APPENDIX - A

Double-Time Temperature-Dependent Green's Functions

The elementary excitation spectrum of a system can be obtained by double time Green's function technique. The retarded double time Green's function for any two arbitrary Heisenberg operators $A(t)$ and $B(t')$ is defined as

$$G_r(t - t') = \langle \langle A(t) \ ; \ B(t') \rangle \rangle = -i\theta(t - t')\langle [A(t), B(t')]\rangle_\pm$$

(A.1)

where the $\pm$ signs refer to anticommutators or commutators for fermion or boson operators respectively, $\theta$ is the usual heavyside step function, $\langle \ldots \rangle$ denotes thermodynamic averaging over grand canonical ensemble,

$$\langle \ldots \rangle = \frac{\text{Tr}\{e^{-\beta H} \ldots\}}{\text{Tr}\{e^{-\beta H}\}}$$

(A.2)

$H$ being the Hamiltonian and $\beta = (k_BT)^{-1}$, with the Boltzmann constant $k_B$ and temperature $T$.

The equation of motion for the Green's function is obtained by differentiating eqn. (A.1) with respect to time and is given by

$$i\frac{d}{dt}G_r(t - t') = i\frac{d}{dt}\langle \langle A(t) \ ; \ B(t') \rangle \rangle_\pm$$

$$= \delta(t - t')\langle [A(t), B(t')]\rangle_\pm + \langle \langle [A(t), H]_- \ ; \ B(t') \rangle \rangle_\pm$$

(A.3a)

when differentiation is done with respect to $t$ or equivalently by

$$i\frac{d}{dt'}G_r(t - t') = i\frac{d}{dt'}\langle \langle A(t) \ ; \ B(t') \rangle \rangle_\pm$$

$$= -\delta(t - t')\langle [A(t), B(t')]\rangle_\pm + \langle \langle A(t) \ ; \ [B(t'), H]_- \rangle \rangle_\pm$$

(A.3b)
when differentiation is done with respect to \( t' \). Here \( \delta(t - t') \) is the Dirac delta function

\[
\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}
\]  

(A.4)

Both the equations (A.3a) and (A.3b) are completely equivalent. The Fourier transformed Green's function \( \langle [A; B]\rangle_\omega \) is defined as

\[
G_r(t - t') = \int_{-\infty}^{\infty} d\omega G_r(\omega)e^{-i\omega(t-t')}
\]  

(A.5)

where \( \omega \) is the complex frequency \( \text{Im} \omega > 0 \). The equation of motion for the Fourier transformed Green's function is given by

\[
\omega G_r(\omega) = \frac{1}{2\pi} \langle [A, B]_\pm \rangle + \langle ([A, H]_-; B) \rangle_\omega^\pm
\]  

(A.6a)

or equivalently by

\[
\omega G_r(\omega) = \frac{1}{2\pi} \langle [A, B]_\pm \rangle - \langle (A; [B, H]_-) \rangle_\omega^\pm
\]  

(A.6b)

which corresponds to equations (A.3a) and (A.3b), respectively. The spectral density function \( S(\omega) \) is given by

\[
S(\omega) = \frac{i[G(\omega + i\epsilon) - G(\omega - i\epsilon)]}{\exp[\beta\omega] \pm 1}
\]  

(A.7)

The time correlation function \( \langle B(t')A(t) \rangle \) is given by

\[
\langle B(t')A(t) \rangle = \int_{-\infty}^{\infty} S(\omega) e^{-i\omega(t-t')} d\omega
\]  

(A.8)

Using equ.(A.7) in eqn.(A.8) we have

\[
\langle B(t')A(t) \rangle = \int_{-\infty}^{\infty} d\omega \frac{i[G(\omega + i\epsilon) - G(\omega - i\epsilon)]}{\exp[\beta\omega] \pm 1} e^{-i\omega(t-t')}
\]  

(A.9)

with \( \epsilon \to 0 \)