CHAPTER 4

DEVELOPMENT OF DETERMINISTIC INVENTORY MODEL WITH STOCK DEPENDENT DEMAND

4.1 INTRODUCTION

In Chapter 3, a deterministic inventory model for time dependent demand, partial backlogging shortages under permissible delay was developed. It is usually observed in the super market, that display of consumer goods in large quantities attracts more customers and generates higher demand. Wolfe [120], presented empirical evidence of this relationship, noting that the sales of style merchandise, such as women’s dress or sports clothes, are depending on inventory amount displayed. Thus increased inventory levels give the customer a wider selection and increase the probability of making a sale. More researchers like Backer and Urban, Datta and Pal, Gupta and Vrat, Silver and Peterson [4, 22, 39, 101] considered the situations in which demand rate depends on the stock level. However, to the best of my knowledge the situation where the demand and deterioration, which are non-linear functions (explanation is given in page 3) of stock under permissible delay in payments is not addressed. This chapter is arranged as follows. In the next section, we present problem description, assumptions and notations that are employed for the development of the model. Then our model formulation is given in section 3. In the penultimate section numerical examples and sensitivity analysis of parameters are shown. In section 5, we conclude with our results.
4.2 PROBLEM DESCRIPTION

An inventory model is developed for the situation where the demand and deterioration are non-linear functions of stock. Some part of the shortage demand is met and the remaining is lost. That is shortages are partially backlogged. The backlogged rate is inversely proportional to the waiting time for the next replenishment. The vital roles of permissible delay in payments and backlogging rate are clearly discussed. The total annual inventory cost is minimized.

4.2.1 ASSUMPTIONS

We use the notations and first four assumptions as in section 3.2.1. The remaining assumption is given below:

The demand rate function \( R(t) \) is deterministic and is a known function of instantaneous stock level \( I(t) \). The functional \( R(t) \) is given by

\[
R(t) = \begin{cases} 
\alpha + \beta \sqrt{I(t)} & \text{if } 0 \leq t < T_1 \\
\alpha & \text{if } T_1 \leq t < T
\end{cases}
\]

where \( \alpha > 0 \) and \( 0 < \beta < 1 \).

4.2.2 NOTATIONS

We use the notations as in section 3.2.2.

4.3 MODEL FORMULATION

The inventory is exhausted due to the combined effects of demand and deterioration in the interval \([0, T_1]\). The excess demand is partially backlogged in the interval \([T_1, T]\) as shown in the figures 3a and 3b.
Hence the change in the inventory level, $I(t)$, with respect to time can be described by the following differential equation.

$$
\frac{dI(t)}{dt} = \begin{cases} 
-\alpha - \beta \sqrt{I(t)} - \theta \sqrt{I(t)} & \text{if } 0 \leq t < T_1 \\
-\frac{\alpha}{1 + \delta(T - t)} & \text{if } T_1 \leq t < T
\end{cases}
$$  \hspace{1cm} (26)

with the boundary condition $I(T_1) = 0$.

Consider the first case, i.e., $0 \leq t < T_1$,

$$
\frac{dI(t)}{dt} = -\alpha - \beta \sqrt{I(t)} - \theta \sqrt{I(t)}
$$

$$
\frac{-\alpha - (\beta + \theta) \sqrt{I(t)}}{\alpha} = dt
$$

Integrating both sides, we get

$$
-\frac{2\alpha}{(\beta + \theta)^2} - \frac{I(t)}{\alpha} = t + c_1 \quad \text{where } c_1 \text{ is the constant of integration.}
$$

Using the boundary condition $I(T_1) = 0$ and simplifying we get

$$
I(t) = \alpha \left( T_1 - t \right).
$$

When $T_1 \leq t < T$,

$$
\frac{dI(t)}{dt} = -\frac{\alpha}{1 + \delta(T - t)}
$$

Its solution is

$$
I(t) = -\frac{\alpha}{\delta} \left\{ \log \left[ 1 + \delta(T - T_1) \right] - \log \left[ 1 + \delta(T - t) \right] \right\}
$$

$$
I(t) = \begin{cases} 
\alpha(T_1 - t) & \text{if } 0 \leq t < T_1 \\
-\frac{\alpha}{\delta} \left\{ \log \left[ 1 + \delta(T - T_1) \right] - \log \left[ 1 + \delta(T - t) \right] \right\} & \text{if } T_1 \leq t < T
\end{cases}
$$  \hspace{1cm} (27)

The stock holding cost in the interval $[0, T_1)$ can be written as

$$
\text{Holding Cost} = \int_0^{T_1} I(t) dt
$$
\[ \text{Holding Cost} = \frac{h \alpha T_1^2}{2} \] (28)

Deterioration cost in \([0, T_1]\) is given by

\[ \text{Deterioration Cost} = P \theta \int_0^{T_1} I(t) \, dt \]

\[ \text{Deterioration Cost} = \frac{P \alpha \theta T_1^2}{2} \] (29)

We have to consider two costs in the shortage period. The first is to find the shortage cost for the backlogged items and the second is to find the opportunity cost due to lost sales.

The shortage cost in the interval \([T_1, T]\) is given by

\[ \text{Shortage Cost} = S \int_{T_1}^{T} -I(t) \, dt \]

\[ = \frac{S \alpha}{2} \int_{T_1}^{T} \left\{ \log \left[ 1 + \delta \left( T - T_1 \right) \right] - \log \left[ 1 + \delta \left( T - t \right) \right] \right\} \, dt \]

\[ \text{Shortage Cost} = \frac{S \alpha}{2} \left\{ \delta \left( T - T_1 \right) - \log \left[ 1 + \delta \left( T - T_1 \right) \right] \right\} \] (30)

The opportunity cost due to lost sales is given by

\[ \text{Opportunity Cost} = \pi \int_{T_1}^{T} \alpha \left[ \frac{1}{1 + \delta (T - t)} \right] \, dt \]

\[ = \frac{\pi \alpha}{\delta} \left\{ \delta (T - T_1) - \log \left[ 1 + \delta (T - T_1) \right] \right\} \] (31)

\[ \text{Opportunity Cost} = \frac{\pi \alpha}{\delta} \left\{ \delta (T - T_1) - \log \left[ 1 + \delta (T - T_1) \right] \right\} \]

Now, consider the supplier's credit period \(M\) in settling the accounts. There are two cases to be considered. Case 1: \(M \leq T_1\) or case 2: \(M > T_1\). We discuss these two cases one by one.
Case 1: $M \leq T_1$

Since the length of the period with positive stock of the items is greater than the credit period, the retailer can use the sale revenue with an annual rate $I_c$ in $[0, T_1)$.

The interest earned is

$$Interest\ Earned = PI_c \int_0^{T_1} (T_1 - t) R(t) dt$$

$$= PI_c \int_0^{T_1} (T_1 - t) \left[ \alpha + \beta \sqrt{I(t)} \right] dt$$

$$= PI_c T_1^2 \left\{ \frac{\alpha}{2} + \frac{2\beta \sqrt{\alpha}}{5} \right\}^{T_1}$$

(32)

After the credit period the retailer has to pay the interest for the goods still in stock with annual rate $I_r$. We can find the interest payable as follows.

$$Interest\ Payable = PI_r \int_M^{T_1} I(t) dt$$

$$= \frac{P\alpha I_r (T_1 - M)^2}{2}$$

(33)

Therefore, the total annual inventory cost in case 1 is given by

$$TVC_1(T_1, T) = \frac{1}{T} \left\{ K + \frac{aT_1}{2} + \frac{P\alpha T_1^2}{2} + \frac{S\alpha}{\delta^2} \{\delta(T - T_1) - \log[1 + \delta(T - T_1)]\} \right\}$$

$$+ \frac{\pi \alpha}{\delta} \{\delta(T - T_1) - \log[1 + \delta(T - T_1)]\} + \frac{P\alpha I_r (T_1 - M)^2}{2}$$

$$- \frac{PI_c \alpha T_1^2}{2} - \frac{2PI_c \beta \sqrt{\alpha T_1}}{5}$$

$$TVC_1(T_1, T) = \frac{1}{T} \left\{ K + \frac{a(h + P\theta) T_1^2}{2} + \frac{\alpha(S + \pi \delta)}{\delta^2} \{\delta(T - T_1) - \log[1 + \delta(T - T_1)]\} \right\}$$
\[
\begin{align*}
+ \frac{P al_1 (T_1 - M)^2}{2} - \frac{P al_e T_1^2}{2} - \frac{2 PI_e \beta \sqrt{\alpha T_1^{3/2}}}{5} \Bigg) \Bigg] \tag{34}
\end{align*}
\]

Our objective is to minimize the total annual inventory cost. For this, we have to find the optimal solutions of \(T_1\) and \(T\) (say \(T_1^*\) and \(T^*\)). They can be found by solving the following equations simultaneously.

\[
\frac{\partial TVC_1(T_1, T)}{\partial T_1^*} = 0 \quad \text{and} \quad \frac{\partial TVC_1(T_1, T)}{\partial T} = 0
\]

provided they satisfy the sufficient conditions

\[
\left[ \frac{\partial^2 TVC_1(T_1, T)}{\partial T_1^2} \right]_{(T_1^*, T^*)} > 0, \quad \left[ \frac{\partial^2 TVC_1(T_1, T)}{\partial T^2} \right]_{(T_1^*, T^*)} > 0 \quad \text{and}
\]

\[
\left\{ \frac{\partial^2 TVC_1(T_1, T)}{\partial T_1^2} \left[ \frac{\partial^2 TVC_1(T_1, T)}{\partial T^2} \right]^2 - \frac{\partial TVC_1(T_1, T)}{\partial T_1^2} \frac{\partial TVC_1(T_1, T)}{\partial T} \right\} > 0
\]

\[
\frac{\partial TVC_1(T_1, T)}{\partial T_1} = 0 \Rightarrow
\]

\[
\frac{1}{T} \left\{ \alpha h T_1 - \frac{\alpha (S + \pi \delta) (T - T_1)}{1 + \delta (T - T_1)} + P al_1 T_1 P al_e T_1 - P al_1 T_1^{3/2} - P al_e \alpha M \right\} = 0
\]

\[
\tag{36}
\]

\[
\frac{\partial TVC_1(T_1, T)}{\partial T} = 0 \Rightarrow
\]

\[
\frac{1}{T} \left\{ \frac{\alpha (S + \pi \delta) (T - T_1)}{1 + \delta (T - T_1)} \right\} - \frac{1}{2} \left\{ K + \frac{\alpha (h + \pi \delta) T_1^2}{2} + \frac{\alpha (S + \pi \delta) \{ \delta (T - T_1) - \log [1 + \delta (T - T_1)] \}}{\delta^2} \right\} + \frac{P al_1 (T_1 - M)^2}{2} - \frac{P al_e T_1^2}{2} - \frac{2 PI_e \beta \sqrt{\alpha T_1^{5/2}}}{5} = 0
\]

\[
\tag{37}
\]

58
TVC\(_1(T_1, T)\) is convex with respect to \(T_1\) and \(T\) (Appendix 4.1).

To obtain the optimal values of \(T_1\) and \(T\), we propose the following algorithm.

**Algorithm – 1**

**Step 1: Obtaining \(T_1\) and \(T\)**

Perform (i) – (iv)

(i) Start with \(T_{1, (1)} = M\)

(ii) Substitute \(T_{1, (1)}\) in equation (36) and find \(T_{(1)}\)

(iii) Using \(T_{(1)}\) in equation (37) we can find \(T_{1, (2)}\)

(iv) Repeat (ii) and (iii) until no change occurs in the values of \(T_1\) and \(T\).

**Step 2: Comparing \(T_1\) and \(M\) and finding \(T_1^*, T^*\)**

(i) If \(M \leq T_1\), \(T_1\) is feasible then go to step 3.

(ii) If \(M > T_1\), \(T_1\) is not feasible. Setting \(T_1 = M\) in the equation (37) to calculate \(T\). These values of \(T_1\) and \(T\) are \(T_1^*\) and \(T^*\) respectively. Go to step 3.

**Step 3: Calculate TVC\(_1\) (\(T_1^*, T^*\))**

**Case 2: \(M > T_1\)**

In this case, the retailer need not pay interest and he earns the interest with annual rate \(I_e\). The interest earned is given by

\[
\text{Interest Earned} = \left[ \int_0^{T_1} (M - t) R(t) \, dt \right] I_e \quad \text{(37)}
\]

\[
\text{Interest Earned} = \left[ \left( \frac{\alpha}{2} + \frac{2\beta \sqrt{\alpha T_1}}{5} \right) + \left( M - T_1 \right) T_1 \left( \frac{\alpha + 2\beta \sqrt{\alpha T_1}}{3} \right) \right] I_e \quad \text{(38)}
\]
The total annual inventory cost in this case is given by

\[
TVC_2(T_1, T) = \frac{1}{T} \left\{ \frac{K + \frac{\alpha (h + P \theta) T_1^2}{2} + \frac{\alpha (s + \pi \delta)}{2} \{ \delta (T - T_1) - \log [1 + \delta (T - T_1)] \}}{2} \right. \\
- \frac{P \alpha I_e T_1^2}{2} - \frac{2 P I_e \beta \sqrt{\alpha} T_1^{5/2}}{5} - PI_e \alpha \left( M T_1 - T_1^2 \right) \\
- \frac{2 PI_e \beta \sqrt{\alpha}}{3} \left[ M T_1^{3/2} - T_1^{5/2} \right] \right\}
\]

(39)

Our aim is to minimize the total annual inventory cost. To achieve this we have to find the optimal values of \( T_1 \) and \( T \) which are the solutions of the following equations.

\[
\frac{\partial TVC_2(T_1, T)}{\partial T_1} = 0 \quad \text{and} \quad \frac{\partial TVC_2(T_1, T)}{\partial T} = 0
\]

(40)

provided they satisfy the sufficient conditions

\[
\left[ \frac{\partial^2 TVC_2(T_1, T)}{\partial T_1^2} \right]_{(T_1^*, T^*)} > 0, \quad \left[ \frac{\partial^2 TVC_2(T_1, T)}{\partial T^2} \right]_{(T_1^*, T^*)} > 0 \quad \text{and}
\]

\[
\left\{ \begin{array}{l}
\left[ \frac{\partial^2 TVC_2(T_1, T)}{\partial T_1^2} \right] \left[ \frac{\partial^2 TVC_2(T_1, T)}{\partial T^2} \right] - \left[ \frac{\partial^2 TVC_2(T_1, T)}{\partial T_1 \partial T} \right]^2 \\
\end{array} \right\} > 0.
\]

\[
\left[ \frac{\partial TVC_2(T_1, T)}{\partial T_1} \right] = 0 \Rightarrow
\]

60
\[
\frac{1}{T} \left\{ \alpha (h + P \theta) T_1 - \frac{\alpha (S + \pi \delta) (T - T_1)}{1 + \delta (T - T_1)} + P \alpha I_e T_1 - \frac{P \alpha I_e M - P I_e \beta \sqrt{\alpha} M \sqrt{T_1}}{3} \right\} = 0
\]

(41)

\[
\frac{\partial TVC_2(T_1, T)}{\partial T} = 0 \implies

\frac{1}{T} \left[ \alpha (S + \pi \delta) (T - T_1) \right]
\]

\[
- \frac{1}{T^2} \left\{ K + \alpha \left( \frac{h + P \theta) T_1}{2} + \frac{\alpha (S + \pi \delta)}{\delta^2} \right\} \left[ \delta(T - T_1) - \log [1 + \delta(T - T_1)] \right] \right\}
\]

\[
+ \frac{P \alpha I_e T_1}{2} - \frac{P \alpha I_e M T_1}{3} - \frac{2 P I_e \beta \sqrt{\alpha} M T_1}{3} 
\]

\[
+ \frac{4 P I_e \beta \sqrt{\alpha} T_1}{15} \right\} = 0
\]

(42)

TVC_2(T_1, T) is convex with respect to T_1 and T (Appendix 4.2).

To find the optimal values T_{1*} and T* we propose the following algorithm.

Algorithm - 2

Step 1: Obtaining T_1 and T

Perform (i) – (iv)

(i) Start with T_{1,(1)} = M
(ii) Substitute T_{1,(1)} in equation (41) and find T_{(1)}
(iii) Using T_{(1)} in equation (42) we can find T_{1,(2)}
(iv) Repeat (ii) and (iii) until no change occurs in the values of T_1 and T
Step 2: Comparing \( T_1 \) and \( M \)

(i) If \( T_1 < M \), \( T_1 \) is feasible. Then go to step 3.
(ii) If \( T_1 \geq M \), \( T_1 \) is not feasible. Set \( T_1 = M \) in the equation (42) to calculate \( T \). Thus we obtain \( T_1^* \) and \( T^* \) and go to step 3.

Step 3: Calculate TVC\(_2\) (\( T_1^* \), \( T^* \))

Our main aim is to find the optimal values of \( T_1 \) and \( T \) which minimize the total annual cost TVC (\( T_1 \), \( T \)), we find that

\[
\text{TVC}(T_1^*, T^*) = \text{Min}[\text{TVC}_1(T_1^*, T^*), \text{TVC}_2(T_1^*, T^*)].
\]

4.4 NUMERICAL EXAMPLES

Numerical examples are given to discuss the sensitivity of the parameters \( M \) and \( \delta \). Computational works are carried out using MATLAB 7.0. Some data input values were taken from earlier published work [17].

Example 1

Let \( K=200 \), \( \alpha=1000 \), \( \beta=0.3 \), \( P=20 \), \( h=1.2 \), \( S = 30 \), \( \pi=15 \), \( L_t=0.13 \), \( L_r=0.15 \), \( \theta=0.08 \), \( M=20/365 \), \( \delta=3 \). We get the optimal values as \( T_1^*=0.1633 \), \( T^*=0.1681 \), TVC \((T_1^*, T^*) \) = 1314.46.

Example 2

Let \( K=200 \), \( \alpha=1000 \), \( \beta=0.3 \), \( P=20 \), \( h=1.2 \), \( S = 30 \), \( \pi=15 \), \( L_t=0.13 \), \( L_r=0.15 \), \( \theta=0.08 \), \( M=60/365 \), \( \delta=5 \). We get the optimal values as \( T_1^*=0.1486 \), \( T^*=0.1522 \), TVC\((T_1^*, T^*) \) = 1188.96.
Example 3

Let $K=200$, $\alpha=1000$, $\beta=0.3$, $P=20$, $h=1.2$, $S=30$, $\pi=15$, $I_c=0.13$, $I_r=0.15$, $\theta=0.08$, $M$ is taken as $5/365$ for different values of $\delta$ say $\delta=1,2,3,10,25$ and $50$. The optimal values obtained are shown in Table 4.i. From Table 4.i we see that the total annual inventory cost increases to a greater extent for large values of $\delta$. So in order to minimize the total annual inventory cost the retailer should improve the backlogging rate.

Example 4

Let $K=200$, $\alpha=1000$, $\beta=0.3$, $P=20$, $h=1.2$, $S=30$, $\pi=15$, $I_c=0.13$, $I_r=0.15$, $\theta=0.08$, $\delta$ is taken as 4 for different values of $M$ say $M = 5/365$, $15/365$, $30/365$ $35/365$, $45/365$, $55/365$. The results are given in Table 4.ii. From Table 4.ii we can say that for a fixed value of $\delta$ the total annual inventory cost decreases as $M$ increases. Hence the retailer can minimize his total annual inventory cost if he gets longer permissible delay period from the supplier.

Table 4.i Sensitivity analysis on optimal values of total annual inventory cost for various values of $\delta$

<table>
<thead>
<tr>
<th>$M$</th>
<th>Inventory Measures</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>TVC($T_1^<em>, T^</em>$)</td>
<td>1312.74</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.1748</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.1864</td>
</tr>
</tbody>
</table>
Table 4.ii Sensitivity analysis on optimal values of total annual inventory cost for various values of M

<table>
<thead>
<tr>
<th>M</th>
<th>Inventory Measures</th>
<th>$\delta=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>4800.35</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.0274</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.0284</td>
</tr>
<tr>
<td>15</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>4198.07</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.0474</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.0477</td>
</tr>
<tr>
<td>30</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>2126.13</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.1438</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.1502</td>
</tr>
<tr>
<td>35</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>1079.66</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.6590</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.6808</td>
</tr>
<tr>
<td>45</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>1101.88</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.1878</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.1903</td>
</tr>
<tr>
<td>55</td>
<td>TVC($T_1^<em>$, $T^</em>$)</td>
<td>914.79</td>
</tr>
<tr>
<td></td>
<td>$T_1^*$</td>
<td>0.2321</td>
</tr>
<tr>
<td></td>
<td>$T^*$</td>
<td>0.2353</td>
</tr>
</tbody>
</table>
4.5 CONCLUSION

In this chapter, we developed an inventory model with stock-dependent demand and shortages under the condition of permissible delay in payments and also we assumed that both demand and deterioration are non-linear functions of stock. Further we assumed that the backlogging rate is a decreasing function of the waiting time for the next replenishment. This assumption is more practical in real life. Replacing $T$ by $T_1$ and $\beta$ by zero, the model is similar to the model by Goyal [36] although he did not consider the interest earned for the rest of the period in the cycle when the credit period was less than cycle length and the deterioration was taken as a linear function of the stock.

Furthermore, the results of the sensitivity analysis are also consistent with the economic incentives. For fixed $M$ the total annual inventory cost increases with increasing value of $\delta$. Hence in order to minimize total annual inventory cost the retailer should increase the backlogging rate. For fixed $\delta$, there is a significant decrease in the optimal annual inventory cost as $M$ increases.