APPLICATIONS OF MULTIVARIABLE $H$-FUNCTION OF SRIVASTAVA-PANDA AND THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA-DAOUST IN TWO BOUNDARY VALUE PROBLEMS
APPLICATIONS OF MULTIVARIABLE
H-FUNCTION OF SRIVASTAVA-PANDA AND
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BOUNDARY VALUE PROBLEMS

ABSTRACT

In the present chapter, we shall employ, multivariable $H$-function of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]) and multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; also see Srivastava and Karlsson [9]) in two boundary value problems. First we shall evaluate an integral involving the product of multivariable $H$-function of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]) and the generalized multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; also see Srivastava and Karlsson [9]) and then we make its applications to solve

I. a boundary value problem on heat conduction in a finite bar and to establish an expansion formula involving the product of the above multivariable $H$-function and generalized multiple hypergeometric function.

II. another boundary value problem on electrostatic potential in spherical region.

Problem-1 A Problem on Heat Conduction in a Finite Bar.

7.1. Introduction. Chandel and Yadava [1] have discussed a
problem on heat conduction involving the multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; also see Srivastava and Karlsson [9]). Srivastava, Gupta and Goyal [13] have discussed a problem on heat conduction in a finite bar using $H$-function of two variables of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]). Further Chandel and Gupta [2] have discussed this problem involving multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; also see Srivastava and Karlsson [9]).

Here in the present chapter, we discuss the same problem by employing the product of multivariable $H$-function of Srivastava-Panda [10],[11],[12] and multiple hypergeometric function of Srivastava-Daoust [6],[7],[8]. First we evaluate the integral involving the product of these functions and then we make its applications to solve the problem on heat conduction in a finite bar and to establish an expansion formula involving the product of the above functions.

7.2. Main Integral. In this section, we evaluate the following integral very useful in our further investigations:

\[
(7.2.1) \quad \int_{-1}^{1} (1-x)^{\alpha-1}(1+x)^{\beta-1} P_m(x) F_{\alpha,\beta,\gamma,\delta}^{\mu,\nu,\xi,\eta}(\ldots; \theta, \phi; \psi, \delta; \ldots)
\]

\[
\left(\begin{array}{c}
(f^r); \eta^{(r)}; \ldots; \eta^{(r)}; \\
(h^r), e^{(r)}; \ldots; h^{(r)}
\end{array}\right), y_1(1-x)^{\alpha}, ..., y_r(1-x)^{\gamma}(1+x)^{\delta}
\]

\[
H^{\theta, \phi}_{\alpha, \beta, \gamma, \delta} \left(\begin{array}{c}
(\theta); \phi; \psi; \delta; \ldots
\end{array}\right)
\]
\[
\int z_{1}(1-x)^{\eta_{1}}(1+x)^{-\eta_{1}} \ldots z_{n}(1-x)^{\eta_{n}}(1+x)^{-\eta_{n}} \, dx
\]

\[
= 2^{\alpha+\beta-1} \sum_{k=0}^{m} \frac{(-m)_{k}(m+k)_{k}}{k!(m+k)_{k}} \sum_{m_{1}, \ldots, m_{n}=0}^{\infty} \frac{\prod_{j=1}^{K}(e_{j}, m_{j}, \xi_{j}^{(r)}, \ldots + m_{j}, \xi_{j}^{(r)}) \prod_{j=1}^{L}(f_{j}, m_{j}, \eta_{j}^{(r)})}{\prod_{j=1}^{L}(g_{j}, m_{j}, \xi_{j}^{(r)}, \ldots + m_{j}, \xi_{j}^{(r)}) \prod_{j=1}^{L}(h_{j}, m_{j}, \eta_{j}^{(r)})} \cdot \frac{\prod_{j=1}^{L}(f_{j}^{(r)}, m_{j}, \eta_{j}^{(r)})}{\prod_{j=1}^{L}(h_{j}^{(r)}, m_{j}, \eta_{j}^{(r)})}
\]

\[
\frac{\prod_{j=1}^{L}(f_{j}^{(r)}, m_{j}, \eta_{j}^{(r)})}{\prod_{j=1}^{L}(h_{j}^{(r)}, m_{j}, \eta_{j}^{(r)})}
\]

\[
H^{0, \lambda+2}_{\mu+2, \lambda+2}[\gamma^{(s)}, \gamma^{(t)}] = \left[\begin{align*}
(a) \colon \Theta^{(r)}, \ldots, \Theta^{(r)} \colon \\
(c) \colon \Psi^{(t)}, \ldots, \Psi^{(t)} \colon
\end{align*}\right]
\]

\[
[1 - \alpha - k - \lambda_{r} m_{r} \ldots - \lambda_{r} m_{r} : \xi_{1}^{(r)}, \ldots, \xi_{n}^{(r)}]^{L} [1 - \beta - \mu_{r} m_{r} \ldots - \mu_{r} m_{r} : \eta_{1}^{(r)}, \ldots, \eta_{n}^{(r)}]^{L} [1 - \alpha - \beta - k - (\lambda_{r} + \mu_{r}) m_{r} \ldots - (\lambda_{r} + \mu_{r}) m_{r} : \xi_{1}^{(r)}, \ldots, \xi_{n}^{(r)}]^{L}
\]

\[
\left[\begin{align*}
(b^{(r)}), \phi^{(s)} \ldots [b^{(r)}), \phi^{(s)} \colon] _{\gamma} \colon z_{1}^{2}, \ldots, z_{n}^{2} ; z_{1}^{2}, \ldots, z_{n}^{2}
\end{align*}\right]
\]

where \(Re(\alpha)>0, Re(\beta)>0\), all \(\lambda_{r}, \mu_{r}, y_{j}(i=1, \ldots, r); \xi_{r}^{(s)} \eta_{r} z_{i}(i=1, \ldots, n)\) are real positive numbers and \(\arg(z_{1}(1-x)^{\eta_{1}}(1+x)^{\eta_{1}}) < \frac{\pi}{2} \Delta_{r}\);

\[
\Delta_{r} = - \sum_{j=1}^{A} \theta_{j}^{(r)} + \sum_{j=1}^{B} \phi_{j}^{(r)} - \sum_{j=1}^{C} \phi_{j}^{(r)} - \sum_{j=1}^{D} \psi_{j}^{(r)} + \sum_{j=1}^{E} \delta_{j}^{(r)} - \sum_{j=1}^{F} \delta_{j}^{(r)} > 0
\]
and

\[ 1 + \sum_{j=1}^{G} \xi_j^k + \sum_{j=1}^{E} \xi_j^{(k)} \geq \sum_{j=1}^{F} \eta_j^{(k)} - \sum_{j=1}^{G} \eta_j^k > 0, k = 1, \ldots, r. \]

Here \( F_{G,H^{(r)}}^{k,...,k^{(r)}} \) stands for multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; see also Srivastava and Karlsson[9])

while \( H_{A,C}^{0,\lambda,...,\lambda^{(r)}} \left[ \begin{array}{c} \alpha^{(r)} \end{array} \right]^m \) stands for multivariable \( H \)-function of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]).

**Proof.** The left hand side of (7.2.1)

\[
= \sum_{m_1, \ldots, m_r = 0}^{\infty} \prod_{j=1}^{J} \left( e_j, m_j \xi_j \right) \frac{\prod_{j=1}^{J} \left( f_j, m_j \eta_j \right)^{m_j}}{\prod_{j=1}^{J} \left( g_j, m_j \xi_j \right) \prod_{j=1}^{J} \left( h_j, m_j \eta_j \right)^{m_j}} \frac{y_1^{m_1} \cdots y_r^{m_r}}{m_1! \cdots m_r!}
\]

\[
\frac{1}{(2\pi i)^n} \left[ \int_{L_1} \cdots \int_{L_n} \prod_{j=1}^{n} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_j) \prod_{j=1}^{n} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_j) \right]
\]

\[
= \frac{1}{\prod_{j=\lambda+1}^{A} \Gamma \left( a_j - \sum_{i=1}^{n} \theta_j^{(i)} s_i \right)} \frac{1}{\prod_{j=\lambda+1}^{A} \Gamma \left( 1 - \left( a_j + \sum_{i=1}^{n} \theta_j^{(i)} s_i \right) \right)} \prod_{j=\lambda+1}^{A} \Gamma \left( b_j - \sum_{i=1}^{n} \phi_j^{(i)} s_i \right) \frac{z_1^{u_1} \cdots z_n^{u_n}}{ds_1 \cdots ds_n}
\]
\[
\int_{-1}^{1} (1-x)^{\alpha-1} x^{\beta-1} P_n(x) dx = 2^{\alpha+\beta-1} B(\alpha, \beta) \frac{\Gamma(\alpha+k)}{k! \Gamma(\alpha+\beta+k)} \left[ \begin{array}{c} -n, n+1, \alpha; \\ 1, 1+\beta; \end{array} \right] 
\]

gives right hand side of (7.2.1).

**7.3 Problem-1.** In this section, we consider the problem of determining a function \(u(x,t)\) representing the temperature in a non-homogeneous bar with ends at \(x=-1\) and \(x=1\) in which the thermal conductivity is proportional to \((1-x^2)\). Let the lateral surface of the bar be insulated. Thus our problem reduces to solve the equation of heat conduction in one dimension,

\[
(7.3.1) \quad \frac{\partial u}{\partial t} = k \frac{\partial}{\partial x} \left[ (1-x^2) \frac{\partial u}{\partial x} \right],
\]

where \(k\) and thermal coefficients both are constants.

**7.4. Solution of the Problem.** Boundary conditions of the problem are that both ends of the bar at \(x=\pm 1\) are insulated so that conductivity vanishes there and the initial condition is

\[
(7.4.1) \quad u(x,0)=f(x).
\]

Here we may assume the solution of the problem (7.3.1) in the form:

\[
(7.4.2) \quad u(x,t) = \sum_{n=0}^{\infty} A_n e^{-kn(\alpha+\beta)k} P_n(x),
\]
which is quite justified and for \( t=0 \), reduces to

\[
(7.4.3) \quad u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n P_n(x).
\]

Now we may choose

\[
(7.4.4) \quad f(x) = (1-x)^{\alpha-1} (1+x)^{\beta-1} F_{\gamma; \delta; \mu; \nu}^{\alpha; \beta; \gamma; \delta; \mu; \nu}(\begin{array}{c}
(e): \xi^1, \ldots, \xi^r \\
\end{array} ; \begin{array}{c}
(g): \zeta^1, \ldots, \zeta^r \\
\end{array} ; \begin{array}{c}
[f^{(r)}], \eta^1, \ldots, f^{(r)} \eta^r \\
[h^{(r)}], \epsilon^1, \ldots, h^{(r)} \epsilon^r \\
\end{array});
\]

\[
\begin{aligned}
&y_1(1-x_i)^{\gamma_1}(1+x)^{\gamma_i}, \ldots, y_r(1-x_i)^{\gamma_r}(1+x)^{\gamma_r} \\
&z_1(1-x)^{\gamma_1}(1+x)^{\gamma}, \ldots, z_n(1-x)^{\gamma_n}(1+x)^{\gamma_n},
\end{aligned}
\]

where \( F_{\gamma; \delta; \mu; \nu}^{\alpha; \beta; \gamma; \delta; \mu; \nu}(a; \theta^1, \ldots, \theta^a \mid b; \phi^1, \ldots, \phi^b \mid c; \psi^1, \ldots, \psi^c \mid d; \delta^1, \ldots, \delta^d) \) stands for multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; see also Srivastava and Karlsson[9])

and \( H_{\gamma; \delta; \mu; \nu}^{\alpha; \beta; \gamma; \delta; \mu; \nu}(a; \theta^1, \ldots, \theta^a \mid b; \phi^1, \ldots, \phi^b \mid c; \psi^1, \ldots, \psi^c \mid d; \delta^1, \ldots, \delta^d) \) stands for multivariable \( H \)-function of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]).

Now by (7.4.3) and (7.4.4), we have

\[
\begin{aligned}
f(x) &= (1-x)^{\alpha-1} (1+x)^{\beta-1} F_{\gamma; \delta; \mu; \nu}^{\alpha; \beta; \gamma; \delta; \mu; \nu}(\begin{array}{c}
(e): \xi^1, \ldots, \xi^r \\
\end{array} ; \begin{array}{c}
(g): \zeta^1, \ldots, \zeta^r \\
\end{array} ; \begin{array}{c}
[f^{(r)}], \eta^1, \ldots, f^{(r)} \eta^r \\
[h^{(r)}], \epsilon^1, \ldots, h^{(r)} \epsilon^r \\
\end{array});
\end{aligned}
\]
\[
\left[ (f^r), \eta^l \right] : \ldots \left[ (f^{(r)}), \eta^{(l)} \right] ; \\
\left[ (h^r), \varepsilon^l \right] : \ldots \left[ (h^{(r)}), \varepsilon^{(l)} \right] ;
\]
\[
y_1 (1-x_1)^{\varepsilon_1} (1+x_1)^{\eta_1} \ldots y_r (1-x_1)^{\varepsilon_r} (1+x_1)^{\eta_r} \]

\[
H_{\mathcal{A}, \mathcal{C} \langle \mathcal{B}^r, \mathcal{D}^l \rangle} \left( \left[ (a) : \Theta^r \ldots \Theta^{(r)} ; \left[ (b^r) : \phi^l \ldots \phi^{(l)} \right] ; \\
\left[ (c) : \psi^l \ldots \psi^{(l)} \right] ; \left[ (d^l) : \delta^r \ldots \delta^{(r)} \right] \right) ;
\]
\[
z_1 (1-x)^{\varepsilon_1} (1+x)^{\eta_1} \ldots z_n (1-x)^{\varepsilon_n} (1+x)^{\eta_n} \]

\[
= \sum_{n=0}^{\infty} A_n P_n(x) .
\]

Therefore,
\[
\int_{-1}^{1} (1-x)^{\varepsilon_1} (1+x)^{\eta_1} \ldots \int_{-1}^{1} \ldots \int_{-1}^{1} F_{\mathcal{G}, \mathcal{H} \langle \mathcal{P}^r, \mathcal{Q} \rangle} \left( \left[ (e) : \xi^r \ldots \xi^{(r)} \right] ; \\
\left[ (g) : \zeta^l \ldots \zeta^{(l)} \right] \right) ;
\]

\[
\left[ (f^r), \eta^l \right] : \ldots \left[ (f^{(r)}), \eta^{(l)} \right] ; \\
\left[ (h^r), \varepsilon^l \right] : \ldots \left[ (h^{(r)}), \varepsilon^{(l)} \right] ;
\]
\[
y_1 (1-x_1)^{\varepsilon_1} (1+x_1)^{\eta_1} \ldots y_r (1-x_1)^{\varepsilon_r} (1+x_1)^{\eta_r} \]

\[
H_{\mathcal{A}, \mathcal{C} \langle \mathcal{B}^r, \mathcal{D}^l \rangle} \left( \left[ (a) : \Theta^r \ldots \Theta^{(r)} ; \left[ (b^r) : \phi^l \ldots \phi^{(l)} \right] ; \\
\left[ (c) : \psi^l \ldots \psi^{(l)} \right] ; \left[ (d^l) : \delta^r \ldots \delta^{(r)} \right] \right) ;
\]
\[
z_1 (1-x)^{\varepsilon_1} (1+x)^{\eta_1} \ldots z_n (1-x)^{\varepsilon_n} (1+x)^{\eta_n} \right) P_m(x) dx
\]

\[
= \sum_{n=0}^{\infty} \int_{-1}^{1} A_n P_n(x) P_m(x) dx
\]
\[
= A_m \int_{-1}^{1} P_m^2(x) \, dx \quad \text{(by orthogonal property of Legendre polynomials Eldélyi [4], p. 277 (13))}
\]

\[
= \frac{2A_m}{2m+1}.
\]

Thus

\[
(7.4.5) \quad A_m = (2m+1)2^{m+\beta-2} \sum_{j=0}^{m} \frac{(-m)^{j} (m+1)^{j}}{(j!)^2} \sum_{m_1, m_2, m_3, m_4 = 0}^\infty \frac{\prod_{j=1}^{M} \prod_{i=1}^{k} (f_j, m_i \xi_j^i + ... + m_r \xi_j^{r(i)})}{\prod_{j=1}^{H} \prod_{i=1}^{k} (g_j, m_i \xi_j^i + ... + m_r \xi_j^{r(i)})} \cdot
\]

\[
\frac{\prod_{j=1}^{r} (f_j^r; m_1 \eta_j^r) \cdot \prod_{j=1}^{r} (h_j^r; m_1 \xi_j^r) \cdot y_1^{m_1} \cdots y_r^{m_r}}{\prod_{j=1}^{r} (h_j^r; m_1 \eta_j^r) \cdot \prod_{j=1}^{r} (h_j^r; m_1 \xi_j^r) \cdot m_1! \cdots m_r!} 
\]

\[
H^{0, \lambda+2(\nu_1, \nu_2), ..., [\nu^{(n)}_1, \nu^{(n)}_2]}_{A+2, C+1} \cdot \left[ \begin{array}{c}
(a): \theta', ..., \theta^{(n)} \\
(c): \psi', ..., \psi^{(n)}
\end{array} \right] 
\]

\[
[1 - \alpha - p - \lambda, m_1 - ... - \lambda, m_r : \xi_1, ..., \xi_n], [1 - \beta - \mu, m_1 - ... - \mu, m_r : \eta_1, ..., \eta_n];
\]

\[
[1 - \alpha - \beta - p - (\lambda_1 + \mu_1) m_1 - ... - (\lambda_r + \mu_r) m_r : \eta_1 + \xi_1, ..., \eta_n + \xi_n];
\]

\[
[(b^s) : \phi^s]; ..., [(b^{(n)}) : \phi^{(n)}]; 2^{(\xi_1 + \eta_1)} z_1, ..., 2^{(\xi_n + \eta_n)} z_n
\]
following solution of the problem:

\[ (7.46) u(x, t) = 2^{\alpha + \beta - 2} \sum_{m=0}^{\infty} (2m+1)e^{-m(m+1)t} P_n(x) \sum_{p=0}^{\infty} \sum_{m_1, \ldots, m_r}^\infty \prod_{j=1}^E (e_j, m_1 \xi_j^1 + \cdots + m_r \xi_j^r) \prod_{j=1}^F (g_j, m_1 \xi_j^1 + \cdots + m_r \xi_j^r) \]

\[ \prod_{j=1}^{F'} (f_j', m_1 \eta_j') \prod_{j=1}^{F''} (f_j^{(r)}, m_r \eta_j^{(r)}) \prod_{j=1}^{H'} (h_j', m_1 \varepsilon_j') \prod_{j=1}^{H''} (h_j^{(r)}, m_r \varepsilon_j^{(r)}) \]

\[ \prod_{j=1}^{H'} (h_j', m_1 \varepsilon_j') \prod_{j=1}^{H''} (h_j^{(r)}, m_r \varepsilon_j^{(r)}) \]

\[ \int_{A+2, C+1} [\nu^{(s)}, \rho^{(s)}, \tau^{(s)}] \left( [\alpha]: \Theta_1, \ldots, \Theta_n \right) \left( [\Psi_1, \ldots, \Psi_m] \right) \left( [\beta]: \xi_1, \ldots, \xi_n \right) \left( [\gamma]: \eta_1, \ldots, \eta_n \right) \]

\[ [1 - \alpha - p - \lambda, m_1 - \cdots - \lambda, m_r : \xi_1, \ldots, \xi_n]; [1 - \beta - \mu, m_1 - \cdots - \mu, m_r : \eta_1, \ldots, \eta_n]; [1 - \gamma - 1, m_1 - \cdots - 1, m_r : \xi_1, \ldots, \xi_n] \]

\[ (b') : \phi'; \ldots; (b^{(m)} : \phi^{(m)}), \]

\[ (d') : \delta'; \ldots; (d^{(m)} : \delta^{(m)}) 2^m (\delta_i + \eta_i) z_1, \ldots, 2^m (\delta_i + \eta_i) z_n \]

valid if \( m \) is a positive integer and all conditions of (7.2.1) are satisfied.

### 7.5. Expansion Formula

Making an appeal to (7.4.3) and (7.4.4), we establish

\[ (7.5.1) \quad (1 - x)^{\alpha - 1} (1 + x)^{\beta - 1} F_{E,F \rightarrow G,H}^{(E,F) \rightarrow (G,H)} \left( [e]: \xi^1, \ldots, \xi^r \right); \left( [g]: \xi^1, \ldots, \xi^r \right) \]
\[
\left[ (f^1) : \eta^1 \right] \ldots \left[ (f^r) : \eta^r \right] ;
\left[ (h^1) : e^1 \right] \ldots \left[ (h^s) : e^s \right] ;
\gamma_1 (1 - x_1) \gamma_1^r (1 + x)^{\mu_1} \ldots \gamma_r (1 - x)^{\gamma_r} (1 + x)^{\gamma_r} \\

H_{0, \lambda + 2 (\alpha + \gamma), \ldots \lambda} \left[ (a) : \theta^r, \ldots \theta^{(a)} \right] ;
\left[ (b') : \phi^1 \right] \ldots \left[ (b^{(a)}) : \phi^{(a)} \right] ;
\left[ (c) : \psi^1, \ldots \psi^{(a)} \right] ;
\left[ (d') : \delta^1 \right] \ldots \left[ (d^{(a)}) : \delta^{(a)} \right] ;
\]

\[
z_1 (1 - x_1) \gamma_1^r (1 + x)^{\nu_1} \ldots z_n (1 - x)^{\nu_n} (1 + x)^{\nu_n} \\

= 2^{\alpha + \beta - 2} \sum_{m=0}^{\infty} (2m + 1) P_m (x) \sum_{p=0}^{M} \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{\prod_{j=1}^{s} (e_j, m_j, \xi_j^1 + \ldots + m_r, \xi_j^r)}{\prod_{j=1}^{s} (g_j, m_j, \xi_j^1 + \ldots + m_r, \xi_j^r)} \\
\prod_{j=1}^{s} \left( f^1_j, m_1, \eta_1^1 \right) \ldots \prod_{j=1}^{s} \left( f^r_j, m_r, \eta_r^r \right) \prod_{j=1}^{s} \left( h^1_j, m_1, \xi_1^1 \right) \ldots \prod_{j=1}^{s} \left( h^r_j, m_r, \xi_r^r \right) \frac{\gamma_1^m \ldots \gamma_r^m}{m_1! \ldots m_r!} \gamma_1 \gamma_2 \ldots \gamma_r \\

H_{0, \lambda + 2 (\alpha + \gamma), \ldots \lambda} \left[ (a) : \theta^r, \ldots \theta^{(a)} \right] ;
\left[ (c) : \psi^1, \ldots \psi^{(a)} \right] ,
\]

\[
[1 - \alpha - p - \lambda_1, m_1 \ldots - \lambda_r, m_r : \xi_1, \ldots, \xi_n] [1 - \beta - \mu_1, \ldots - \mu_r, m_r : \eta_1, \ldots, \eta_n] ;
[1 - \alpha - \beta - p - (\lambda_1 + \mu_1) m_1 \ldots - (\lambda_r + \mu_r) m_r : \eta_1 + \xi_1, \ldots, \eta_n + \xi_n] ;
\]

\[
\left[ (b') : \phi^1 \right] \ldots \left[ (b^{(a)}) : \phi^{(a)} \right] \left[ (d') : \delta^1 \right] \ldots \left[ (d^{(a)}) : \delta^{(a)} \right] 2^{\xi_1 + \eta_1} z_1 \ldots 2^{\xi_n + \eta_n} z_n
\]
provided that all conditions of (2.1) are satisfied and \( m \) is positive integer.

**Problem 2.**

7.6. A Problem on Electrostatic Potential in Spherical Regions. Chandel, Agrawal and Kumar [3] have discussed a problem involving multivariable \( H \)-function of Srivastava and Panda ([10],[11],[12]; on electrostatic potential in spherical regions. Here in this section, we shall make applications of the multivariable \( H \)-function of Srivastava and Panda ([10], [11],[12]; also see, Srivastava, Gupta and Goyal [13]) and the generalized multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; also see Srivastava and Karlsson [9]) to obtain the harmonic function \( V \) representing the electrostatic potential in the domain \( R < c \) such that \( V \) assumes a prescribed value \( F(\theta) \) on the spherical surface \( R = c \), where \( R, \theta, \phi \) are the spherical polar coordinates and \( V \) is independent of \( \phi \). Thus \( V \) satisfies Laplace equation

\[
(7.6.1) \quad R \frac{\partial^2 (rV)}{\partial R^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0
\]

in the domain \( R < c \), \( 0 < \theta < \pi \) and under the condition

\[
(7.6.2) \quad \lim_{R \to c} V = F(\theta), (0 < \theta < \pi, R = c)
\]

where \( V \) and its derivatives of first and second orders are assumed to be continuous throughout the interior of the sphere :

\[
0 \leq R < c, 0 \leq \theta < \pi.
\]

Physically, the function \( V \) may represent steady temperature in a solid sphere \( R \leq c \), whose surface temperature depends only on \( \theta \) i.e. the surface temperature is uniform over each circle \( \theta = \theta_0, R = c \).
Here we also consider electrostatic potential in the surface \( R < c \) free of charges.

If we take \( \cos \theta = x \ (0 \leq \theta < p) \),

the equation (6.1) reduces to

\[
(7.6.3) \quad R \frac{\partial^2 (RV)}{\partial R^2} + \frac{\partial}{\partial x} \left[ (1-x^2) \frac{\partial V}{\partial x} \right] = 0, \quad R < c, -1 < x < 1.
\]

If we further take \( F(\theta) = f(\cos \theta) = f(x) \),

then \( V(R, x) \) satisfies the transformed equation (7.6.3), with the boundary conditions

\[
(7.6.4) \quad \lim_{R \to c} V(R, x) = f(x) \quad (R < c, -1 < x < 1),
\]

where \( V \) is continuous every where interior to the sphere and bounded when \( 0 \leq R < R_0 < c \),

\[
(7.6.5) \quad \lim_{R \to \infty} w(R, x) = 0,
\]

where \( w \) is harmonic function in the bounded domain \( R > c \), exterior to the spherical surface and \( RV \) is bounded for large value of \( R \) and for all \( x (-1 \leq x \leq 1) \).

**7.7. Formal Solution of the Problem.** We shall determine formal solution of the above boundary value problem for

\[
(7.7.1) \quad f(x) = (1-x)^{\alpha-1}(1+x)^{\beta-1} \sum_{G,H,...,H^{(r)}}^E_{E^{(r)},...} \left[ (e): \xi_{(-)}, \ldots, \xi^{(r)}_{(-)} ; (e): \xi^{(r)}_{(-)} \right] ;
\]

\[
[(f^{(r)}), \eta_{(-)}^{(r)}] ; \ldots; [(f^{(r)}), \eta^{(r)}] ; y_{(-)}^{(r)}(1-x_{(-)}^{(r)})(1+x_{(-)}^{(r)})^{\mu_{(-)}^{(r)}}, \ldots, y_{(-)}^{(r)}(1-x_{(-)}^{(r)})(1+x_{(-)}^{(r)})^{\nu_{(-)}^{(r)}}
\]

\[
[(h^{(r)}), \epsilon_{(-)}^{(r)}] ; \ldots; [(h^{(r)}), \epsilon^{(r)}] ;
\]

\[
H_{\alpha, \epsilon^{(r)}, \rho^{(r)}}^{\beta, \mu^{(r)}, \nu^{(r)}} \left[ (a): \theta_{(-)}, \ldots, \theta^{(r)}_{(-)} ; (b^{(r)}): \phi^{(r)}_{(-)} ; \ldots; (b^{(r)}): \phi^{(r)} ; \right] ;
\]

\[
(c): \psi_{(-)}, \ldots, \psi^{(r)}_{(-)} ; [(d^{(r)}): \delta_{(-)}^{(r)} ; \ldots; (d^{(r)}): \delta^{(r)}] ;
\]
\[ z_1(1-x)^{k_1}(1+x)^{n_1}, \ldots, z_n(1-x)^{k_n}(1+x)^{n_n}, \]

where \( F_{G,H,v_{i}^{(i)},v_{j}^{(j)}}^{\ell_{1},\ell_{2},\ldots,\ell_{r}} \) is the generalized multiple hypergeometric function of Srivastava and Daoust ([6],[7],[8]; see also Srivastava and Karlsson[9]) and \( H_{A,C[B,D],[b^{(a)},p^{(a)}]}^{0,\lambda,(\mu^1,\nu^1),\ldots,(\mu^r,\nu^r)} \) stands for the multivariable \( H \)-function of Srivastava and Panda ([10],[11],[12]; also see Srivastava, Gupta and Goyal [13]), \( \Re(\alpha)>0, \Re(\beta)>0 \), all \( \lambda, \mu, \nu \) \((i=1,\ldots,n)\) are positive real numbers; and \( f(x) \) and \( f'(x) \) both are assumed to be sectionally continuous over the interval \((-1,1)\).

**Case 1. Solution for** \( V(R,x) \) **when** \( R<c \). (Interior to the sphere).

In this case, solution of the problem is given by

\[
(7.7.2) \quad V(R,x) = \sum_{m=0}^{\infty} A_{m} (R/c)^{m} P_{m}(x) \quad (R < c),
\]

which by applying (7.6.4) reduces to

\[
(7.7.3) \quad f(x) = \sum_{m=0}^{\infty} A_{m} P_{m}(x) \quad (R < c),
\]

where \( A_{m} \) is given by (7.4.5).

Therefore, substituting the value of \( A_{m} \) in (7.6.2), we derive the following required solution of the problem:

\[
(7.7.4) \quad V(R,x) = 2^{\alpha+\beta-2} \sum_{m=0}^{\infty} (2m+1) (R/c)^{m} P_{m}(x) \sum_{p=0}^{m} \frac{(-m)_{p} (m+1)_{p}}{(p!)^2} 2^{p}
\]
\[
\sum_{m_1,\ldots,m_r=0}^{\infty} \prod_{j=1}^{E} (e_j, m_j \xi_j + \cdots + m_r \xi_j^{(r)}) \prod_{j=1}^{F} (f_j', m_1 \eta_j') \cdots \prod_{j=1}^{P} (f_j^{(r)}, m_r \eta_j^{(r)}) \prod_{j=1}^{H} (g_j, m_1 \xi_j' + \cdots + m_r \xi_j'^{(r)}) \prod_{j=1}^{H'} (h_j', m_1 \epsilon_j') \cdots \prod_{j=1}^{H'} (h_j'^{(r)}, m_r \epsilon_j'^{(r)}) \frac{m_1^{m_1} \cdots m_r^{m_r}}{m_1! \cdots m_r!} \\
2^{(\alpha+\alpha_k) m_1 + \cdots + (\alpha_n+\alpha_k) m_n} \\
2^{(\alpha_1+\alpha_k) m_1 + \cdots + (\alpha_n+\alpha_k) m_n} \\
(1 - \alpha - 1 \lambda_m - \cdots - \lambda_r, m_r : \xi_1, \ldots, \xi_n) [1 - \beta - \mu_1 m_1 - \cdots - \mu_r m_r : \eta_1, \ldots, \eta_n] \\
[1 - \alpha - \beta - p - (\lambda_1 + \mu_1) m_1 - \cdots - (\lambda_r + \mu_r) m_r : \eta_1 + \xi_1, \ldots, \eta_n + \xi_n] \\
\begin{bmatrix} (b_i) : \phi_i' \cdots \phi_i^{(n)} \\
(b_i^{(r)} : \delta_i' \cdots \delta_i^{(n)} \\
(c_i) : \psi_i' \cdots \psi_i^{(n)} \end{bmatrix} \\
\begin{bmatrix} (a) : \Theta' \cdots \Theta^{(n)} \\
(a_i) : \phi_i' \cdots \phi_i^{(n)} \\
(c_i) : \psi_i' \cdots \psi_i^{(n)} \end{bmatrix} \\
2^{(\xi_1+\eta_1) z_1 + \cdots + (\xi_n+\eta_n) z_n} \\
\]
\[2^{(\lambda_i + \mu_i) m_i + \ldots + (\lambda_r + \mu_r) m_r} H^{0, \alpha+2; \mu_i, \ldots, \mu_r; m_i, \ldots, m_r}_{a+2, C+2; \mu_i, \ldots, \mu_r; m_i, \ldots, m_r} \left( \begin{array}{c} (a) \colon \theta', \ldots, \theta^{(a)}_i \\ (c) \colon \psi', \ldots, \psi^{(a)}_i \end{array} \right) \]

\[\left[ 1 - \alpha - p - \lambda_i m_i - \ldots - \lambda_r m_r : \xi_1, \ldots, \xi_n \right] \left[ 1 - \beta - \mu_i m_i - \ldots - \mu_r m_r : \eta_1, \ldots, \eta_n \right] \left[ 1 - \alpha - \beta - p - (\lambda_1 + \mu_i) m_i - \ldots - (\lambda_r + \mu_r) m_r : \eta_1 + \xi_1, \ldots, \eta_n + \xi_n \right] \]

\[\left[ (b') : \phi'_i \right] ; \ldots ; \left[ (b^{(a)}_i) : \phi^{(a)}_i \right] ; \left[ (a') : \delta' \right] ; \ldots ; \left[ (a^{(n)}_i) : \delta^{(a)}_i \right] ; 2^{(\xi_i + \eta_i) z_i} ; \ldots ; 2^{(\xi_n + \eta_n) z_n} \]

provided that \( m \) is a positive integer and all conditions of (7.2.1) are satisfied.

REFERENCES


