APPLICATIONS OF
HERMITE
POLYNOMIALS,
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OF SEVERAL
VARIABLES ALONG
WITH MULTIVARIABLE
$H$-FUNCTION OF
SRIVASTAVA-PANDA IN
A PROBLEM OF
HEAT CONDUCTION
CHAPTER -V

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ABSTRACT
Motivated by the work of Chandel and Sengar ([7],[8]) on heat conduction and recent work due to Chandel and Singh [9] on two boundary value problems on
1. heat conduction in a rod
2. deflection of vibrating string under certain conditions
and frequent requirement of various properties of special functions which play a vital role in the study of potential problems, heat conduction and other allied problems in quantum mechanics. in the present Chapter, first we evaluate an integral involving the product of Hermite polynomials, multivariable $H$-function of Srivastava-Panda ([27],[28],[29]) and generalized polynomials of several variables due to Srivastava [21], and then we make its applications in solving a problem on heat conduction given by Bhonsle [1]. Finally, we also derive an interesting expansion formula.

5.1. Introduction . Appell’s functions and the functions related to them have many applications in mathematical physics ([13],[14],[15]). Srivastava, Gupta and Goyal [28] have discussed a problem on heat conduction in a finite bar using $H$-function of two variables of Srivastava and Panda ([25],[26],[27]). Singh [17] used
generalized hypergeometric function in a problem of cooling of a heated cylinder. Further Singh [8] evaluated some integrals involving Kampé de Fériet’s function and one of them was used to obtain a solution of a problem in heat conduction given by Bhonsle [1]. Chandel and Yadava [3] have evaluated certain integrals involving multiple hypergeometric function of Srivastava and Daoust ([21], [22], [23]; also see Srivastava and Karlsson [24; p.37, eqns. (2.1) to (2.3))], and their applications have been made in solving the same problem on heat conduction. Chandel and Bhargava [2] have used generalized kampé de Fériet function of two variables of Srivastava and Daoust ([21],[22],[23]), while Chandel and Gupta [5] have used multivariable $H$-function of Srivastava and Panda ([25],[26],[27]; also see Srivastava, Gupta and Goyal [28]) in a cooling of a heated cylinder.


Recently, Chandel and Sengar [7] have discussed two boundary value problems on heat conduction involving the product of multivariable $H$-function of Srivastava and Panda ([25],[26],[27]) and several generalized polynomials of Srivastava [19] and their special cases have been discussed. Further Chandel and Sengar [8] have discussed a problem of heat conduction in a rod under the Robin Condition involving the product of above multivariable $H$-function ([25],[26],[27]) and several generalized polynomials of Srivastava [19].
Very recently, the authors Chandel and Singh [9] have discussed two boundary value problems on
(i) heat conduction in a rod
(ii) deflection of vibrating string, under certain conditions.

In the continuation of the above study, the present Chapter is motivated by the frequent requirement of various properties of special functions which play a vital role in the study of potential theory, heat conduction and other allied problems in Quantum Mechanics.

First we evaluate an integral involving the product of multivariable $H$-function of Srivastava and Panda ([25],[26],[27]; also see Srivastava, Gupta and Goyal [28]), Hermite polynomials (Rainville [16]) and generalized polynomials of several variables of Srivastava [20, p. 185, eqn (7)] defined by

\[(5.1.1) \quad S_{N_1,\ldots,N_r}^{M_1,\ldots,M_r}(x_1,\ldots,x_r) = \sum_{s_1=0}^{[N_1/M_1]} \ldots \sum_{s_r=0}^{[N_r/M_r]} (-N_1)^{s_1} \cdots (-N_r)^{s_r}, \]

where $N_1,\ldots,N_r$; $M_1,\ldots,M_r$ are arbitrary positive integers and coefficients $A[N_1,s_1;\ldots;N_r,s_r]$ are arbitrary parameters real or complex independent of $x_1,\ldots,x_r$.

Then we shall make its applications in solving a problem on heat conduction given by Bhosle [1] and to establish an expansion formula.

5.2. **Main Integral.** In this section, we shall evaluate the integral

\[(5.2.1) \quad \int_{-\infty}^{\infty} z^2 e^{-z^2} H_2(\bar{z}) H_2(\bar{\omega}) \ldots \left[ H_{\theta(\delta)}(z) \right]^{(a)} \left[ \theta(\delta) \right]^{(a)} : \]

where $\theta(\delta)$.
\[
\left[\begin{array}{c}
b^{(s)}_1 \\
b^{(s)}_2 \\
\vdots \\
b^{(s)}_n
\end{array}\right] \cdot \left[\begin{array}{c}
\phi^{(s)}_1 \\
\phi^{(s)}_2 \\
\vdots \\
\phi^{(s)}_n
\end{array}\right] ;
\left[\begin{array}{c}
d^{(s)}_1 \\
d^{(s)}_2 \\
\vdots \\
d^{(s)}_n
\end{array}\right] \cdot \left[\begin{array}{c}
\delta^{(s)}_1 \\
\delta^{(s)}_2 \\
\vdots \\
\delta^{(s)}_n
\end{array}\right] ; \ x_1z^{2a_1}, \ldots, x_nz^{2a_n} \right] \cdot \mathcal{S}_{y_1, \ldots, y_r}^{M_1, \ldots, M_r} \left( y_1z^{2b_1}, \ldots, y_rz^{2b_r} \right) \, dz
\]

\[
= \sqrt{\pi} 2^{2(v-\rho)} \sum_{s_1=0}^{N_1/M_1} \ldots \sum_{s_r=0}^{N_r/M_r} \left( \begin{array}{c}
N_1/M_1 \\
\vdots \\
N_r/M_r
\end{array} \right) \\
\cdot \left( \begin{array}{c}
-1 \\
\vdots \\
-1
\end{array} \right)_{s_1! \ldots s_r!} \cdot \mathcal{A}[N_1, s_1; \ldots; N_r, s_r] \cdot \frac{y_1^{s_1}}{s_1!} \ldots \frac{y_r^{s_r}}{s_r!}
\]

\[
H_{v, \rho}^{\lambda, \mu, \nu, \delta} \\
\cdot \mathcal{A}[N_1, s_1; \ldots; N_r, s_r]
\]

where \(H_{v, \rho}^{\lambda, \mu, \nu, \delta}\) are Hermite polynomials (Rainville [16]), \(\mathcal{S}_{y_1, \ldots, y_r}^{M_1, \ldots, M_r} (x_1, \ldots, x_n)\) are generalized polynomials of several variables due to Srivastava [20, p.185, eqn. (7)], \(H_{v, \rho}^{\lambda, \mu, \nu, \delta} \cdot \mathcal{A}[N_1, s_1; \ldots; N_r, s_r]\) is multivariable \(H\)-function of Srivastava and Panda ([25],[26],[27]; also see Srivastava, Gupta and Goyal [28]),

\[
|\arg x_1z^{2a_1}| < \pi/2 \Delta_i,
\]

\[
\Delta_i = \sum_{j=1}^{\lambda} \theta^{(j)}_i - \sum_{j=1}^{\nu} \phi^{(j)}_i - \sum_{j=1}^{\rho} \psi^{(j)}_i + \sum_{j=1}^{\gamma} \mu^{(j)}_i - \sum_{j=1}^{\delta} \phi^{(j)}_i - \sum_{j=1}^{\delta} \psi^{(j)}_i - \sum_{j=1}^{\delta} \psi^{(j)}_i + \sum_{j=1}^{\delta} \delta^{(j)}_i > 0; \ i = 1, \ldots, n
\]

\(\rho = 0, 1, 2, \ldots, N_j, M_j\) are positive integers and the coefficients \(\mathcal{A}[N_1, s_1; \ldots; N_r, s_r]\) are arbitrary parameters real or complex independent of \(y_i, \beta_i, z (i = 1, \ldots, r)\)

The above integral will be quite useful in our further investigations.

Proof. Multiplying both side of Lebedev eqn. [14, (4.16.1)] by \(e^{-z^2} H_{2v} (z)\) and making an appeal to orthogonal property of Hermite
polynomials (Rainville [16]), we have

\[ \int_{-\infty}^{\infty} e^{-z^2} H_{2\nu}(z) dz = \frac{\sqrt{\pi} 2^{2\nu} \Gamma(2\nu + 1)}{\Gamma(\nu + 1)}, \rho = 0, 1, 2, \ldots \]

Now left hand side of (5.2.1)

\[ = \sum_{k_1=0}^{\lfloor \lambda / M \rfloor} \cdots \sum_{k_r=0}^{\lfloor \lambda / M \rfloor} \frac{(-N_1)_{M; k_1} \ldots (-N_r)_{M; k_r}}{k_1! \ldots k_r!} A[N_1, k_1; \ldots; N_r, k_r] \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_r^{k_r}}{k_r!} \]

\[ \frac{1}{(2\pi \nu)^n} \prod_{j=1}^{\lambda} \prod_{i=1}^{n} \frac{\Gamma[(d_j^{(i)} - \delta_j^{(i)} s_i)] \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)} \]

\[ \prod_{j=\lambda+1}^{\lambda-n} \Gamma \left( a_j - \sum_{i=1}^{n} \theta_j^{(i)} s_i \right) \prod_{j=\lambda-n+1}^{\lambda} \Gamma \left( 1 - a_j - \sum_{i=1}^{n} \theta_j^{(i)} s_i \right) \]

\[ \left( \int_{-\infty}^{\infty} e^{-z^2} H_{2\nu}(z) e^{2z^2 + 2\beta_1 k_1 + \ldots + 2\beta_r k_r + 2a_1 s_1 + \ldots + 2a_r s_n} dz \right) ds_1 \ldots ds_n \]

= right hand side of (5.2.1)  (By employing (5.2.2).

5.3. Application to Heat conduction. Bhonsle [1] has employed Hermite polynomials in solving the partial differential equation

\[ \frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial z^2} - K \phi z^2, \]

where \( \phi(z, t) \) tends to zero for large value of \( t \). When \( |z| \to \infty \), the above equation is related to the problem of heat conduction due to Churchill [11]
(5.3.2) \[ \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial z^2} h_i(\phi - \phi_0), \]

provided that \( \phi_0 = 0 \) and \( h_i = Kz^2. \)

The solution of (3.1) given by Bhsle (1) is

\[ \phi(z, t) = \sum_{s=0}^{\infty} A_s e^{-(1+2s)Kt-z^2/2} H_s(z). \]

Now we consider the problem of determining \( \phi(z, t) \), when for \( t = 0, \)

\[ (5.3.4) \phi(z, 0) = f(z) = z^{2\rho} e^{-z^2} H_{\alpha, \lambda, \mu, \nu}^{n, \nu} \left( \begin{array}{c} (a): \theta', ..., \theta^{(n)} \\ (c): \psi', ..., \psi^{(n)} \end{array} \right), \]

\[ \begin{bmatrix} (b') \phi' \\ \vdots \\ (b^{(n)}): \phi^{(n)} \end{bmatrix}; \begin{bmatrix} (d') \delta' \\ \vdots \\ (d^{(n)}): \delta^{(n)} \end{bmatrix}; x_i z^{2a_i}, ..., x_n z^{2a_n} \right) S_{M, N, N, \nu}^{(a_i, ..., a_n)} (y_i z^{2\beta_i}, ..., y_n z^{2\beta_n}). \]

Making an appeal to (5.3.3) and (5.3.4), we derive

\[ \int_0^\infty e^{-z^2} z^{2\rho} H_{2\nu}(z) H_{\alpha, \lambda, \mu, \nu}^{n, \nu} \left( \begin{array}{c} (a): \theta', ..., \theta^{(n)} \\ (c): \psi', ..., \psi^{(n)} \end{array} \right), \]

\[ \begin{bmatrix} (b') \phi' \\ \vdots \\ (b^{(n)}): \phi^{(n)} \end{bmatrix}; \begin{bmatrix} (d') \delta' \\ \vdots \\ (d^{(n)}): \delta^{(n)} \end{bmatrix}; x_i z^{2a_i}, ..., x_n z^{2a_n} \right) S_{M, N, N, \nu}^{(a_i, ..., a_n)} (y_i z^{2\beta_i}, ..., y_n z^{2\beta_n}) dz \]

\[ = \sum_{s=0}^{\infty} A_s \int_0^\infty e^{-z^2/2} H_s(z) H_{2\nu}(z) dz \]

\[ = \sqrt{2\pi}(2\nu)! A_{2\nu} \text{ (By orthogonal property of Hermite polynomials Erdélyi [12,p.289]).} \]

Thus
\[(5.3.5) \quad A_s = \frac{2^{s-2s-1/2} \prod \left( N_{i,M_s} \right) \prod \left( -N_{r,M_s} \right)}{s!} \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} (-N_{1,M_s})_{k_1} \cdots (-N_{r,M_s})_{k_r} A[N_1,k_1;\ldots;N_r,k_r] \]

\[
\left( \frac{y_1}{4^{b_1}} \right)^{k_1} \cdots \left( \frac{y_r}{4^{b_r}} \right)^{k_r} H_{A+1,n+1}^{0,1;n,\nu_1;\ldots;\nu_{2s};\nu_{2s+1};\ldots;\nu_{2s+1}} \left[ \left( \alpha \right); \theta,\ldots,\theta^{(n)} \right], \\
\left( \frac{\nu}{4^{b_1}} \right)^{k_1} \cdots \left( \frac{\nu}{4^{b_r}} \right)^{k_r} H_{A+1,n+1}^{0,1;n,\nu_1;\ldots;\nu_{2s};\nu_{2s+1};\ldots;\nu_{2s+1}} \left[ \left( c \right); \psi,\ldots,\psi^{(n)} \right], \\
[-2p-2\beta_1 k_1 - \ldots - 2\beta_r k_r : 2\alpha_1,\ldots,2\alpha_n] ; \\
\left( b' \right) ; \ldots ; \left( b^{(n)} \right) ; \\
s/2 - p - \beta_1 k_1 - \ldots - \beta_r k_r : \alpha_1,\ldots,\alpha_n] ; \\
\left( d' \right) ; \ldots ; \left( d^{(n)} \right) ; \]

valid if all conditions of (5.2.1) are satisfied.

Now substituting the value of \( A_s \) from (5.3.5) in (5.3.3), the solution of the main problem is given by

\[(5.3.6) \quad \phi(z,t) = \frac{e^{-z^2/2}}{2^{(\nu+1)/2}} \sum_{s=0}^{\infty} H_s(z) e^{-i\nu s t} \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} (-N_{1,M_s})_{k_1} \cdots (-N_{r,M_s})_{k_r} \]

\[
A[N_1,k_1;\ldots;N_r,k_r] \left( \frac{y_1}{4^{b_1}} \right)^{k_1} \cdots \left( \frac{y_r}{4^{b_r}} \right)^{k_r} H_{A+1,n+1}^{0,1;n,\nu_1;\ldots;\nu_{2s};\nu_{2s+1};\ldots;\nu_{2s+1}} \left[ \left( \alpha \right); \theta,\ldots,\theta^{(n)} \right], \\
\left( \frac{\nu}{4^{b_1}} \right)^{k_1} \cdots \left( \frac{\nu}{4^{b_r}} \right)^{k_r} H_{A+1,n+1}^{0,1;n,\nu_1;\ldots;\nu_{2s};\nu_{2s+1};\ldots;\nu_{2s+1}} \left[ \left( c \right); \psi,\ldots,\psi^{(n)} \right], \\
[-2p-2\beta_1 k_1 - \ldots - 2\beta_r k_r : 2\alpha_1,\ldots,2\alpha_n] ; \\
\left( b' \right) ; \ldots ; \left( b^{(n)} \right) ; \\
s/2 - p - \beta_1 k_1 - \ldots - \beta_r k_r : \alpha_1,\ldots,\alpha_n] ; \\
\left( d' \right) ; \ldots ; \left( d^{(n)} \right) ;
\]

where all conditions of (5.2.1) are satisfied.
### 5.4. Expansion Formula

By an appeal to (5.3.4) and (5.3.6), we derive the expansion formula

\begin{align*}
2^{(4p+1)/2}z^{2\rho}e^{-z^2/2}H_{\lambda(c\ell)}^{\lambda(c\ell)}\left[b^{(a)} : \phi^{(a)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} \right] \\
= \sum_{s=0}^{\infty}H_{\lambda(c\ell)}(z) \left[ \frac{\left[ y_{r/4}^{1/2} \right]_{k_1} \ldots \left[ y_{r/4}^{1/2} \right]_{k_r}}{k_1! \ldots k_r!} \right] \left[ e^{2\rho(z) - \beta k_1 - \ldots - \beta k_r} : \left[ b^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} : \phi^{(n)} \right] \right]
\end{align*}

provided that all conditions of (5.2.1) are satisfied.

Specializing the parameters of \( H \)-funciton and Srivastava's generalized polynomials of several variables, we can derive several results for different special functions.

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