theory has a single doublet of fermions \([14]\) or two doublets of fermions \([15]\) appears controversial. If a single doublet of fermions is appropriate, then the long-wavelength theory contains electric charge neutral \( J = \frac{1}{2} \) solitons, pairs of which behave like electric charge neutral fermions. If, however, the appropriate model has two doublets of fermions, then the long-wavelength limit has electric charge neutral \( J = \frac{1}{2} \) vortices pairs of which behave like bosons. Superconductivity is expected to occur via Bose condensation of charged bosons which occur when electrons are removed in the vicinity of the vortices.

We conclude by recapitulating the main points made in this letter. We showed that vortex excitations in Higgs models interacting with fermions were in one-to-one correspondence with vortices in the Higgs model with a Chern–Simons term. In the non-abelian theory, we showed that a single fermion can be integrated out and then be realised in the effective Chern–Simons theory as a pair of \( Z_2 \) solitons. Finally, we commented on the connection between the
quasi-particle excitations in this model and the quasi-particles appearing in the $U\to\infty$ limit of the half-filled Hubbard model. However, much further study is required before a deep connection can be made between the field theory models that we addressed here and the condensed matter system.

We would like to thank Dr. Avinash Khare for useful discussions.

References

A QUASI-EXACTLY SOLVABLE PROBLEM WITHOUT Sl(2) SYMMETRY

Dileep P. JATKAR, C. NAGARAJA KUMAR and Avinash KHARE
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, India

In the last few years a number of potentials have been discovered in quantum mechanics which are quasi-exactly solvable (QES) in the sense that for them only $N$ eigenvalues and corresponding eigenfunctions are known analytically [1]. The reason for the existence of these QES problems was discovered recently [2,3] (an excellent review is given by Shifman [4]). It was pointed out that for the known one-dimensional problems the quasi-exactly solvability is due to the existence of hidden dynamical $Sl(2)$ symmetry [3,4]. In particular, for the known cases the Hamiltonian $H$ can be cast in bilinear form in terms of the generators of the $Sl(2)$ group,

$$H = \sum_{j=0}^{2j} a_j J_j + \sum_{j=0}^{2j} b_j J_j,$$

where

$$J_+ = -2jZ + Z^2 \frac{dZ}{dZ},$$
$$J_0 = -j + Z \frac{dZ}{dZ},$$
$$J_- = \frac{dZ}{dZ}.$$  

Here $Z = Z(x)$ and $j(j+1)$ is the eigenvalue of the Casimir operator. The Hamiltonian $H$ in QES case belongs to the finite dimensional representations of $Sl(2)$ with spin $j$ which essentially implies only $2j+1$ energy levels.

Following Zamolodchikov’s conjecture Turbiner [3] has posed the question: Are there QES problems which cannot be represented in terms of $Sl(2)$ generators? In this Letter we show that the answer to the question is yes, and in particular we discuss a problem in one-dimensional quantum mechanics and show that it cannot be represented by the quadratic generators of $Sl(2)$ algebra. Possible reasons for this are also pointed out. Finally we show that using SUSY we can appreciably expand the class of QES problems in quantum mechanics. In particular we show that corresponding to a QES problem one can generate families of isospectral as well as strictly isospectral potentials [5].

We shall consider the potential ($\hbar = 2m = 1$)

$$V(x) = \frac{g^2 (1 + e^2)}{[(1 + e^2}/\epsilon^2 + \sinh^2(\mu x/2)]^2$$

$$\times \left( 8 \sinh^4(\mu x/2) + \frac{4(\epsilon^2 - 5)}{\epsilon^2} \sinh^2(\mu x/2) + 2(1 + \epsilon^2)(1 - 2\epsilon^2) \right),$$

where $g, \epsilon$ are real dimensionless constants while $\mu$ has dimensions of mass. This potential arises in the context of the stability analysis around the kink solution for $\phi^6$-type field theory in 1+1 dimensions [5,7] characterized by

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi),$$

where
\( u(\phi) = A(g^2\phi^2 + \epsilon^2)(1 - g^2\phi^2)^2, \)

\[ A = \frac{\mu^2}{8g^2(1 + \epsilon^2)}. \]  

(5)

The kink solution to the field equation has been shown to be [5,7]

\[ \phi(x) = g^{-1}\sigma(x), \]

\[ \sigma(x) = \frac{\sinh(\mu x/2)}{[(\epsilon^2 + 1)/\epsilon^2 + \sinh^2(\mu x/2)]^{1/2}}. \]  

(6)

As is well known, when one performs stability analysis around any kink solution one obtains a Schrödinger-like equation

\[ [-d^2/dx^2 + V(x)]\psi_n(x) = E_n\psi_n(x), \]  

(7)

where \( V(x) = d^2u(\phi)/d\phi^2\Big|_{\phi(0)} \) (is given by eq. (3) in our case) and \( E_n \geq 0 \). It is also known from the kink stability analysis that due to translational invariance, eq. (7) is always an example of unbroken supersymmetry. In particular the ground state energy \( E=0 \) while the ground state eigenfunction \( \psi_0(x) \) is given by

\[ \psi_0(x) = \sigma_0(x)/dx, \]  

(8a)

which in our case takes the form

\[ \psi_0(x) = N[1 - \sigma^2(x)\{\sigma^2(x) + \epsilon^2\}^{1/2}, \]  

(8b)

with \( \sigma(x) \) given by eq. (6) and \( N \) is a normalization constant. What about the other bound state solutions of eq. (7)? While they do exist for \( E_n < \mu^2 \) [6] no analytic results are available except at \( \epsilon^2 = 1/2 \) when one also knows the second excited state energy eigenvalue and eigenfunction,

\[ E_2 = \frac{1}{4}\mu^2, \]

\[ \psi_2(x) \propto [1 - \sigma_2^2(x)]^{1/2}[\sigma_2(x) - \frac{i}{2}], \]  

(9)

where \( \sigma_2(x) \) is again as given by eq. (6). Thus at \( \epsilon^2 = 1/2 \) one knows both the ground state and second excited state energies and eigenfunctions as given by eqs. (8) and (9) and hence potential (3) is partially algebraically solvable. Hence if the conjecture of Turbiner [3] is correct then for \( \epsilon^2 = 1/2 \) we should be able to express the Hamiltonian in terms of the quadratic generators of \( \text{Sl}(2) \) algebra.

Following refs. [3,4], since the Schrödinger-like equation (7) is an example of unbroken supersymmetry, we write \( \psi_n(x) \) in the form

\[ \psi_n(x) = \psi_0(x)\chi_n(x), \]  

(10)

so that \( \chi_n(x) \) carries the information about the nodes of the wavefunction. On using eqs. (5) to (10) it follows after a lengthy but straightforward algebra that in terms of the variables \( Z = g^2\phi^2(x) \) the Schrödinger equation (7) takes the form (\( \epsilon^2 = 1/2 \))

(11)

This is of the form

\[ [-P_a(Z) d^4/dZ^4 + P_3(Z) d/dZ + P_2(Z) - E]\chi_n = 0 \]  

(12)

of Turbiner [3] (see his eqs. (11) and (12)) where in his notation

\[ P_4 = a_++a_0 Z^4+(a_++a_0)Z^2 \]

\[ +a_0 Z+a_-- \]  

(13a)

\[ P_3 = 2(2j-1)a_+ Z^3 + [(3j-1)a_+b_+] Z^2 \]

\[ +[2j(a_++a_0)+a_0+b_0]Z+j a_--b_-- \]  

(13b)

\[ P_2 = 2j(2j-1)a_+ Z^2+2jZ(ja_++b_+) \]

\[ +a_0 j^2 + b_0 j \]  

(13c)

On comparing eqs. (11) and (13) we find that the two equations are inconsistent with each other. For example, whereas the \( Z^3 \) coefficient in \( P_3 \) tells us that \( j = \frac{1}{4} \) (which is impossible) the \( Z^2 \) coefficient in \( P_2 \) gives \( j = 0 \) or \( j = 1/2 \). Thus we conclude that the potential (3) even though partially algebraically solvable (for \( \epsilon^2 = 1/2 \)) cannot be expressed in terms of the quadratic generators of \( \text{Sl}(2) \).

What could be the possible reason for this? As pointed out by Christ and Lee [6] the Schrödinger-like equation in this case can be converted into Heun’s equation which has four regular singularities. On the other hand if one goes through the known QES problems [3,4] then one finds that they all have three regular singularities and are of hypergeometric type [8]. This suggests that QES problems with more than three regular singular points may not be represented

201
in terms of $Sl(2)$ generators. This is the main result of the work. Clearly it would be interesting to find the hidden dynamical symmetry in these cases.

Before ending we would like to note that given a QES problem one can generate a whole class of isospectral QES potentials by using SUSY. In particular if $V(x)$ is a given QES potential with normalized ground state wave function $\psi_0(x)$ then the one-parameter family of strictly isospectral (same bound state energies and same $S$-matrix) potentials $\tilde{V}(x)$ are given by [5]

$$\tilde{V}(x) = V(x) - 2 \frac{d^2}{dx^2} \ln[I(x) + \lambda],$$

where

$$I(x) = \int_{-\infty}^{x} \psi \delta(y) dy$$

and $\lambda > 0$ or $\lambda < -1$. In many cases if a QES problem with $N$ states is known then one could in fact construct an $N$-parameter family of strictly isospectral potentials and a countable infinity of isospectral potentials [5]. Shifman [9] also has investigated this aspect where he expands the class of QES problems belonging to an $Sl(2)$ algebra while here we will be looking at the QES problems without $Sl(2)$ symmetry. The isospectral aspect of the potentials has been extensively studied over the years [5]. We will follow the conventional method here. As an illustration we shall now construct a one-parameter family of strictly isospectral potentials corresponding to the potential (3). Using eq. (6b) (with $\epsilon = \frac{1}{2}$) it is easy to calculate $I(x)$. We find

$$I(x) = \int_{-\infty}^{x} \left[1 - \sigma^2(y)\right]^{1/2} \left[\sigma^2(y) + \frac{1}{2}\right] dy$$

$$= \left\{i\sigma(\sigma + \frac{1}{2})^{1/2}(\frac{1}{2} - \sigma^2)\right\}$$

$$+ i\ln\left[\sigma + (\sigma^2 + \frac{1}{2})^{1/2}\right] + c_1/c_2,$$

where

$$c_1 = i\sqrt{2} - \frac{\sqrt{2}}{2} \ln(\sqrt{2} - 1),$$

$$c_2 = i\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \ln(\sqrt{2} + \sqrt{2}).$$

Using this $I(x)$, eq. (14) now gives us the desired one-parameter family of strictly isospectral potentials corresponding to $V(x)$ as given by eq. (3).

References


W.Y. Keung, U.P. Sukhatme, Q. Wong and T.D. Imbo, University of Illinois at Chicago preprint, UICHEP/89-2;


PECULIAR CHARGED VORTICES IN HIGGS MODELS WITH PURE CHERN–SIMONS TERM

Dileep P. JATKAR and Avinash KHARE
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India

Received 11 September 1989; revised manuscript received 28 November 1989

We show that abelian as well as nonabelian Higgs models in (2 + 1) dimensions with the pure Chern–Simons term possess peculiar charged vortex solutions of finite energy. For all of them the magnetic field vanishes not only at infinity but also at the origin. Such objects can also be shown to exist in an abelian Higgs model without the Chern–Simons term but with non-minimal coupling.

There has been considerable interest in topologically massive gauge theories [1] which give rise to massive gauge fields without breaking the gauge symmetry. More recently, deep connections have been established [2] between the pure Chern–Simons (C–S) action in (2 + 1) dimensions and rational conformal field theories in (1 + 1) dimensions. It has also led to an intrinsically three-dimensional definition of the Jones polynomials [2].

Over the years, a detailed study of abelian [3] as well as nonabelian [4] Higgs models with the C–S term has been made and charged vortex solutions of finite energy and angular momentum have been obtained. Recently, Fröhlich and Marchetti [5] have shown the existence of quantum charged vortices. It has been argued that these charged vortices play a crucial role in the fractional quantum Hall effect [6], high-Tc superconductors [7] and also in the A phase of liquid 3He [8]. In most of these applications what is relevant is the long wavelength limit of the above theories i.e. the C–S term. In this limit the gauge field kinetic energy term becomes irrelevant and can then be dropped from the action. The question we address in this paper is the effect of dropping the gauge field kinetic energy term on the charged vortex solutions. In particular, we discuss abelian and nonabelian Higgs models with pure C–S action and show that in both cases one now has peculiar charged vortices of finite energy and angular momentum. The peculiarity is there in the sense that unlike the usual vortices, in this case the magnetic field vanishes not only at large distances but also at the origin. Finally, we also consider the possibility of generating such peculiar vortices even without the presence of the C–S term.

The langrangian for an abelian Higgs model with C–S term is given by [3]
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial \phi) (\partial^2 + c_4^2) \phi + \frac{1}{2} \mu e^{a\phi} F_{\mu\nu} A^a, \]
where \( \mu \) is the C–S mass of the gauge field, while \( \beta \) is an arbitrary parameter which has been introduced in the gauge field kinetic energy so as to see the behaviour of the vortex solutions in the limit of vanishing gauge field kinetic energy (\( \beta \to 0 \)). The analysis of ref. [3] still holds for any finite value of \( \beta \) while the analysis for \( \beta = 0 \) needs some special care. We use the same ansatz as in ref. [3] i.e. \( \rho \geq 0, 0 \leq \theta \leq 2\pi \)
\[ A(\rho) = -\frac{\epsilon}{\rho} A(\rho) / \rho, \quad A_0(\rho) = A_0(\rho), \]
\[ \phi(\rho) = \exp(i\rho) f(\rho), \]
The boundary conditions for the finite energy vortex solution are
\[ \lim_{\rho \to \infty} A(\rho) = -n/e, \quad A_0(\rho) \to 0, \]
\[ f(\rho) \to \sqrt{c_2}/2c_4, \]
\[ \lim_{\rho \to 0} A(\rho) \to 0, \quad A_0(\rho) \to c, \quad f(\rho) \to 0, \]
where \( c \) is any nonzero constant \([9,10]\). Using ansatz (2) and eq. (1) one obtains the following field equations:

\[
\beta \left( \frac{d^2 A}{dp^2} - \frac{1}{\rho} \frac{dA}{dp} \right) - e(n + eA)f^2 = \mu \rho \frac{dA_0}{d\rho},
\]

(3a)

\[
\beta \left( \frac{d^2 A_0}{dp^2} + \frac{1}{\rho} \frac{dA_0}{dp} \right) - e^2 A_0 f^2 = \frac{\mu A}{\rho d\rho},
\]

(3b)

\[
\frac{d^2 f}{dp^2} + \frac{1}{\rho \rho} + e^2 A_0 f - \frac{(n + eA)^2 f}{\rho^2} + 2c_2 f - 4c_4 f^3 = 0.
\]

(3c)

We have not been able to solve these coupled nonlinear equations. However, for large \( \rho \) it can be done. In particular, on substituting the ansatz

\[ A(\rho) = -n/e + a \exp(-m_\rho \rho), \]

\[ A_0(\rho) = e^2 \phi_0^2 a_2 \exp(-m_\rho \rho), \]

\[ f(\rho) = \sqrt{c_2/2a_4 + a_2} \exp(-m_\rho \rho) \]

in eq. (3) we find that

\[ m_\pm^2 = \frac{\sqrt{\mu^2 + 4 \beta e^2 \phi_0^2} \pm \mu}{2\beta}, \]

\[ m^- = \sqrt{4c_2}, \]

(4)

where

\[ \phi_0 = c_2/2c_4. \]

(5)

Since \( m^- \) has two values naively it would appear that there are two topologically nontrivial solutions in each sector but this is not so. As has been shown by Inozemtsev \([9]\) and Lozano et al. \([10]\) \( \Phi' \) the solution with higher mass \( (m^+ \rho) \) does not exist for all \( \rho \). The acceptable vector meson mass is \( m^+_\rho \) which depends on three parameters \( \mu, \beta \) and \( \phi_0 \). From eqs. (3) and (4) we observe that as \( \beta \) (or \( \mu \)) decreases, \( m^- \) increases while \( m^+ \) is unchanged. Thus adding gauge field kinetic energy or C-S mass tends to favour the formation of type-II vortices. What happens if one of these parameters \( (\beta, \mu, \phi_0) \) is set to zero?

(i) As is well known, if \( \mu = 0 \) (i.e. no C–S term) then as shown by Nielsen and Olesen one has neutral vortices \([11]\) with \( m_n = e\phi_0/\sqrt{\beta} \).

(ii) If \( \phi_0 = 0 \) (i.e. no spontaneous symmetry breaking) then as has been shown in ref. \([12]\) one can still obtain finite energy, charged vortex solutions. However, these are now nontopological in nature and can be termed as Q-balls in local gauge field theories with a gauge field mass \( m_n = \mu/\beta \) \([12]\). Note that this mass can be obtained from eq. (4) only in the limit \( \phi_0 \to 0 \) from \( m^+_\rho \) and not \( m^-_\rho \).

(iii) If we set \( \beta = 0 \) (i.e. no gauge field kinetic energy) then we find from eq. (4) that \( m^- = e^2 \phi_0^2 / \mu \).

It is also clear from here that at least two out of the three parameters \( \phi, \mu \) and \( \beta \) must be nonvanishing in order to have nonzero, finite \( m^-_\rho \). Thus it is obvious that no Q-ball type solution \( (\phi_0 = 0) \) would exist in the absence of the gauge field kinetic energy \( (\beta = 0) \).

Let us now concentrate on the solutions of the lagrangian (1) with \( \beta = 0 \). Firstly, notice that as with usual vortices again we have quantization of flux in units of \( 2\pi/e \), i.e.

\[ \Phi = \int d^2 x \, e^2 \partial_\rho A = - \int d^2 x \frac{1}{\rho} \frac{dA}{d\rho} = \frac{2\pi n}{e}. \]

(7)

Hence integrating eq. (3) we find that the electric charge of these objects is also nonzero and quantized in units of \( 2\pi e/\mu \)

\[ Q = \int e^2 A_0 f^2 d^2 \rho = - \mu \int d^2 x \frac{1}{\rho} \frac{dA}{d\rho} = \mu \Phi = \frac{2\pi \mu n}{e}. \]

(8)

Following (3) it is then easy to see that these objects are anyons i.e. they have fractional angular momentum

\[ J = \int d^2 x \, e^2 \partial_\rho T_{00} = \frac{nQ}{2e} = \frac{\pi \mu n^2}{e^2}. \]

(9)

However, there is a very important difference between the \( \beta = 0 \) objects and the usual charged vortices which has not been noticed before. This follows from the solutions of eq. (3) for small \( \rho \) and \( \beta = 0 \). We find that as \( \rho \to 0 \)

\[ \Phi = \int d^2 x \, e^2 \partial_\rho A = - \int d^2 x \frac{1}{\rho} \frac{dA}{d\rho} = \frac{2\pi n}{e}. \]

(ii) If \( \phi_0 = 0 \) (i.e. no spontaneous symmetry breaking) then as has been shown in ref. \([12]\) one can still obtain finite energy, charged vortex solutions. However, these are now nontopological in nature and can be termed as Q-balls in local gauge field theories with a gauge field mass \( m_n = \mu/\beta \) \([12]\). Note that this mass can be obtained from eq. (4) only in the limit \( \phi_0 \to 0 \) from \( m^+_\rho \) and not \( m^-_\rho \).

(iii) If we set \( \beta = 0 \) (i.e. no gauge field kinetic energy) then we find from eq. (4) that \( m^- = e^2 \phi_0^2 / \mu \).

It is also clear from here that at least two out of the three parameters \( \phi, \mu \) and \( \beta \) must be nonvanishing in order to have nonzero, finite \( m^-_\rho \). Thus it is obvious that no Q-ball type solution \( (\phi_0 = 0) \) would exist in the absence of the gauge field kinetic energy \( (\beta = 0) \).

Let us now concentrate on the solutions of the lagrangian (1) with \( \beta = 0 \). Firstly, notice that as with usual vortices again we have quantization of flux in units of \( 2\pi/e \), i.e.

\[ \Phi = \int d^2 x \, e^2 \partial_\rho A = - \int d^2 x \frac{1}{\rho} \frac{dA}{d\rho} = \frac{2\pi n}{e}. \]

(7)

Hence integrating eq. (3) we find that the electric charge of these objects is also nonzero and quantized in units of \( 2\pi e/\mu \)

\[ Q = \int e^2 A_0 f^2 d^2 \rho = - \mu \int d^2 x \frac{1}{\rho} \frac{dA}{d\rho} = \mu \Phi = \frac{2\pi \mu n}{e}. \]

(8)

Following (3) it is then easy to see that these objects are anyons i.e. they have fractional angular momentum

\[ J = \int d^2 x \, e^2 \partial_\rho T_{00} = \frac{nQ}{2e} = \frac{\pi \mu n^2}{e^2}. \]

(9)

However, there is a very important difference between the \( \beta = 0 \) objects and the usual charged vortices which has not been noticed before. This follows from the solutions of eq. (3) for small \( \rho \) and \( \beta = 0 \). We find that as \( \rho \to 0 \)
\[ A(\rho) = -\frac{c d^2 e^2}{\mu(2|n| + 2)} \rho^{2|n| + 2} + o(\rho^{2|n| + 4}), \]  
\[ A_0(\rho) = c - \frac{ned^2}{2|n| \mu} \rho^{2|n| + 2} + o(\rho^{2|n| + 4}), \]  
\[ f(\rho) = \frac{d \rho^{2|n| + 2}}{\rho^{2|n| + 2}}, \]

so that the magnetic field (and also the electric field) vanishes at \( \rho = 0 \) like \( \rho^{2|n|} \) for an \( n \)-vortex. This to be contrasted with the case of nonzero \( \beta \) where the magnetic field is nonzero and finite at \( \rho = 0 \).

Thus for \( \beta = 0 \) we have solutions where the magnetic field starts from zero at \( \rho = 0 \) and then increases as \( \rho \) increases and finally falls off exponentially as \( \rho \to \infty \).

What then could be the physical interpretation of such objects and could there be systems in which they exist? We feel that these objects can be interpreted as vortices without any singularity at the origin. Such type of non-singular, topologically stable solutions have presumably been observed in cholesteric liquid crystals [14] where the vortices (\( \chi \) lines) are formed by helical ordering of the directions \( d \) of the molecules. Our solutions may also play an important role in the context of high-\( T_c \) superconductivity since Shraiman and Siggia [15] have argued that for a low density of vacancies (\( \chi \ll 1 \)), the commensurate Néel order in a spin-1, two-dimensional antiferromagnet is unstable for any \( x \) leading to a spiral state.

The whole discussion can be trivially extended to nonabelian gauge theories with the C–S term. One can run through the argument of ref. [4] and show that for the SU(2) case one will have \( Z_2 \) objects with flux \( \Phi = 2\pi n/e \), charge \( Q = 2\pi n/e = \pm ne \) (\( n = \pm 1, \pm 2, \ldots \)), and \( J = \pm n \). The behaviour of the magnetic field (\( B(\rho) \)) here is identical to its abelian \( n = 1 \) counterpart (i.e. \( B(\rho) \to \pm \rho^2 \)).

Thus we have shown that both abelian and nonabelian Higgs models in (2 + 1) dimensions with the C–S term lead to the peculiar vortices. At this stage, it is interesting to enquire if such objects can be generated, even without the C–S term. We shall now show that this is possible if one considers an abelian Higgs model with two Higgs fields, one of which is non-minimally coupled. Consider the lagrangian

\[ L' = -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} (\partial_\mu - ie A_\mu) \phi^* (\partial^\mu + ie A^\mu) \phi 
+ \frac{1}{2} (\partial_\mu \chi)^* \partial^\mu \chi + V(\phi, \chi), \]

where \( \phi \) is minimally coupled, while there is a non-minimal coupling for \( \chi \) given by

\[ \partial_\mu \chi = \partial_\mu \phi + ie A_\mu \phi + j \epsilon_{\mu \nu \rho} F^{\nu \rho} \frac{\chi}{|\chi|}. \]

Let us choose

\[ V(\phi, \chi) = \frac{1}{2} m^2 |\chi|^2 - \frac{1}{4} \lambda_1 |\chi|^4 + c_4(|\phi|^2 - c_2/2 c_4)^2. \]

In the special case of \( \phi = 0 \) and \( c_2 = 0 \) it has been shown [16] that this model generates the C–S term spontaneously and has neutral vortex solutions. We have therefore included the second Higgs field \( \phi \) so as to get the desired peculiar charged vertex solutions. We now choose the ansatz

\[ \chi(\rho) = \sqrt{m^2/\lambda_1}, \]

in eqs. (11)–(13). We find that the \( \chi \)-field decouples from the theory and we get the model as given by eq. (1) with \( \beta = 0, \mu = 4\pi \sqrt{m^2/\lambda_1} \) but now an extra gauge field mass term \( (e^2 m^2/2 \lambda_1) A_\mu A^\mu \) appears. One can now run through the earlier arguments and obtain the peculiar vortex solutions with vanishing magnetic field at \( \rho = 0 \).

It is tempting to think that an exact solution can be found of the field equations (3a)–(3c) in the \( \beta = 0 \) limit since that case eqs. (3a) and (3b) reduce to the simple form

\[ -e^2 A_0 f^2 = \frac{\mu}{\rho} \frac{dA_0}{d\rho}, \]

From here it follows that

\[ a(\rho) a'(\rho) = \rho^2 A_0(\rho) A_0(\rho), \]

where

\[ a(\rho) = A(\rho) + n/e. \]
We thank S.N. Behera, S.M. Bhattacharjee and S. Rao for helpful discussions and D.M. Gaitonde for carefully reading the manuscript.

References

[7] R.B. Laughlin, Why I think high $T_c$ superconductivity and the fractional quantum Hall effect are related, Stanford preprint;
SU(2) WZW correlators and the KdV equation

Dileep P. Jatkar
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751005, India

Received 27 May 1990; revised manuscript received 29 September 1990

We show the relation between the second Poisson structure of higher spin generalizations of the KdV field and the differential equations of two-point correlators of SU(2) WZW theory on the torus.

Classical nonlinear systems possessing soliton solutions have been studied in great detail \[1,2\]. These systems are integrable, i.e. they have an infinite number of conserved quantities associated with them and consequently a hierarchy of equations. The second hamiltonian structure of these systems is related to conformal field theories. More specifically, the second Poisson structure corresponding to the generalised KdV is naturally related to non-linearly extended Virasoro algebras (W-algebras).

Rational conformal field theories (RCFT) \[3,4\] are characterised by rational values of the central charge \(c\) and conformal dimensions \(h\) \[5,6\]. They have a finite number of primary fields under some chiral algebra \[7,8\] which always contains the Virasoro algebra as a subalgebra. Correlation functions of these primary fields can be obtained by using null vector relations. Recently, a more general method was proposed \[6\] for deriving correlation functions in any RCFT based on the condition of modular invariance and analyticity on the torus.

In this letter, we will use the method of ref. \[6\] to derive differential equations for the two-point correlators of isospin \(k/2\) fields which are the highest isospin fields in the SU(2) WZW model at level \(k\). We then show that the same equations can be derived from having Poisson operators act on densities of degree \(-k/2\). We explicitly show this identification up to \(k=4\). It can also be checked for all higher values of \(k\).

In the SU(2) WZW model at any level \(k\) there are \(k+1\) primary fields \(\phi_j\) where \(j\) denotes isospin \((j=0, 1/2, ..., k/2)\), with dimension \[7,9\]

\[
h_j = \frac{j(j+1)}{k+2},
\]

and the value of the central charge \(c\) is given by

\[
c = \frac{3k}{k+2}.
\]

Here we will consider only two-point correlators of the highest isospin, i.e. isospin \(k/2\) at level \(k\), on the torus. From eq. (1) it is easy to see that \(h_{k/2}=k/4\). The fusion rule for the field \(\phi_{k/2}\) decomposes only over the identity, i.e.

\[
[\phi_{k/2}] \otimes [\phi_{k/2}] \sim [I].
\]

Thus \(k+1\) current blocks will flow on the torus through the identity and we will get a \((k+1)\)th order differential equation for the correlator. Using the method of ref. \[6\] we write down the first few differential equations as:

(i) \(k=1\), Isospin = 1/2, \(h_{1/2} = 1/4\),
\[
\partial^2 f - \frac{1}{2} \partial \phi f = 0,
\]

(ii) \(k=2\), Isospin = 1, \(h_1 = 1/2\),
\[
\partial^2 f - 3 \partial \phi \partial f - \frac{2}{3} \partial \phi f = 0,
\]

(iii) \(k=3\), Isospin = 3/2, \(h_{3/2} = 3/4\),
\[
\partial^2 f - \frac{1}{2} \partial \phi \partial^2 f - \frac{1}{2} \partial \phi \partial f - \frac{2}{3} \partial \phi \partial f - \frac{1}{2} (\partial^2 \phi + 18 \alpha_4) f = 0,
\]

(iv) \(k=4\), Isospin = 2, \(h_2 = 1\),
\[
\partial^2 f - 15 \partial \phi \partial^2 f - \frac{45}{2} \partial \phi \partial^2 f - \frac{27}{4} (\partial^3 \phi + 24 \alpha_4) \partial f = 0,
\]

0370-2693/91/$03.50 © 1991 - Elsevier Science Publishers B.V. (North-Holland)
where
\[ a_4 = \frac{1}{2} \partial^2 \rho - \frac{3}{2} \rho^2. \]

Now, let us consider the KdV equation
\[ u_t = u_{xxx} + 3u u_x, \tag{8} \]
which is probably the simplest non-linear integrable equation. Its integrability is manifest in the infinite number of conserved quantities which are in involution. This system is also known to have bi-hamiltonian structure \[2,10\], i.e. we can define two independent Poisson brackets and corresponding hamiltonians to get the equation of motion (8). But here we will only consider the second hamiltonian structure and the second Poisson bracket which is defined as
\[ \{u(x), u(y)\}_2 = \left[ \partial_x^2 + 2u(x) \partial_x + u_x \right] \delta(x-y). \tag{9} \]

We will call the differential operator on the RHS of eq. (9) the Poisson operator and for the hamiltonian
\[ H_2[u] = \frac{1}{2} \int_{-\infty}^{\infty} u^2 \, dx \tag{10} \]
the equation of motion is given by
\[ \{u(x), H_2[u]\} = \left[ \partial_x^2 + 2u(x) \partial_x + u_x \right] \frac{\delta H_2}{\delta u(x)}. \tag{11} \]

The second Poisson bracket is relevant from the point of view of conformal field theory. Recently, Gervais \[11\] has shown that the second Poisson bracket, i.e. eq. (9), is closely related to the classical Virasoro algebra. Similarly, many investigators \[12,13\] have shown that the second Poisson structure corresponding to the generalized KdV equation is naturally related to the non-linearly extended Virasoro algebras (W-algebras).

It is well known that any RCFT has a finite number of primary fields under some chiral algebra which necessarily contains the Virasoro algebra as a subalgebra. Since the KdV field \( u(x) \) naturally provides a representation of the Virasoro algebra, we will consider it as an essential field which will be accompanied by auxiliary fields \( v_n(x) \). In the terminology of conformal field theory, we consider the fields \( v_n \) to be conformal primaries with conformal dimension \( n \). The corresponding Poisson operator is
\[ \mathcal{P} = \frac{\partial^2}{\partial x^2} + 2u(x) \partial_x + u_x + n v_n \partial_x + (n-1) (v_n)_x \]
where \( \mathcal{P} \) is the most general \((2n-1)\)th order differential operator whose form is restricted by the constraint that \( \mathcal{P} \) is an odd operator when \( n \) is an integer, while it is an even operator when \( n \) is a half-odd integer. There are further unknown coefficients in \( \mathcal{P} \), which can be fixed by demanding that the Jacobi identity be satisfied \[12,14\]. Using this method we get the Poisson operator \( \mathcal{P} \) for different values of \( n \) as
\[ \{v_{3/2}(x), v_3(y)\} = \left[ \partial_x^2 + \frac{1}{2} u(x) \right] \delta(x-y), \tag{13} \]
\[ \{u(x), u(y)\} = \left[ \partial_x^2 + 2u(x) \partial_x + u_x \right] \delta(x-y), \tag{14} \]
\[ \{v_{3/2}(x), v_{3/2}(y)\} = \left[ \partial_x^4 + 5u(x) \partial_x^2 + 5u_x \partial_x \right] \delta(x-y) \]
\[ + \frac{2}{3} u^2(x) \partial_x \delta(x-y), \tag{15} \]
\[ \{v_3(x), v_3(y)\} = \left[ \partial_x^2 + 2u(x) \partial_x + 9u_x \partial_x \right] \delta(x-y) \]
\[ + 16u^2(x) \partial_x \delta(x-y). \tag{16} \]

\( v_{3/2} \) in eq. (15) does not define the complete Poisson structure in the sense that not all Jacobi identities are satisfied \[12\]. We choose the Weierstrass \( \rho \) function as the basis of meromorphic functions on torus. Since \( \rho \) function has double poles in the complex plane, it behaves like a density of weight two. We can, therefore, express \( u(x) \) as a linear function of \( \rho \) with an appropriate choice of proportionality constant, i.e.
\[ u(x) = -\frac{1}{2} \rho . \tag{17} \]

With this choice, the Poisson operators of eqs. (13)–(16) precisely match with the differential equations of the SU(2) WZW model precisely match with the differential equations of the SU(2) WZW model respectively. Even though we have explicitly shown only up to \( k=4 \) that the equations of SU(2) WZW model are identical to the equations from the Poisson operators, this is in fact true for all higher values of \( k \).

Now let us try to understand why these two theo-
It is well known that for all KdV-like equations there exists a Lax operator \( L \) of the form

\[
L = \partial_x^n + u_1 \partial_x^{n-2} + \ldots + u_{n-1},
\]

under the transformation \( x \to \tilde{x} \). The precise transformation law of \( u_i \) is determined by allowing the Lax operator to act on densities of degree \( (1 - n)/2 \). Using this method for covariantising the \( n \)th order differential operator \( \partial_x^n \) we get

\[
\partial_x^2 - 2Ax - A_x, \tag{19}
\]

\[
\partial_x^2 - 10A \partial_x^2 - 15A_x \partial_x^2 - 9A_{xx} \partial_x^2 + 16A^2 \partial_x^2 + 16A_x - 2A_{xxx}, \tag{20}
\]

where

\[
A = \frac{e_x}{e} - \frac{1}{2} \left( \frac{e_x^2}{e} \right), \quad e = e(x) \, dx^{-1}
\]

and

\[
\partial_x^2 - \frac{1}{2} B, \tag{21}
\]

\[
\partial_x^2 - 5B \partial_x^2 - 5B_x \partial_x^2 + \frac{3}{2}B^2 - \frac{3}{2}B_{xx}, \tag{22}
\]

where,

\[
B = 2 \frac{b_{xx}}{b}, \quad b = b(x) \, dx^{-1/2}.
\]

Under the substitution \( A = -u \) and \( B = -u \) \([16]\), we get the Poisson operator \( \mathcal{Q} \) (refer to eqs. (13)–(16)). Both \( A \) and \( B \) obey the KdV equation

\[
A_{xxx} - 8AA_x = 0, \quad B_{xxx} - 8BB_x = 0.
\]

Thus \( A \) and \( B \) both are densities of weight two. Since we have chosen the \( \varphi \) function as the basis of meromorphic functions on a torus, \( A \) and \( B \) are linear functions of \( \varphi \) and for the choice \( A = B = \frac{1}{2} \varphi \) we find that the \( \varphi \) function obeys its equation

\[
\partial_x^2 \varphi = 12 \varphi \partial_x \varphi = 0.
\]

Hence, it is appropriate to choose the connection to be \( \frac{1}{2} \varphi \). Thus we have shown that this operator can be "covariantised" and that the connection can be chosen to be \( -\frac{1}{2} \varphi \). In that case, we can again see that the "covariant" differential operator is precisely the Poisson operator.

We have established an intriguing identification between some differential operators of SU(2) WZW theory and the Poisson operators of generalised KdV.

However this identification is by no means complete, because it does not work for \( j < k/2 \). So it is clear that it is not just the naive connection between the full SU(2) symmetry of the WZW model and the symmetry of the generalised KdV system. A more subtle connection has to exist. Along these lines we have found a connection between an exactly solvable quantum mechanical problem with reflectionless potential, treated as a one-dimensional analogue of rational conformal field theories \([17]\), and SU(2) symmetry. It is well known that reflectionless potentials correspond to soliton solutions of the KdV equation. Here also we have found that the quantum mechanical problem does not completely map onto the SU(2) problem in the spirit of the WZW model. Details are being worked out and will be published elsewhere \([18]\). Another interesting question is whether further similar connections can be established between SU(\( N \)) WZW theories for \( N \geq 3 \) and other integrable models. We hope to return with an answer to some of these questions in the future.

I thank Dr. A. Khare and P. Durganandini for many useful discussions. I also thank Dr. S. Rao and Dr. A. Sen for useful discussions and encouragement.

Note added. After completing this work, we received a preprint by Park \([19]\) in which a connection has been found between KdV-like equations and gauged WZW models. This is different from what we have done in this paper.

References

We identify the spin of the anyons with the holomorphic dimension of the primary fields of a Gaussian conformal field theory. The angular momentum addition rules for anyons go over to the fusion rules for the primary fields and the $r \leftrightarrow 1/2r$ duality of the Gaussian CFT is reproduced by a charge-flux duality of the anyons. For a U(1) Chern-Simons theory with topological mass parameter $k = 2n$, N-anyon states on the torus have $2n$ components, which correspond to the $2n$ conformal blocks of an N-point function in the Gaussian conformal field theory.

Chern-Simons (CS) theories, of late, have evoked wide interest in physics. CS gauge theories in three dimensions are closely related to rational conformal field theories (RCFT) in two dimensions. This relation is expected to shed some light on the structure and classification of RCFT. Recently the abelian CS theory was studied in relation to linking numbers. Besides CS theories have also led to an intrinsic 3-dimensional definition of Jones polynomials.

Even earlier, it was known that exotic (fractional) spin and statistics in 2 + 1 dimensions could be implemented by CS gauge theories. Anyons, particles with fractional spin and statistics have come into the limelight in the last couple of years, mainly due to their applicability in theories of fractional quantum Hall effect and high-Tc superconductivity. Both of these are purely planar phenomena which are expected to be manifestations of fractional spin excitations in the system.

Witten has recently shown that expectation values of Wilson line operators in CS gauge theories give a natural representation of holomorphic blocks of some RCFT. An explicit relation between U(1) CS theory and conformal blocks in $c = 1$ Gaussian CFT was also established recently. Since a U(1) CS theory coupled to sources has anyon excitations, it is natural to ask how anyons are manifested in the CFT, in the limit where the dynamical degrees of freedom of the anyons are frozen. In this letter, we identify static anyons of a CS parameter $k = 2n$, $n = \text{integer}$, U(1) CS theory with the holomorphic primary fields of a $c = 1$ CFT. We show that the rules for forming composite anyons are identical to the fusion rules of the CFT. We derive a charge-flux, $k/2 \leftrightarrow 2/k$ duality for anyons, which is analogous to the $r \leftrightarrow 1/2r$ duality for the CFT. We introduce a single anyon in a U(1) CS theory via a source, with a unit charge. By forming all possible composite objects with different spins from this anyon until we reach a state where the flux is an integer multiple of $2\pi$, we construct the complete set of anyons that occur in the theory for each value of the topological mass parameter. We then identify the
appropriate $c = 1$ CFT, whose complete set of primaries are analogous to those anyons. On the torus, the space of $2n$ states for each $N$ anyon configuration then goes over to the space of $2n$ conformal blocks of the $N$-point function.

A U(1) CS theory coupled to a static source is described by the Lagrangian

$$L = \frac{k}{4\pi} \epsilon_{\mu\nu\rho} a^\mu \partial^\nu a^\rho + j_\rho a^\rho$$

(1)

with $j_\rho = \delta(x)$, since we normalize the charge to unity. The modified Gauss' law $kb = 2\pi j_\rho$ implies that the charge is also accompanied by a flux of $2\pi/k$. The angular momentum and statistics factor of this object is given by

$$J = \frac{\theta}{2\pi} = \frac{1}{2k} Q^2 = \frac{k\Phi^2}{8\pi^2},$$

(2)

where $Q$ is the charge and $\Phi$ is the flux. It is clear that Eq. (2) is invariant under the transformation

$$\frac{Q}{2} \leftrightarrow \frac{\Phi}{2\pi} \text{ and } \frac{k}{2} \leftrightarrow \frac{2}{k}.$$  

(3)

We shall refer to this as the charge-flux duality. The spin and statistics factor is invariant under this duality transformation.

Composite anyons may be formed from two elementary anyons, with the spin of the composite given by

$$J_{12} = J_1 + J_2 + \frac{1}{2} (Q_1 \Phi_2 + Q_2 \Phi_1) = \frac{1}{2k} [Q_1^2 + Q_2^2 + 2Q_1Q_2].$$

(4)

The extra term, besides naive angular momentum addition, comes from the charge of each anyon seeing the flux of the other anyon. The charge and flux of the composite are merely additive and are given by

$$Q_{12} = Q_1 + Q_2 \quad \text{and} \quad \Phi_{12} = \Phi_1 + \Phi_2.$$  

(5)

When the total flux of an $N$-anyon composite is an integer multiple of $2\pi$, since all the charges in the theory are integer, the phase of the composite which is $\exp \left( iQ_{\text{coul}} \times \Phi_{\text{coul}} / 4\pi \right)$ will be either fermionic or bosonic. Furthermore, the flux is unobservable and the composite is not an anyon and this happens when

$$N\Phi = 2\pi \Rightarrow N = k.$$  

(6)

Hence, we require $k$ to be an integer and the distinct charges in the theory are measured modulo $k$. Note that it is possible for certain anyons to be bosonic or fermionic if their charges are $q$ and their fluxes are $2\pi q$, where $q$ is any integer. But these fermions and bosons are distinguishable from genuine fermions and bosons with no fluxes attached, by their interaction with other anyons. Hence, for a CS theory, with topological mass parameter $k$, we have $k-1$ distinct anyons and hence $k$ fields including the final boson.

A single free boson $\phi$ compactified on a circle of radius $r$ with $2r^2 = n, n = \text{integer},$
Anyons & Gaussian Conformal Field Theories

has holomorphic primary fields \( V_\alpha(z) = \exp(ia\phi(z)/\sqrt{2n}) \) where \( \alpha \) is the charge and the conformal dimensions \( h_\alpha \) of these fields are given by

\[
    h_\alpha = \begin{cases} 
        \frac{\alpha^2}{4n} & \alpha = 0, \ldots, n \\
        \frac{(2n-\alpha)^2}{4n} & \alpha = n + 1, \ldots, 2n-1. 
    \end{cases}
\]

The fusion rule \( V_\alpha \otimes V_\beta \) translates in terms of conformal dimensions to

\[
    \frac{\alpha^2}{4n} \otimes \frac{\beta^2}{4n} = \frac{(\alpha + \beta)^2}{4n}.
\]

Comparing Eqs. (4) and (8), we see that the spin \( J \) may be directly identified with the conformal dimension \( h \) provided that \( Q \) of the anyon is identified with \( \alpha \) of the primary and the topological mass parameter, \( k \), is identified with the radius of compactification as

\[
    k = 2n = 4r^2.
\]

The identity in the CFT with \( h = 0 \) is identified with the no anyon state. Note that the conformal dimensions for the complex conjugate primaries are the same as for the original field. Hence, the conformal dimensions of all but the identity and the self conjugate field are doubly degenerate. This feature does not seem to appear in the anyon spin case, but in fact it does, because spin \( J \) and spin \( (\text{integer} + \frac{1}{2} J) \) are the same measurable quantity.

The CFT is known to have an electric ↔ magnetic duality defined by

\[
    r \leftrightarrow \frac{1}{2r}.
\]

Under this duality \( n \leftrightarrow 1/n \), but the number of primaries in a CFT with \( 2r^2 = p/q \) is given by \( 2pq = I \) and the conformal dimensions are still given by Eq. (7) with \( n \) replaced by \( I \). Hence, there is no change in the conformal dimensions or the number of primaries when \( n \leftrightarrow 1/n \). From Eq. (9) we see that \( r \leftrightarrow 1/2r \) corresponds to \( k^2 \leftrightarrow 2/k \). Hence, this duality is the analog of the charge-flux duality for anyons.

So far, we have only established the one-to-one correspondence between the local properties of anyons and the local properties of primary fields. This connection is not unexpected, since Witten has shown that Wilson lines in a CS theory correspond to vertex operator insertions in the underlying CFT that exists at every time slice. The Wilson lines correspond to static charges in the 3-dimensional theory and the connection of static charges to anyons is well-known.

Witten also showed that the Hilbert space of a Riemann surface with \( N \) punctures can be identified with space of conformal blocks of the underlying CFT. In the language of anyons, this implies that the Hilbert space of \( N \) anyon wavefunctions corresponds to the space of conformal blocks of \( N \)-point functions of the Gaussian CFT with \( 2r^2 = n \), i.e., the properties of the \( N \)-anyon wavefunction under exchange
of any two anyons is completely determined by the behavior of the conformal blocks of the \( N \)-point function under the exchange of any two points. Hence, just as fermion or boson wavefunctions have to be symmetrized or antisymmetrized under exchange of particles, the symmetry properties of anyon wavefunctions are completely determined by the underlying CFT.

(a) Anyons and conformal primaries on a sphere:

Lee\textsuperscript{12} has observed that for the Dirac string to be unobservable, the number of anyons on a sphere has to be an integral multiple of \( p \), where the spin of the anyon is \( J = q/2p \), with \( p, q \) coprime integers. Hence, to have a finite number of anyons on a sphere, \( J \) has to be rational. Similarly for a conformal primary with conformal dimension \( h = q/2p \), the only non-zero \( N \)-point functions are those that conserve charge, i.e., when \( N \) is an integral multiple of \( p \), since charge is measured modulo \( 2p \). The anyons and the conformal primaries for which we have established a correspondence have \( q - 1 \) and \( p = 2n \), even integer. In this case, as we have said earlier, the space of anyon states is one-component and corresponds to the single conformal block describing the sphere with \( N \) vertex operator insertions.

(b) Anyons and conformal primaries on a torus:

The Dirac string has to be unobservable even on a torus. Hence, the condition that \( J \) has to be an integral multiple of \( p \), where the spin of the anyon is \( J = q/2p \) still holds. However, more constraints develop on the torus.\textsuperscript{12-14} As shown in Eq. (1), anyons are charged particles interacting with a CS gauge field. The equations of motion determine flux in terms of charge, but they determine the gauge potential only up to a constant, i.e.,

\[
a(x) = \bar{a}(x) + \bar{a}.
\]

This constant is irrelevant on a plane or sphere, but on a torus, the variables

\[
\theta_1 = \oint_a \bar{a} \cdot dl, \quad \theta_2 = \oint_b \bar{a} \cdot dl,
\]

where \( a \) and \( b \) form the homology basis on a torus, are physically relevant observables. In fact, for a pure CS theory, the phases \( \exp(i\theta_1) \) and \( \exp(i\theta_2) \) are the only physical gauge degrees of freedom and in fact, are canonically conjugate variables, i.e., \([\theta_1, \theta_2] = 2\pi i/\kappa\).\textsuperscript{4-13}

Under large gauge transformations, \( \theta_1 \to \theta_1 + 2\pi m_1 \), \( m_1 = \text{integer} \). The unitary operators that generate the transformations \((2\pi, 0) \) and \((0, 2\pi) \) in the \( \theta_1, \theta_2 \) space are given by

\[
V_i = \exp(i k e_i \theta_j) \quad i, j = 1, 2.
\]

The large gauge transformations commute only for \( k = \text{integer} \).\textsuperscript{13} Also modular invariance in the \((\theta_1, \theta_2)\) space requires that \( k \) be an even integer.\textsuperscript{4} As we have seen earlier, the connection with CFT was established only in this case.

Recently, it has been shown\textsuperscript{12,14} that the anyon states on a torus are not uniquely
determined by the position of the anyons but that there is a degeneracy which is related to its spin. For spin \( J = 1/2n \), the degeneracy factor is \( 2n \): This can be easily seen as follows. Let \( T_a \) and \( T_b \) be the operators that take an anyon around the \( a \) and \( b \) cycles of the torus respectively, and bring it back to its original position. Consider a sequence of operations \( T_b^{-1} T_a^{-1} T_b T_a \). These paths can be deformed into two linked loops. Hence, the fractional statistics of the anyons implies that

\[
T_b^{-1} T_a^{-1} T_b T_a = \exp(-2i\theta) = \exp(-4\pi i J) = \exp\left(-\frac{2\pi i}{2n}\right).
\]

Hence, any \( N \) anyon state on the torus has to form a representation of this algebra and has \( 2n \) components.

This has also been given\(^{15} \) a more mathematical foundation by saying that these anyon states are \( 2n \)-dimensional representations of the 'unpermuted braid group' \( L_{2n} \), a subgroup of \( B_{2n} \) (\( B_{2n} \) is the \( N \)-string braid group of the torus) where the elements of \( L_{2n} \) take particles around \( a \) and \( b \) cycles of the torus. But all \( 2n \) components transform the same way under the \( \Sigma_{2n} \) subgroup of \( B_{2n} \) whose elements exchange identical particles and hence all \( 2n \) components are 'statistically equivalent'.

The correspondence between the Hilbert space of the \( (J = 1/4n) \) \( N \) anyon states and the space of conformal blocks of the Gaussian CFT with \( 2r^2 = n \) and \( 2n \) conformal blocks still holds because the fusion rules of the Gaussian CFT allows precisely \( 2n \) conformal blocks for any \( N \)-point function on the torus. In fact this correspondence could have been used to predict that anyon states on the torus have to be multi-component. For other non-Gaussian CFT's, the fusion rules are less trivial and the number of conformal blocks on a torus depends on \( N \). The connection with anyons in such theories is more involved.

In conclusion, we have studied the one-to-one correspondence between the kinds of anyons in a CS theory with topological parameter \( k = 2n = \text{even integer} \) and the primaries of a \( c = 1 \) CFT with \( 2r^2 = n \). The angular momentum addition rules of the anyons are identified with the fusion rules of the CFT, and the space of \( N \) anyon wavefunctions is identified with the space of conformal blocks of the \( N \)-point function. While this connection has been implicitly made in Refs. 3–5, in this letter, we explicitly show that the recently discovered multi-component wavefunctions for anyons on a torus are precisely what is expected from the topological field theory-conformal field theory connection. Hence, static properties of the anyons can be obtained by studying the appropriate underlying CFT. But whether this correspondence can be used to study dynamical anyons is still an open question. Another interesting question which is currently being explored is the relationship of \( c \neq 1 \) models with fractional statistics.\(^{16} \)

Acknowledgment

We thank C. S. Aulakh, S. M. Bhattacharjee, and A. Sen for helpful discussions.
References

We consider small perturbations around the self-dual Chern-Simons as well as Maxwell Chern-Simons vortices. For topological case, we show that there exist perturbations for which, at least to the leading order, the vortices remain noninteracting. On the other hand for the non-topological case we show that for most of the perturbations the vortices are unstable against decay to the elementary excitations.
In last few years the charged vortex solutions [1] of the abelian Higgs model with Chern-Simons (C-S) term [2] have received considerable attention in the literature. Some time ago we showed that these charged vortex solutions continue to exist even in the absence of the gauge field kinetic energy term [3]. Later significant advance was made by Hong et al. [4] as well as by Jackiw and Weinberg [5] who showed that self-dual C-S vortices can be obtained in case the Higgs potential is

\[ V(\phi) = \frac{e^4}{8\mu^2} |\phi|^2 (|\phi|^2 - v^2)^2 \]  

which corresponds to being at the first order transition point. It has also been shown that these vortices are noninteracting and that there is a \( N = 2 \) supersymmetry in the problem [6]. Further in this case one also has self-dual nontopological vortices [7,8]. These nontopological vortices are at the threshold of their stability against the decay to the elementary excitations. Sum rules have recently been derived by one of us for both types of self-dual vortices and using them the magnetic moment has been computed exactly [8,9]. Finally, Lee et al. [10] have shown that self-dual vortices can also be obtained in Maxwell C-S theory if one couples an additional neutral scalar field to the Higgs field.

The purpose of this paper is to consider small perturbations around self-dual C-S as well as Maxwell C-S vortices. In particular we consider small perturbations in the Higgs potential. In the topological case we show that there exists perturbation, for which at least to the leading order, the vortices remain noninteracting. On the other hand, for the nontopological case we show that for most of the perturbations the vortices are unstable against decay to the elementary excitations.

We shall first consider perturbations around the self-dual C-S vortices. One starts with,

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu - ieA_\mu) \phi^* (\partial^\mu + ieA^\mu) \phi + \frac{e^4}{4} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda - V(\phi) \]  

where \( V(\phi) \) is given in eq. (1). On using the ansatz \( (\rho \geq 0, 0 \leq \theta \leq 2\pi) \)

\[ \tilde{A}(\tilde{\rho}) = -e\phi \frac{ev^2}{r\mu} (\rho(r) - n) \quad , \quad A_0(\tilde{\rho}) = \frac{ev^2}{\mu} h(r) \]
\[ \phi(\vec{r}) = \exp(i n \theta) v f(r), \quad \rho = \frac{\mu r}{e^2 v^2} \] (3)

where \( r, g, h \) and \( f \) are dimensionless variables, the energy of the \( n \)-vortex can be shown to be

\[ E_n = \pi v^2 \int_0^\infty r dr \left[ \left( \frac{df}{dr} - \frac{g f}{r} \right)^2 + \left( \frac{1}{r f} \frac{df}{dr} - \frac{f}{2} (f^2 - 1) \right)^2 + \frac{1}{r} \frac{d}{dr} (g(f^2 - 1)) \right] \] (4)

where use has been made of Gauss' law

\[ -h f^2 = \frac{1}{r} \frac{d}{dr} g. \] (5)

The self-dual equations (for \( n \geq 0 \)) which have minimum energy emerge from here. They are

\[ \frac{df}{dr} = \frac{g f}{r} \] (6a)

\[ \frac{1}{r} \frac{d}{dr} g = \frac{f^2}{2} (f^2 - 1). \] (6b)

The self-dual equations for \( n < 0 \) can be trivially obtained by letting \( g \to -g, f \to f \) and \( h \to -h \). We shall discuss topological and nontopological cases separately.

(a) Topological case: In this case energy of the self-dual \( n \)-vortex is

\[ E_n = \pi v^2 n \] (7)

from where it follows that the vortices are noninteracting \((E_n = n E_1)\). Their flux \( \Phi \), charge \( Q \), angular momentum \( J \) (which is in general fractional) and magnetic moment \( K_z \) is given by

\[ \Phi = \frac{2\pi}{e} n, \quad Q = \mu \Phi, \quad J = -\frac{\pi \mu}{e^2 v^2} n^2, \quad K_z = \frac{2\pi \mu^2}{e^2 v^2} (n^2 + n) \] (8)

The magnetic moment has been calculated by making use of the following two sum rules [8]

\[ n = \frac{1}{2} \int_0^\infty f^2 (1 - f^2) r dr \] (9)
\[ n^2 = \frac{1}{2} \int_0^\infty (1 - f^2)^2 r \, dr \]  \hspace{1cm} (10)

Let us now study the effect of perturbing the topological self-dual vortex by

\[ -\mathcal{L}_{\text{pert}} = V_{\text{pert}} = \alpha \frac{e^4 v^2}{\mu^2} |\phi|^2 (v^2 - |\phi|^2) \]  \hspace{1cm} (11)

where \( \alpha \ll 1 \). On using eqs. (1) and (11) it easily follows that, irrespective of the sign of \( \alpha \), the system now corresponds to being below the first order transition point. In general if

\[ V(f) = B f^2 - |A| f^4 + C f^6 \]  \hspace{1cm} (12)

then it can be shown that \( 4BC \leq \langle > A^2 \) corresponds to \( T \leq \langle > T_c^I \) \cite{11}. Thus \( f = 0 \) is now a local minimum while the two absolute degenerate minima to \( O(\alpha) \) are at

\[ f = \pm (1 + 2\alpha). \]  \hspace{1cm} (13)

However, due to the topological nature of the vortex solution, \( g(0) \), \( g(\infty) \) and \( f(0) \) must remain unaltered to all orders in \( \alpha \) i.e.

\[ g(0) = n \quad , \quad g(\infty) = 0 \quad , \quad f(0) = 0. \]  \hspace{1cm} (14)

The profiles of \( g \), \( h \) and \( f \) will however get modified from their self-dual configurations. On substituting

\[ f(r) = f_{sd}(r) + \alpha f_1(r) + O(\alpha^2) \]
\[ g(r) = g_{sd}(r) + \alpha g_1(r) + O(\alpha^2) \]
\[ h(r) = h_{sd}(r) + \alpha h_1(r) + O(\alpha^2) \]  \hspace{1cm} (15)

in the energy expression (4) we find that the self-dual \( n \)-vortex energy is unaltered to order \( \alpha \). This is because in view of eq. (6) the first two terms in eq. (4) do not contribute to order \( \alpha \) while the last term in eq. (4) always contribute \( \pi v^2 n \). Hence, the energy of the perturbed \( n \)-vortex to \( O(\alpha) \) is given by

\[ E_n = \pi v^2 n + 2\alpha \pi v^2 \int_0^\infty r \, dr \, f_{sd}^2(1 - f_{sd}^2). \]  \hspace{1cm} (16)
On using the sum rule (9) we find then that

$$E_n = \pi v^2 n (1 + 4\alpha).$$

(17)

Thus due to the perturbation, the n-vortex energy increases or decreases depending on the sign of $\alpha$. However, in both the cases one finds that to $O(\alpha)$ the vortices are still noninteracting ($E_n = nE_1$), even though in either case one is now away from the self-dual point.

Let us consider another perturbation given by

$$-L_{pert} = V_{pert} = \alpha \frac{e^4 v^2}{\mu^2} (v^2 - |\phi|^2)^2$$

(18)

which has recently been discussed by Bezeia [12]. As has been shown by him, if $\alpha > 0$ then the situation corresponds to being below the first order transition point (and hence one has topological vortices). Using sum rule (10) one then recovers his result

$$E_n = \pi v^2 n (1 + 4\alpha).$$

(19)

Thus in this case

$$E_n - nE_1 = 4\pi v^2 \alpha n (n - 1) > 0$$

(20)

so that the vortex-vortex interaction is now repulsive in nature. Not surprisingly one finds that in this case the Ginzburg-Landau parameter

$$K_{GL} = \frac{m_s}{m_v} = \frac{(1 + 28\alpha)e^2 v^2/\mu}{(1 + 16\alpha)e^2 v^2/\mu} > 1$$

(21)

so that one is in the type-II region.

Clearly one can consider several other perturbations of the form $\alpha(1 - f^2)(a + bf^2)$ for which energy can be computed exactly to $O(\alpha)$ with the help of the sum rules (9) and (10).
(b) Nontopological case: In this case the energy of the self-dual nontopological n-vortex is
\[ E_n = \pi v^2 (n + \beta) \]  
(22)
where \( g(\infty) = -\beta \) with \( \beta \) being any positive number satisfying \( \beta > n + 2 \) [9]. This has been proved by one of us using the following sum rules [9]
\[ n + \beta = \frac{1}{2} \int_0^\infty f^2(1 - f^2) r \, dr \]  
(23)
\[ \beta^2 - n^2 = \frac{1}{2} \int_0^\infty (2f^2 - f^4) r \, dr. \]  
(24)
The flux \( \Phi \), charge \( Q \), angular momentum \( J \) and magnetic moment \( K_z \) of these nontopological vortices are given by
\[ \Phi = \frac{2\pi}{e} (n + \beta) \quad Q = \mu \Phi \quad J = \frac{\pi \mu}{e^2} (\beta^2 - n^2) \quad K_z = -\frac{2\pi \mu^2}{e^3 v^2} (\beta + n)(\beta - n - 1). \]  
(25)
Further, the mass of the elementary excitation in the theory is given by
\[ m_s = \frac{e^2 v^2}{2\mu}, \]  
(26)
so that at the self-dual point the nontopological vortices are at the threshold of their stability against decay to the elementary excitations i.e.
\[ \frac{E_{sd}}{Q_{sd}} = \frac{m_s}{e} = \frac{e v^2}{2\mu}. \]  
(27)
Let us now study the effect of perturbing the nontopological self-dual vortex by
\[ -\mathcal{L}_{\text{pert}} = V_{\text{pert}} = \frac{\alpha e^4 v^4}{\mu^2} |\phi|^2 \]  
(28)
where \( 0 < \alpha \ll 1 \). Using eqs. (1), (12) and (28) it follows that for \( \alpha > 0 \) this situation corresponds to \( T > T_{c1} \) i.e. \( f = 0 \) is now the absolute minimum of the theory so that to all orders in \( \alpha \), \( f(\infty) = 0 \). Similarly it is clear that to all orders in \( \alpha \), \( g(0) = n \).
However, \( g(\infty) \) may get altered from its unperturbed value of \(-\beta\). Hence, the self-dual nontopological vortex energy may get modified to \( O(\alpha) \). For the same reason \( \Phi, Q, J \) and \( K_s \) may also get modified. However, it is important to notice that the ratio \( E_{sd}/Q_{sd} \) is unaltered to \( O(\alpha) \). Hence, the energy per unit charge of the perturbed nontopological vortex to \( O(\alpha) \) is given by

\[
\frac{E}{Q} = \frac{e\nu^2}{2\mu} + \frac{\alpha e\nu^2}{(n + \beta)\mu} \int_0^\infty r^2 dr f_{sd}^2.
\]  

(29)

On using sum rules (23) and (24) this simplifies to

\[
\frac{E}{Q} = \frac{e\nu^2}{2\mu} [1 + 4\alpha(\beta - n - 1)].
\]

(30)

Because of the perturbation (28) the scalar meson mass \( m_s \) is also increased and is given by

\[
m_s = (1 + 4\alpha)\frac{e\nu^2}{2\mu}.
\]

(31)

However, since \( \beta > n + 2 \) \([9]\) it immediately follows that \( E/Q > m_s/e \) so that with this perturbation the nontopological vortex is unstable against decay into the elementary excitations.

Let us consider another perturbation given by

\[
-L_{pert} = V_{pert} = \frac{\alpha e^4}{\mu^2} |\phi|^4
\]

(32)

where \( 0 < \alpha \ll 1 \). This situation again corresponds to \( T > T^c \). Following the arguments of the previous perturbation it follows that to \( O(\alpha) \), the energy per unit charge of the perturbed nontopological vortex is given by

\[
\frac{E}{Q} = \frac{e\nu^2}{2\mu} + \frac{\alpha e\nu^2}{(n + \beta)\mu} \int_0^\infty r^2 dr f_{sd}^4
\]

\[
= \frac{e\nu^2}{2\mu} [1 + 4\alpha(\beta - n - 2)].
\]

(33)

Since the \( |\phi|^4 \) perturbation does not change the mass of the scalar field, \( m_s/e \) continues to have the value \( e\nu^2/2\mu \). Therefore even with this perturbation the nontopological
vortex is unstable against decay into the elementary excitations. It also follows that for any perturbation of the form \( \alpha f^4 + \gamma f^6 \) \((\alpha > 0, \gamma > 0)\), the vortex will be unstable.

Finally we consider the perturbation

\[
-\mathcal{L}_{\text{pert}} = V_{\text{pert}} = -\frac{\alpha e^4}{8\mu^2} |\phi|^2 |(\phi|^2 - v^2)^2
\]

\((0 < \alpha \ll 1)\). Unlike the previous perturbations this situation still corresponds to \( T = T_0' \). In this case using eq. (4) one can show that

\[
\frac{E}{Q} > \frac{ev^2}{2\mu}(1 - \frac{\alpha}{2})
\]

while

\[
\frac{m^2}{e} = \frac{ev^2}{2\mu}(1 - \frac{\alpha}{2})
\]

so that even in this case the nontopological vortex is unstable.

Finally let us consider perturbations around the self-dual Maxwell C-S vortices. We start with

\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi + \frac{1}{8}\partial_\mu N \partial^\mu N
\]

\[+\frac{\mu}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda - \frac{1}{8}e^2N^2|\phi|^2 - \frac{1}{8}(e|\phi|^2 - ev^2 + \mu N)^2
\]

(37)

where \( N \) is a neutral scalar field. On using the ansatz

\[
\tilde{A}(\rho) = -e\theta \frac{A(\rho)}{\rho} \quad A_0(\rho) = A_0(\rho) \quad \phi(\rho) = \exp(i\theta)f(\rho) \quad N(\rho) = N(\rho)
\]

(38)

the energy of the \( n \)-vortex can be shown to be

\[
E_n = \pi \int_0^\infty \rho d\rho [\frac{\rho}{2}(\frac{dA_0}{d\rho} - \frac{N}{2\rho})^2 + (\frac{1}{\rho} \frac{dA}{d\rho} - \frac{1}{2}(ef^2 - ev^2 + \mu N))^2]
\]

\[+\pi \int_0^\infty \rho d\rho [\frac{df}{d\rho} - (n + eA)f^2 + e^2f^2(A_0 - \frac{N}{2})^2 + \frac{1}{\rho} \frac{d}{d\rho}((n + eA)(f^2 - v^2))].
\]

(39)
The self-dual equations which follow from here are

\[ A_0 = \frac{N}{2} \quad \frac{df}{d\rho} = (n + eA)f \rho \quad \frac{dA}{\rho d\rho} = \frac{1}{2}(e f^2 - \nu^2 + \mu N). \quad (40) \]

Further one has the Gauss law equation

\[ \frac{d^2 A_0}{d\rho^2} + \frac{1}{\rho} \frac{dA_0}{d\rho} - e^2 A_0 f^2 = \frac{\mu}{\rho} \frac{dA}{d\rho}. \quad (41) \]

From here we obtain the following two sum rules

\[ n + \beta = \frac{e^2}{2} \int_0^\infty \rho d\rho (\nu^2 - f^2) - \frac{\mu e}{2} \int_0^\infty \rho d\rho N \]

\[ n + \beta = \frac{e^3}{2\mu} \int_0^\infty \rho d\rho N f^2 \quad (43) \]

where \( \beta = 0 \) for topological vortices.

Using these two sum rules and the discussion about the C-S vortices one can consider the effects of various perturbations. In particular if

\[ -\mathcal{L}_{pert} = V_{pert} = \alpha e^2 (e | \phi |^2 - \nu^2 + \mu N) \quad (44a) \]

\[ -\mathcal{L}_{pert} = V_{pert} = \alpha e^2 N | \phi |^2 \quad (44b) \]

then one can show that in either case irrespective of the sign of \( \alpha \) the vortices are still topological and noninteracting.

If on the other hand one considers the perturbation

\[ -\mathcal{L}_{pert} = V_{pert} = \alpha \nu^2 | \phi |^4 \quad (45) \]

then one can show that for \( \alpha > 0 \) the theory has absolute minima at \( \phi = 0, N = ev^2/\mu \). Following the arguments of the C-S vortices one can show that in this case the nontopological vortices are unstable against decay to the charged scalars. This is because
at the self-dual point $E_{sd}/Q_{sd} = m_s/e$ while with this perturbation $E/Q > m_s/e$.

Similarly the perturbation

$$-\mathcal{L}_{pert} = V_{pert} = -\alpha e^2 N^2 |\phi|^2 - \alpha(e |\phi|^2 - ev^2 + \mu N)^2$$

also leads to nontopological vortices for which $E/Q > m_s/e$.

More results could be derived here if one could obtain the analog of the $n^2$ sum rule as in the C-S vortex case. Even there more results could be obtained if one could derive any other sum rule. In particular one may then be able to improve the bound on $\beta$. 
References


