CHAPTER 10

DEVELOPMENT OF TWO ECHELON SYSTEM FOR DIFFERENTIAL ITEMS UNDER SINGLE MANAGEMENT

10.1 INTRODUCTION

In the previous Chapter, a coordinated supply chain model, with order cost reduction and credit period incentives, was developed. It is difficult to produce or purchase items with 100% good quality. Among the products, those which are within the allowable limits of the specification are termed as non defective units and other items as defective units. Jayanta Kumar Dey et. al., [50] presented an inventory model with differential items (both perfect and defective).

Many researchers like Donaldson, Papachristos and Skouri, Silver and Meal [24, 86, 102] investigated the demand’s dependence on time. Time proportional demand at both primary and secondary shops is considered. Moreover, in the present competitive market, the selling price of a product is one of the decisive factors in selecting the item for use. In practice, higher selling price of a product negates the demand whereas reasonable or low price has the reverse effect. This argument is more appropriate for defective units whose demand is always price dependent. Subramanyam and Kumaraswamy, Wee and Law [107, 118] studied the dependence of demand on pricing.

Furthermore, when shortages occur, some customers are willing to wait for backorders and others would turn to buy from other sellers i.e., some customers are ready to wait for familiar goods like brand gumboots, hi-fi equipments and clothes
till their demands are satisfied. To compensate for the inconvenience of backordering and to secure orders, the suppliers may offer a price discount on the stockout item. By offering price discounts, the supplier can secure more backorders through negotiation. By the higher price discount, a large number of backorder ratios could be fetched by the supplier. Chaung et al., [10] considered an inventory model with backorder price discount.

Most of the classical inventory models did not take into account the effects of inflation. This has happened mostly because of the belief that the inflation would not influence the inventory policy variables to any significant rate. However, most of the countries have suffered from large-scale inflation and sharp decline in the purchasing power of money in the last several years. So, it is important to investigate how inflation influences various inventory policies. Chang, Liao et. al., [18, 63] discussed in their inventory models about the effects of inflation. This study shows that the proposed model is more practical to operate than the traditional one since inflation plays a vital role in economic development of a country. Hence under the consideration of inflationary effect and the backorder price discount this model is more profitable. In this chapter we consider the concepts like partial backlogging of shortages, backorder price discounts, influence of inflation. However, the situation, which involves two echelon system for differential items, different natures of demand at primary and secondary shop, partial backlogging at primary shop, backorder price discount offer at the primary shop and influences of inflation, is not addressed to the best of my knowledge. The proposed model in this chapter is more practical and more profitable. This chapter is arranged as follows. In section 2, problem description, assumptions and notations are given. In section 3, two echelon inventory model is formulated. In
section 4, numerical examples are provided to illustrate the theory and model. In
section 5, we conclude with our results.

10.2 PROBLEM DESCRIPTION

Differential items are received at the primary shop with an infinite rate of
replenishment. At this shop only non-defective units are sold and the defective
units spotted at this shop are continuously transferred to the adjacent secondary
shop. At the secondary shop defective units are sold at a reduced price after some
rework. Time proportional demand at both primary and secondary shops is
considered. The demand for defective items at the secondary shop is depending on
the selling price, which is dependent on the degree of defectiveness of the units
and time.

For each cycle at the primary shop, there will be five different scenarios at
the secondary shop depending upon the relative position of cycle length of primary
shop and that of the secondary shop. For each scenario, profit is maximized and
optimum order quantities are evaluated. The influence of inflation is also
considered in the proposed model. Numerical examples are provided to illustrate
the model.

10.2.1 ASSUMPTIONS

For Primary shop

1. Rate of replenishment of differential items is infinite.
2. The defective units detected during the time period $T$ are continuously
   transferred to the secondary shop.
3. From this shop only non-defective units are sold.
4. Shortages are allowed and partially backlogged at the beginning of the next
cycle, by the non-defective units specially purchased, for this purpose, at the
earlier price, through negotiation.
5. Price discounts are offered at this shop for the backorders to lure the customers.
6. \( B \) is a variable and is in proportion to the price discount \( \pi_x \) offered at this shop
per unit i.e., \( B = \frac{B_0 \pi_x}{\pi_0} \) and \( 0 \leq \pi_x \leq \pi_0 \) (no trader is ready to loose his profit at
any cost. Discounts and reductions will not affect his profit).
7. Demand for the non-defective units \( (R_{ij}(t)) \) at the primary shop is assumed to be
of two parts – one independent of ‘t’ i.e., having a constant demand initially,
and the other part as a function of t and increasing with time during the stock-in
period. Again this part decreases with time during the shortage period. This
implies that due to the non-availability of the item, the demand for the item is
gradually lost and hence slowly declines. Ultimately at the end of the shortage
period it reduces to the original value, which is independent of ‘t’. Hence, the
demand rate at time \( t \) in \( j^{th} \) cycle is given by

\[
R_{ij}(t) = D_0 + D_1(t - C_j) \text{ when } C_j < t < C_j + T_1 \text{ where } D_0, D_1 > 0 \text{ in the no-
shortage period and}
\]

\[
R_{ij}(t) = D_0 - D_2(t - C_{j+1}) \text{ in stock out period. i.e., when } C_j + T_1 < t < C_j + T
\]

here \( D_2 = \frac{D_1 T_1}{T - T_1} \).
8. The inflation rate \( i \) is a constant.
For secondary shop

1. Defective units spotted out at the primary shop are continuously transferred to this adjacent shop at a variable rate $\theta l_{1,j}(t)$ per unit time for $j^{th}$ cycle, $j = 1, 2, ..., n$.

2. Defective units received are reworked and then sold.

3. The selling price of a defective unit depends on the rate of defectiveness. i.e.,

$$P_2^i = \frac{P_1}{r_0^i}$$

where $r_0$ is chosen in such a way that $P_2^i < P_1^i$.

4. The demand for defective units at this shop for $j^{th}$ cycle ($j = 1, 2, ..., n$) depends on time and selling price $P_2^i$.

i.e.,

$$R_{2,j}(t) = a e^{-b t} - cP_2^i$$

where $a, b, c > 0$.

5. Lead time is zero.

6. Shortages are not allowed.

10.2.2 NOTATIONS

For primary shop (during $j^{th}$ cycle, $j = 1, 2, ..., n$)

$n$ Number of replenishments over $[0, H]$.

$T$ The total time length of each cycle i.e., $H = nT$.

$\theta$ Constant rate of defective units out of on hand inventory where $0 < \theta < 1$.

$i$ Constant rate of inflation per unit time where $0 \leq i < 1$.

$C_j$ Time at which $j^{th}$ cycle starts where $C_j = (j-1)T$ and $C_{j+1} = C_j + T$.

$R_{1,j}(t)$ Demand at time $t$ at the primary shop in $j^{th}$ cycle.

$l_{1,j}(t)$ Inventory level at time $t$ at the primary shop in $j^{th}$ cycle where $C_j \leq t \leq C_{j+1}$.

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\( P_i \) The unit purchasing cost at time zero and the unit purchasing cost at time \( t \) is \( P_i e^t \).

\( P_i^l \) The unit selling price at time zero and \( P_i^l e^t \) is the selling price per unit at time \( t \) where \( P_i^l > P_i \).

\( K_i \) The ordering cost per order at time zero and \( K_i e^t \) is the ordering cost per order at time \( t \).

\( c_i \) The holding cost per unit at time zero and \( c_i e^t \) is the holding cost per unit at time \( t \).

\( s_i \) The shortage cost per unit short at time zero and \( s_i e^t \) is the shortage cost per unit short at time \( t \).

\( \pi_x \) Backorder price discount per unit at time zero and \( \pi_x e^t \) is the backorder price discount per unit at time \( t \).

\( W_j \) Total number of defective units during \( j^{th} \) cycle.

\( R_j \) Backlogged demand.

\( Z_j(T_j) \) Profit at the primary shop.

**For Secondary shop (during \( j^{th} \) cycle \( j = 1, 2, ..., n \))**

\( R_{2j}(t) \) Demand at time \( t \).

\( I_{2j}(t) \) Inventory level at time \( t \), where \( C_j \leq t \leq C_{j+1} \).

\( T_2 \) The time after which the demand is greater than the rate of defective units received, after time \( C_j \).

\( T_3 \) Total time length of each cycle.

\( Q_{1j} \) Inventory level at time \( C_j + T_1 \).

\( Q_{2j} \) Inventory level at time \( C_j + T_2 \).
$Q_M$  Inventory level at time $C_j + T$, inventory to be sold at the time period of the primary cycle.

$p_2'$  The unit selling price at time zero and $p_2' e^{lt}$ is the selling price of defective items per unit at time $t$.

$k_2$  Setup cost per setup at time zero and $k_2 e^{lt}$ is the setup cost per setup at time $t$.

$h_2$  Holding cost per unit at time zero and $h_2 e^{lt}$ is the holding cost per unit at time $t$.

$c_w$  Cost of reworking the defective items per unit at time zero and $c_{we} e^{lt}$ is the cost of reworking the defective units per unit at time $t$.

$Z_2(T_1)$  Profit at the secondary shop.

10.3 MODEL FORMULATION

Primary shop

In the proposed model we assume that a part of the stock out period demand is fulfilled. At $j^{th}$ cycle, after fulfilling a part of stock-out period demand, the on-hand inventory level for differential items is initially at $Q_j$ and upto $t = C_j + T_1$, it gradually declines due to the combined effect of demand and defective units which are continuously transferred to the adjacent secondary shop for sale. The inventory level reaches zero at $t = C_j + T_1$. After this time, shortage starts and continues upto time $t = C_j + T$ when the next lot arrives. At time $t = C_j + T$ the backlogged stock-out period demand is $R_j$. We can represent the system as in Figure.10a.

The change in the inventory level $I_j(t)$ with respect to time can be written as
\[
\frac{dI_{1j}(t)}{dt} = \begin{cases} 
-R_{1j}(t) - \theta I_{1j}(t) & \text{if } C_j \leq t \leq C_j + T_1 \\
-BR_{1j}(t) & \text{if } C_j + T_1 \leq t \leq C_j + T
\end{cases}
\text{ for } j=1,2,\ldots,n
\]

i.e.,
\[
\frac{dI_{1j}(t)}{dt} + \theta I_{1j}(t) = -\left[D_0 + D_1\left(t - C_j\right)\right] \text{ if } C_j \leq t \leq C_j + T_1
\]

and
\[
\frac{dI_{1j}(t)}{dt} = -B\left[D_0 - D_2\left(t - C_{j+1}\right)\right] \text{ if } C_j + T_1 \leq t \leq C_j + T
\]

where \( D_j = \frac{D_j T_j}{T - T_j} \) and \( B \) is the backorder rate subject to the condition that
\[
I_{1j}(t) = Q_j \text{ at } t = C_j
\]
\[
I_{1j}(t) = 0 \text{ at } t = C_j + T_1
\]

Also
\[
I_{1j}(t) = -R_j \text{ at } t = C_j + T
\]

where \( R_j \) is the stockout period demand.
Figure 10a. Pictorial Representation of the inventory level at the primary shop
When $C_j \leq t \leq C_j + T_1$

$$\frac{dI_{1j}(t)}{dt} + \theta I_{1j}(t) = -\left[ D_0 + D_1 \left( t - C_j \right) \right]$$

(101)

By using the boundary condition $I_{1j}(t) = 0$ at $t = C_j + T_1$ we get the solution as

$$I_{1j}(t) = \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) \right] e^{\theta (C_j + T_1 - t)} - \left[ \frac{D_0}{\theta} - \frac{D_1}{\theta^2} + \frac{D_1}{\theta} \left( t - C_j \right) \right]$$

if $C_j \leq t \leq C_j + T_1$

Similarly, we can find $I_{1j}(t)$ when $C_j + T_1 \leq t \leq C_j + T$

$$I_{1j}(t) = \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] e^{\theta (C_j + T_1 - t)} - \left[ \frac{D_0}{\theta} - \frac{D_1}{\theta^2} + \frac{D_1}{\theta} \left( t - C_j \right) \right]$$

if $C_j \leq t \leq C_j + T_1$

$$B \left\{ D_0 \left( C_j + T_1 - t \right) + \frac{D_2}{2} \left[ (C_j + T - t)^2 - (T_1 - T)^2 \right] \right\}$$

if $C_j + T_1 \leq t \leq C_j + T$

(102)

By using the condition $I_{1j}(t) = Q_j$ at $t = C_j$ in the above solution we have

$$Q_j = \left( \frac{D_1}{\theta^2} \frac{D_0}{\theta} \right) + \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) \right] e^{\theta T_1}$$

(103)

By using the condition $I_{1j}(t) = -R_j$ at $t = C_j + T$ we get

$$R_j = B \left\{ D_0 \left( T - T_1 \right) + \frac{D_2}{2} \left( T_1 - T \right)^2 \right\}$$

(104)

The total number of defective units in the $j^{th}$ cycle is

$$W_j = \theta \int_{C_j}^{C_j + T_1} I_{1j}(t) dt$$

$$= \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) \right] e^{\theta T_1} - \left[ \frac{D_0}{\theta} - \frac{D_1}{\theta^2} \right] T_1 - \frac{D_1 T_1^2}{2}$$

(105)

Since the lengths of time intervals (T for each cycle) are all the same, we have

$$I_{1j}(jT + t) = I_{1j}(t) \quad \text{for} \quad 1 \leq j \leq n$$

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Total inventory holding cost = \[ \sum_{j=1}^{n} \left( h_t e^{\theta T_i} \int_{C_i}^{C_{j+1}} I_j(t) \, dt \right) \]

Holding cost

\[
= \left( \frac{h_t e^{\theta T_i}}{\theta} \right) \left( \frac{e^{\theta n T_i} - 1}{e^{\theta T_i} - 1} \right) \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_i - 1) \right] e^{\theta T_i} - 1 \right) \left( D_0 - \frac{D_1}{\theta} \right) T_i - \frac{D_1 T_i^2}{2} \right) \]

(106)

Total Replenishment cost

\[ = K_1 e^{\theta T_i} + K_2 e^{2\theta T_i} + \ldots + K_n e^{n\theta T_i} \]

\[ = e^{\theta T_i} K_1 \left[ \frac{e^{\theta n T_i} - 1}{e^{\theta T_i} - 1} \right] \]

(107)

Total selling price = \[ \sum_{j=1}^{n} (Q_j + R_j) P_t e^{\theta T_i} \]

\[ = P_t e^{\theta T_i} \left[ \frac{e^{\theta H} - 1}{e^{\theta T_i} - 1} \left( \frac{D_1}{\theta^2} - \frac{D_0}{\theta} \right) + \left( \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_i - 1) \right) e^{\theta T_i} + B \left( D_0 (T - T_i) + \frac{D_2}{2} (T - T_i)^2 \right) \right] \]

(108)

Total Purchase cost

\[ = \sum_{j=1}^{n} (Q_j + R_j) P_t e^{\theta T_i} \]

\[ = P_t e^{\theta T_i} \left[ \frac{e^{\theta H} - 1}{e^{\theta T_i} - 1} \left( \frac{D_1}{\theta^2} - \frac{D_0}{\theta} \right) + \left( \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_i - 1) \right) e^{\theta T_i} + B \left( D_0 (T - T_i) + \frac{D_2}{2} (T - T_i)^2 \right) \right] \]

(109)
Total Price discount for backorders

\[ \pi_x B \sum_{j=1}^{n} e^{j(T-T_1)} \frac{C_{j+T}}{C_{j+T_1}} \int_{0}^{T} \frac{R_{T_j}(t)}{t-C_{j+1}} \, dt \]

\[ = \pi_x B \sum_{j=1}^{n} e^{j(T-T_1)} \frac{C_{j+T}}{C_{j+T_1}} \left[ D_0 - D_2 (T-C_{j+1}) \right] \, dt \]

\[ = \pi_x B e^{(T-T_1)} \left[ \frac{e^{in(T-T_1)} - 1}{e^{i(T-T_1)} - 1} \right] \left\{ D_0 (T-T_1) + \frac{D_2}{2} (T-T_1)^2 \right\} \]

Total cost due to lost sales

\[ = (S_1 + \pi_0) (1-B) \sum_{j=1}^{n} e^{j(T-T_1)} \frac{C_{j+T}}{C_{j+T_1}} \int_{0}^{T} \frac{R_{T_j}(t)}{t-C_{j+1}} \, dt \]

\[ = (S_1 + \pi_0) (1-B) e^{(T-T_1)} \left[ \frac{e^{in(T-T_1)} - 1}{e^{i(T-T_1)} - 1} \right] \left\{ D_0 (T-T_1) - \frac{D_2}{2} (T-T_1)^2 \right\} \]

Total stock out cost is the sum of the backorder price discount and the cost due to lost sales

\[ = [\pi_x B + (S_1 + \pi_0)(1-B)] e^{i(T-T_1)} \left[ \frac{e^{in(T-T_1)} - 1}{e^{i(T-T_1)} - 1} \right] \left\{ D_0 (T-T_1) + \frac{D_2}{2} (T-T_1)^2 \right\} \] (110)

Total cost due to defective units = \( \sum_{j=1}^{n} W_j P_j e^{jT_1} \)

\[ = P_j e^{iT_1} \left[ \frac{e^{inT_1} - 1}{e^{iT_1} - 1} \right] \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] \left[ e^{iT_1} - 1 \right] - \left( D_0 - \frac{D_1}{\theta} \right) T_1 - \frac{D_1 T_1^2}{2} \] (111)

Hence the profit at the primary shop is given by

\[ Z_1(T_1) = \text{Total Revenue} - \text{Total Purchase cost} - \text{Total Holding cost} - \text{Total Replenishment cost} - \text{Total stock out cost} - \text{Total cost due to defective units} \] (112)
Secondary shop

Defective units are sold, after rework, at the adjacent secondary shop. The amount of stock is zero initially at this shop. Just after $t = C_j$, due to the defectiveness of the units at the primary shop, the inventory level is raised at the rate $\theta l_1(t) - R_{2j}(t)$ up to $C_j + T_2$, till the arrival of defective units is greater than the demand at the secondary shop (i.e., $\theta l_1(t) > R_{2j}(t)$).

The stock attains a level $Q_{3j}$ at $t = C_j + T_2$.

After $t = C_j + T_2$, demand $R_{2j}(t)$ is greater than the rate of defective units per unit time and the demand is satisfied partly from current defective units transferred from the primary shop and partly from the stock of the secondary shop. This continues up to $t = C_j + T_1$ and the stock attains a level $Q_{2j}$ at $t = C_j + T_1$.

After $t = C_j + T_1$, the supply of defective units stops and the inventory level gradually declines to meet the demand up to $t = C_j + T_3$.

For each cycle of the primary shop, there will be five different scenarios at the secondary shop depending upon the position of $T$ (cycle length of the primary shop) and $T_3$ (cycle length of the secondary shop). Hence for $n$ cycles at the primary shop, there will be $5^n$ scenarios in the secondary shop. Only five scenarios are discussed and other scenarios can be defined in a similar way.

**Scenario I**: $C_j + T_3 = C_j + T$.

Cycle length of secondary shop = Cycle length of primary shop. Defective units are exhausted just at the time of arrival of the next lot of differential units (Figure 10b).
**Scenario II:** \( C_j + T_3 < C_j + T \) for \( j = 1,2,\ldots,n \)

\[ T_3 < T \quad \text{for } j = 1,2,\ldots,n. \]

Defective units are sold before the arrival of a new lot of differential units. In this case, it is assumed that the shop will remain closed for the remaining period till the next lot of items arrives (Figure 10c).

**Scenario III:** \( C_j + T_3 = C_j + T_1 \) for \( j = 1,2,\ldots,n \)

\[ T_3 = T_1 \quad \text{for } j = 1,2,\ldots,n. \]

It is assumed that with the exhaust of differential units at the primary shop, the defective units at the secondary shop are also completely sold (Figure 10d).

**Scenario IV:** \( C_j + T_3 = C_j + T_1' \) where \( T_1' < T_1 \) for \( j = 1,2,\ldots,n \)

\[ T_3 < T_1 \]

It is assumed that before the exhaust of differential units at the primary shop, the defective units at the secondary shop are also completely sold (Figure 10e).

**Scenario V:** \( C_j + T_3 > C_j + T \) for \( j = 1,2,\ldots,n \)

There will be some defective units left to be sold at the time equal to time period of the primary shop. It is assumed that to run along with the primary shop, all the remaining defective units are sold at the throw-away price at the end of the time period \( T \) and there is an infinite demand of defective units at a much reduced price (Figure 10f).
Figure 10 b. Pictorial Representation of the inventory level at the secondary shop: Scenario I

Figure 10 c. Pictorial Representation of the inventory level at the secondary shop: Scenario II
Figure 10 d. Pictorial Representation of the inventory level at the secondary shop:

Scenario III

Figure 10 e. Pictorial Representation of the inventory level at the secondary shop:

Scenario IV
Figure 10 f. Pictorial Representation of the inventory level at the secondary shop: Scenario V

10.3.1 Mathematical Formulation (for Scenario I – V)

The differential equations describing the instantaneous states of inventory $I_{2j}(t)$ in the interval $C_j \leq t \leq C_j + T_3$ (for $j = 1, 2, \ldots, n$) are given by

$$
\frac{dI_{2j}(t)}{dt} = \begin{cases} 
\partial I_{2j}(t) - R_{2j}(t) & \text{if } C_j \leq t \leq C_j + T_2, \text{ for Scenario I – V;} \\
0 & \text{if } C_j + T_2 \leq t \leq C_j + T_1, \text{ for Scenario I, II, III, V} \\
-R_{2j}(t) & \text{if } C_j + T_1 \leq t \leq C_j + T_3, \text{ for Scenario IV} \\
\end{cases}
$$

(113)

with the boundary conditions
\[ I_{ij}(t) = \begin{cases} 
0 & \text{at } t = C_j \quad \text{for Scenario I - V} \\
Q_{ij} & \text{at } t = C_j + T_1 \quad \text{for Scenario I, II, V} \\
0 & \text{at } t = C_j + T_1' \quad \text{for Scenario III} \\
0 & \text{at } t = C_j + T_1' \quad \text{for Scenario IV} 
\end{cases} \]

Also we consider the following conditions

\[ I_{2j}(t) = \begin{cases} 
Q_{2j} & \text{at } t = C_j + T_2 \quad \text{for Scenario I - V} \\
Q_{3j} & \text{at } t = C_j + T \quad \text{for Scenario V} 
\end{cases} \]

\[
\begin{align*}
\frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) & \quad e^{\theta T_1} \left[ 1 - e^{-\theta (C_j - t)} \right] - \left( t - C_j \right) \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] - \frac{D_1}{2} \left( t - C_j \right)^2 \\
& + \frac{a}{b} \left[ e^{-bt} - e^{-bC_j} \right] \quad \text{if } C_j \leq t \leq C_j + T_2 \quad \text{for scenario I - V} \\
\frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) & \quad e^{-\theta T_2} - e^{-\theta (C_j - t)} \left( t - C_j - T_2 \right) \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] \\
& + \frac{D_1}{2} \left[ T_2^2 - (C_j - t)^2 \right] + \frac{a}{b} \left[ e^{-bt} - e^{-b(C_j + T_2)} \right] + Q_{2j} \\
& \text{if } C_j + T_2 \leq t \leq C_j + T_1 \quad \text{for scenario I, II, III.} \\
\frac{D_0}{\theta} + \frac{D_1}{\theta^2} \left( \theta T_1 - 1 \right) & \quad e^{-\theta T_1} - e^{-\theta (C_j - t)} \left( t - C_j - T_1 \right) \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] \\
& + \frac{D_1}{2} \left[ T_1^2 - (C_j - t)^2 \right] + \frac{a}{b} \left[ e^{-bt} - e^{-b(C_j + T_1)} \right] \\
& \text{if } C_j + T_1 \leq t \leq C_j + T_3 \quad \text{for scenario IV} \\
\frac{a}{b} \left[ e^{-bt} - e^{-b(C_j + T_3)} \right] & \quad -cP_2 \left( C_j + T_3 - t \right) \quad \text{if } C_j + T_1 \leq t \leq C_j + T_3 \\
& \text{for scenario I & II} \\
\frac{a}{b} \left[ e^{-bt} - e^{-b(C_j + T)} \right] & \quad -cP_2 \left( C_j + T - t \right) + Q_{3j} \quad \text{if } C_j + T_1 \leq t \leq C_j + T_3 \\
& \text{for scenario V} 
\end{align*}
\]
By considering the inflation rate, the holding cost in the secondary shop can be written as

Holding cost

$$= \sum_{j=1}^{n} h_j e^{\gamma_j T_2} \left[ \frac{C_j + T_2}{C_j} \right] \int_{I_2}^1 f(t) dt + \sum_{j=1}^{n} h_j e^{\gamma_j (T_2 - T_1)} \left[ \frac{C_j + T_2}{C_j + T_2} \right] \int_{I_2}^1 f(t) dt + \sum_{j=1}^{n} h_j e^{\gamma_j (T_3 - T_1)} \left[ \frac{C_j + T_3}{C_j + T_1} \right] \int_{I_2}^1 f(t) dt$$

$$= A_1 + A_2 + A_3 (\text{say}) \quad (114)$$

Now

$$A_1 = h_2 e^{\gamma_2 T_2} \left[ \frac{e^{\gamma_2 T_1} - 1}{e^{\gamma_2 T_1} - 1} \right] \left[ \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] e^{\theta T_1} \left[ T_2 + \frac{1}{\theta} \left( e^{-\theta T_1} - 1 \right) \right] - \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] \frac{T_2}{2} \right.$$

$$- \frac{D_1 T_2^3}{6} + \frac{a}{b^2} e^{-bC_j} \left[ 1 - bT_2 - e^{-bT_2} \right] \right] \text{ for scenario I-IV}$$

$$A_2 = h_2 e^{\gamma_2 (T_1 - T_2)} \left[ \frac{e^{\gamma_2 (T_1 - T_2)} - 1}{e^{\gamma_2 (T_1 - T_2)} - 1} \right] \left[ \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] e^{\theta T_1} \left[ e^{-\theta T_2 (T_1 - T_2)} + \frac{1}{\theta} \left( e^{-\theta T_1} - e^{-\theta T_2} \right) \right] \right.$$

$$- \frac{1}{2} \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] \left[ T_1 - T_2 \right]^2 - \frac{D_1}{2} \left[ T_1^3 - 3T_1^2 T_2 + 2T_2^3 \right] - \frac{a}{b^2} e^{-bC_j} \right.$$ 

$$\left. \left\{ b e^{-b T_2 (T_1 - T_2)} + e^{-b T_1} - e^{-b T_2} + Q_2 f(T_1 - T_2) \right\} \text{ for scenario I,II,IIIIV} \right]$$

$$A_3 = h_2 e^{\gamma_2 (T_1 - T_2)} \left[ \frac{e^{\gamma_2 (T_1 - T_2)} - 1}{e^{\gamma_2 (T_1 - T_2)} - 1} \right] \left[ \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] e^{\theta T_1} \left[ e^{-\theta T_2 (T_1 - T_2)} + \frac{1}{\theta} \left( e^{-\theta T_1} - e^{-\theta T_2} \right) \right] \right.$$

$$+ \frac{1}{2} \left[ D_0 - \frac{D_1}{\theta} - cP_2 \right] \left[ T_1 - T_2 \right]^2 + \frac{D_1}{2} \left[ 2T_1^3 - 3T_1^2 T_2 + 2T_2^3 \right] - \frac{a}{b^2} e^{-bC_j} \right.$$ 

$$\left. \left\{ e^{-b T_1} - e^{-b T_2} + b e^{-b T_1} (T_1 - T_2) \right\} \right\} \text{ for scenario IV}$$

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\[ A_3 = \begin{cases} 
\frac{c P^i_2}{2} (T_3 - T_1)^2 & \text{for scenario II} \\
0 & \text{for scenarios III & IV} \\
\frac{c P^i_2}{2} (T_3 - T_1) (2T - T_3 - T_1) + Q_3 j (T_3 - T_1) & \text{for scenario V} 
\end{cases} \]

We have the boundary conditions,

\[ I_{2j}(t) = Q_{1j} \quad \text{at } t = C_j + T_1 \text{ for scenarios I, II & V} \]

\[ = 0 \quad \text{at } t = C_j + T_1 \text{ for scenario III} \]

\[ = 0 \quad \text{at } t = C_j + T_1' \text{ for scenario IV} \]

\[ \therefore Q_{1j} = \left[ \frac{D_0}{\theta} + \frac{D_1}{\theta^2} (\theta T_1 - 1) \right] e^{\theta t_1} \left[ e^{-\theta T_1} - e^{-\theta t_1} \right] + (T_2 - T_1) \left[ D_0 - \frac{D_1}{\theta} - C P^i_2 \right] + \frac{D_1}{2} (T_2^2 - T_1^2) + \frac{ae^{-bC_j}}{b} \left[ e^{-bT_1} - e^{-bT_2} \right] + Q_{2j} \text{ for scenario I, II & V} \] (115)

We have another boundary condition,

\[ I_{2j}(t) = \begin{cases} 
Q_{2j} & \text{at } t = C_j + T_2 \text{ for scenarios I, II, IV & V} \\
0 & \text{at } t = C_j + T_1 \text{ for scenario III} 
\end{cases} \]
\[
Q_{2j} = \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{1-e^{-\theta T_2}}{1-e^{-\theta T_1}}\right] - T_2 \left[D_0 + \frac{D_1}{\theta^2} e^{-\theta T_2} \right] - \frac{D_1^2}{2} T_2^2 \right] + \frac{a e^{-b c^2}}{b} \left[\frac{e^{-b T_2} - e^{-b T_1}}{1-e^{-\theta T_1}}\right]
\]

for scenario I, II & V

\[
Q_{2j} = \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{1-e^{-\theta T_2}}{1-e^{-\theta T_1}}\right] - (T_2 - T_1) \left[D_0 + \frac{D_1}{\theta^2} e^{-\theta T_2} \right] + \frac{D_1^2}{2} (T_2^2 - T_1^2) \right] + \frac{a e^{-b c^2}}{b} \left[\frac{e^{-b T_2} - e^{-b T_1}}{1-e^{-\theta T_1}}\right]
\]

for scenario III

\[
Q_{2j} = \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{1-e^{-\theta T_2}}{1-e^{-\theta T_1}}\right] - (T_2 - T_1) \left[D_0 + \frac{D_1}{\theta^2} e^{-\theta T_2} \right] + \frac{D_1^2}{2} (T_2^2 - T_1^2) \right] + \frac{a e^{-b c^2}}{b} \left[\frac{e^{-b T_2} - e^{-b T_1}}{1-e^{-\theta T_1}}\right]
\]

for scenario IV

\[(116)\]

Total number of defective units received from the primary shop during the \(j\)th cycle = \(W_j\)

\[
= \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{e^{\theta T_1} - 1}{e^{\theta T_1} - 1}\right] - \left(D_0 + \frac{D_1}{\theta^2}\right) T_1 - \frac{D_1^2 T_1^2}{2}\right]
\]

from equation (105).

By considering the inflation rate, the total cost of reworking the defective units in \(n\) cycles = \(\sum_{j=1}^{n} C_{WE} e^{\theta T_1} \cdot W_j\)

\[
= C_{WE} e^{\theta T_1} \left[\frac{e^{\theta T_1} - 1}{e^{\theta T_1} - 1}\right] \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{e^{\theta T_1} - 1}{e^{\theta T_1} - 1}\right] - \left(D_0 + \frac{D_1}{\theta^2}\right) T_1 - \frac{D_1^2 T_1^2}{2}\right]
\]

(117)

Selling price for the repaired defective units at the secondary shop

\[
= \sum_{j=1}^{n} P_{2e} e^{\theta T_3} \cdot W_j
\]

\[
= P_{2e} e^{\theta T_3} \left[\frac{e^{\theta T_3} - 1}{e^{\theta T_3} - 1}\right] \left[\frac{D_0 + \frac{D_1}{\theta^2} (\theta T_1 - 1)}{\theta^2} \left[\frac{e^{\theta T_1} - 1}{e^{\theta T_1} - 1}\right] - \left(D_0 + \frac{D_1}{\theta^2}\right) T_1 - \frac{D_1^2 T_1^2}{2}\right]
\]

(118)
Setup cost at secondary shop

\[ = \sum_{j=1}^{n} K_2 e^{jT_2} \]

\[ = K_2 e^{jT_2} \left( \frac{e^{nT_2} - 1}{e^{T_2} - 1} \right) \] (119)

Hence the profit at the secondary shop is given by

\[ Z_2(T_i) = \text{Selling price} - \text{Holding cost} - \text{Setup cost} - \text{Rework cost} \] (120)

The cycle length \( T_s \) of \( j \)th cycle at the secondary shop is given by

\[ Q_{1j} = \begin{cases} \frac{a}{b} e^{-bT_1} \left( e^{-bT_3} - e^{-bT_1} \right) - cP_2 \left( T_3 - T_1 \right) & \text{for scenario I & II} \\ \frac{a}{b} e^{-bT_1} \left( e^{-bT} - e^{-bT_1} \right) - cP_2 \left( T - T_1 \right) + Q_{3j} & \text{for scenario V} \end{cases} \] (121)

At \( t = C_j + T_2, \theta I_{1j}(t) = R_{2j}(t) \)

\[ R_{2j} = \left( D_0 + \frac{D_1}{\theta} \left( \theta T_2 - 1 \right) \right) e^{\alpha(t - T_2)} - \left[ D_0 + \frac{D_1}{\theta} \left( \theta T_2 - 1 \right) \right] \] (122)

Therefore, the total profit for the system from two shops for each scenario in time horizon \( H \) is given by

\[ Z(T_i) = Z_1(T_i) + Z_2(T_i) \text{ where } T = H / n \] (123)

It is easy to verify that the profit function is concave. Our problem is to determine the optimal value of \( T_1 \) such that the average profit for the system is maximum and also to determine the corresponding values of \( Q_j, T_2 \) and \( Z \). These optimal values have been determined by using Quasi-Newton Method.
10.4 NUMERICAL EXAMPLES

Numerical examples are given to mention the workability of the proposed theory, model and to discuss the effects of different backorder rates, backorder price discounts and inflation rates. Computational works are carried out using MATLAB 7.0. Some data input values were taken from earlier works [50].

10.4.1 WORKABILITY OF THE THEORY AND MODEL

Example 1

Let $D_0=32$, $D_1=10$, $\theta = 0.15$, $P_1'=11.5$, $P_1 = 5$, $i = 0.05$, $a = 15$, $b = 0.4$, $c = 0.5$, $h_1 = 0.7$, $h_2 = 0.22$, $k_1 = 80$, $k_2 = 15$, $S_1 = 2$, $\pi_0 = 0.21001$, $\pi_x = 0.21$, $C_w = 1$, $H = 12$, $r_0 = 7$, $B_0 = 0.8$ in appropriate units.

The optimum values of $T_1$ along with the maximum profit $Z$ have been calculated for each scenario and the results are displayed in Table 10.i. The values of $Q_{i0}$, $Q_{i1}$, $Q_{i2}$, $T_1$, $T_2$, $T_3$ and $T$ are also shown in this Table 10.i.

10.4.2 SENSITIVITY ANALYSIS

Example 2

For scenario I consider, $D_0 = 32$, $D_1 = 10$, $\theta = 0.15$, $P_1'=11.5$, $P_1 = 5$, $i = 0.05$, $a =15$, $b = 0.4$, $c = 0.5$, $h_1=0.7$, $h_2 = 0.22$, $k_1 = 80$, $k_2 = 15$, $S_1=2$, $\pi_0 = 0.21001$, $\pi_x = 0.21$, $C_w = 1$, $H = 12$, $r_0 = 7$ in appropriate units.

We consider various upper limits for back order rate ($B_0$). Hence $B=(B_0 \pi_x)/\pi_0$ is also varying. The calculated values are presented in Table 10.ii. From this table we observe that if $B$ (backlogging rate at the primary shop) increases, profit also increases. A trader increases the backlogging rate means he
is ready to give backorder price discount for more orders. Even then he can earn more profit through getting more orders. From our model this concept is explicit.

Example 3

For scenario I consider all the data as in Example 1. Here we change $\pi_x$. (Backorder price discount). The computed results are shown in Table 10.iii. From Table 10.iii we come to know that there is an increase in profit with the increase of $\pi_x$. When a trader offers backorder price discount he can secure more backorders. By increasing backorders price discount there will be a reduction in profit per unit. But by securing more customers he can raise his overall profit. Hence a trader can increase backorder price discount to get more profit. That is why traders announce festival offers to attract customers. For this reason we say that our model is more realistic. From the above table we observe that $Q_j$ (Ordering quantity) increases as $\pi_x$ increases. Due to the raise of $Q_j$, holding cost, purchase cost will increase. Even then the trader can earn more profit.

Example 4

For scenario I consider all the data as in Example 1. We change $i$ (rate of inflation). The results are presented in Table 10.iv. From this table, we observe that as $i$ increases $Q_j$, $T_1$ and $T_2$ decrease. Hence the holding cost, purchase cost and the reworking cost of defective units will decrease. So the profit should increase which is obtained in our table. So we come to know that if rate of inflation is higher, the trader should order a lesser quantity to get more profit. Price of the commodities increases due to the effect of inflation. A marginal increase in profit is imminent. Without inflation a country’s economy would not improve. Even though inflation is the necessary evil, it should be under control.
Table 10.i Optimal Results for different scenarios

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$Z_0$ (profit)</th>
<th>$n$</th>
<th>$Q_i$</th>
<th>$Q_{ij}$</th>
<th>$Q_{ij}$</th>
<th>$T_1$</th>
<th>$T_1'$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>2971.9</td>
<td>6</td>
<td>89.79</td>
<td>0.3544</td>
<td>2.0359</td>
<td>-</td>
<td>1.8639</td>
<td>-</td>
<td>0.9641</td>
<td>2</td>
</tr>
<tr>
<td>II</td>
<td>2879.10</td>
<td>6</td>
<td>97.30</td>
<td>6.17</td>
<td>6.70</td>
<td>-</td>
<td>1.9740</td>
<td>-</td>
<td>1.5339</td>
<td>1.8869</td>
</tr>
<tr>
<td>III</td>
<td>2839.90</td>
<td>6</td>
<td>97.79</td>
<td>-</td>
<td>5.2</td>
<td>-</td>
<td>1.9809</td>
<td>-</td>
<td>1.5666</td>
<td>1.5666</td>
</tr>
<tr>
<td>IV</td>
<td>2891.90</td>
<td>6</td>
<td>91.79</td>
<td>-</td>
<td>0.36</td>
<td>-</td>
<td>1.8935</td>
<td>1.0607</td>
<td>1.1266</td>
<td>-</td>
</tr>
<tr>
<td>V</td>
<td>2890.50</td>
<td>6</td>
<td>98.81</td>
<td>1.9723</td>
<td>1.5809</td>
<td>0.56</td>
<td>1.8466</td>
<td>-</td>
<td>0.8661</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 10.ii Optimal Results for different backorder rates

<table>
<thead>
<tr>
<th>$B$</th>
<th>Profit</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>2949.0</td>
<td>1.9780</td>
<td>1.5530</td>
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<tr>
<td>0.75</td>
<td>2956.9</td>
<td>1.9164</td>
<td>1.2473</td>
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<td>0.8</td>
<td>2971.9</td>
<td>1.8639</td>
<td>0.9641</td>
</tr>
<tr>
<td>0.85</td>
<td>2993.4</td>
<td>1.8195</td>
<td>0.7067</td>
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<tr>
<td>0.9</td>
<td>3020.4</td>
<td>1.7815</td>
<td>0.4738</td>
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<tr>
<td>0.95</td>
<td>3052.0</td>
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</table>

Table 10.iii Optimal Results for different backorder price discounts

<table>
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<tr>
<th>$\pi_2$</th>
<th>Profit</th>
<th>$Q_1$</th>
</tr>
</thead>
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<td>0.21</td>
<td>2971.9</td>
<td>89.79</td>
</tr>
<tr>
<td>3.21</td>
<td>2992.1</td>
<td>112.92</td>
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<td>4.21</td>
<td>3050.2</td>
<td>119.31</td>
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</table>
Table 10.iv Optimal Results for different inflation rates

<table>
<thead>
<tr>
<th>i</th>
<th>Profit</th>
<th>Q1</th>
<th>T1</th>
<th>T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>2470.2</td>
<td>93.9127</td>
<td>1.9248</td>
<td>1.2904</td>
</tr>
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<td>2633.3</td>
<td>92.2417</td>
<td>1.9002</td>
<td>1.1620</td>
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<tr>
<td>0.04</td>
<td>2800.6</td>
<td>90.9014</td>
<td>1.8804</td>
<td>1.0557</td>
</tr>
<tr>
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<td>2971.9</td>
<td>89.79</td>
<td>1.8639</td>
<td>0.9641</td>
</tr>
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<td>1.8498</td>
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<td>88.0436</td>
<td>1.8377</td>
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</tr>
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<td>3509.0</td>
<td>87.3448</td>
<td>1.8271</td>
<td>0.7521</td>
</tr>
<tr>
<td>0.09</td>
<td>3695.5</td>
<td>86.7340</td>
<td>1.8179</td>
<td>0.6967</td>
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<tr>
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<td>4889.4</td>
<td>84.2566</td>
<td>1.7800</td>
<td>0.6153</td>
</tr>
</tbody>
</table>

10.5 CONCLUSION

In the present model we have developed an inventory model for differential items which are received at the primary shop. We assume that at the primary shop only non-defective items are sold. Defective items identified at this shop are transferred to the adjacent secondary shop and the items are reworked and then sold. We considered the concepts like partial backlogging, backorder price discount, inflation in our model. We also studied the effects of these concepts through our numerical examples. Increase of backlogging parameter shows the trader’s ability to give backorder price discount for more orders. Through more orders the trader’s profit will increase (shown in Table 10.ii). By offering a larger price discount for backorders a trader can secure more customers and raise his overall profit (shown in Table 10.iii). Classical inventory models failed to take into account the effects of inflation. But our model considers the inflation (a necessary evil) and its effects on inventory policy variables. Our model shows that inflation can cause considerable profit. From Table 10.iv we can say that a higher inflation lowers the inventory policy variables like Q, T1 and T2.
In the proposed model, during the stock out period a fraction $B$ of the demand is backordered, we assume that $0 \leq B \leq 1$. If $B = 1$, we are getting the model by Jayanta Kumar Dey et al., [50] in which there is no price discount and inflation. We also took inflation rate $i$ as $0 \leq i < 1$. If $i = 0$ we get the model by Jayanta Kumar Dey et al., [50] in which only complete backlogging was considered. The endeavour of our model is to maximize the profit even though we consider the additional expenditure and losses (backorder price discount, partial backlogging and lost sales). Numerical examples proved this.