CHAPTER 4

SPATIAL STABILITY OF STRATIFIED DUSTY SHEAR FLOWS

4.1. Introduction.

Theoretical and experimental investigations for spatially growing waves for inviscid, parallel shear flows have extensively been made by Michalke (1966), Fremuth (1966), Moslowe and Thompson (1971), Maltingly and Criminale (1971) and Dauey (1971) etc.

Yadav, K.L. (1982) discussed the spatial stability of stratified parallel shear flow of a dusty gas. He obtained the spectrum of eigen values and established some stability theorems for $k_r > 0$.

In this chapter, the same problem has been discussed for $k_r < 0$. An attempt has been made to obtain the spectrum of eigen values.

4.2. Basic Equation and Boundary Conditions.

Let an incompressible, inviscid, dusty fluid of density $\rho (y)$ be confined between two horizontal and parallel rigid planes situated at $y = 0$ and $y = h$. The x-axis is taken along the direction of the flow in the lower plane and y-axis is taken perpendicular to the planes. The planes are
taken at a distance \( h \) with each other. \( U(y) \) is the basic velocity of flow.

The dependence of any perturbation quantity \( f \) on \( x, y \) and \( t \) is taken of the form

\[
f(y) \exp \left\{ i (kx - \omega t) \right\}
\]

...(4.2.1)

The linearised governing stability equation for Boussinesq fluid is given by

\[
(\bar{U} - c) \left[ \phi'' - k^2 \phi \right] - \bar{U}'' \phi + \frac{\beta g \phi}{U - c} = 0
\]

where

\[
\bar{U} - c = (U - c) \left( 1 + \frac{f}{A} \right)
\]

...(4.2.2)

\[
f = \frac{mN}{\rho}
\]
is the mass concentration of the dust

\[
A = 1 + i k \tau (U - c)
\]

and \( \beta = -\frac{\rho'}{\rho} > 0 \).

The associated boundary conditions are

\[
\phi(y) = 0 \text{ at } y = 0, \ h
\]

...(4.2.3)

For spatially growing fluctuations the eigen values of the system are given by the complex wave number \( k (= k_r + i k_i) \) as a function of real and positive frequency \( \omega \).

\[
c = \frac{\omega}{k}
\]

is the complex wave velocity so that

\[
c_r = \frac{\omega k_r}{|k|^2} \quad \text{and} \quad c_i = \frac{-\omega k_i}{|k|^2}
\]
The phase velocity $c_p$ is defined as $c_p = \frac{\omega}{k_r}$.

4.3 Result Obtained by Yadav, K.L.

(a) Case of Fine Dust.

**Lemma 1.** If $U''$ is positive throughout the flow domain and $2a > b$, then stable modes lie in the region

$$\frac{\omega}{2a} < k_r < \frac{\omega}{b}$$

**Lemma 2.** The unstable modes for $k_r > 0$ lie inside the curve given by

$$k_r^3 - \frac{\omega}{a} k_r^2 + k_r k_i^2 - \frac{\omega}{a} \left( \frac{U''^2}{16\beta g} \right)_{\text{max}} (1 + f) = 0$$

**Lemma 3.** The bounds for unstable modes are given by

$$\left( \frac{2a k_r}{\omega} - 1 \right) k_r^2 < \frac{\left( \frac{1}{4} - \frac{J_{\min}}{1 + f} \right) U''^2_{\text{max}}}{a^2}$$

This shows the destabilizing role of fine dust.

**Theorem 1.** A necessary condition for instability with $k_r > 0$ is that the complex wave number $k$ must lie in a region bounded by the circle

$$k_r^2 + k_i^2 - \frac{\omega k_r}{a} = 0$$

and the curve
\[ k_r^3 - \frac{\omega}{a} k_r^2 + k_r^2 k_r - \frac{\omega}{a} \left( \frac{U''}{16 \beta g} \right)_{\text{max}} (1 + f) = 0 \]

and the left side of the straight line \( k_r = \frac{\omega}{a} \).

**Theorem 2.** If \( U'' \) is negative throughout the flow domain then a necessary condition for instability with \( k_r > 0 \) is that the complex wave number \( k \) must lie in a region bounded by the circles

\[ k_r^2 + k_i^2 - \frac{\omega k_r}{b} = 0 \]

and \[ k_r^2 + k_i^2 - \frac{\omega k_r}{a} = 0 \]

and the right hand side of the straight line \( k_r = \frac{\omega}{2b} \).

**(b) Case of Coarse Dust.**

**Theorem 1.** If \( U'' \) is negative throughout the flow domain, then for spatially decaying stable modes with \( k_r > 0 \), \( (k_r, k_i) \) must lie outside the circle those centre is \( \left( \frac{\omega}{2b}, 0 \right) \) and radius \( \frac{\omega}{2b} \).

**Theorem 2.** If the unstable modes occur when the conditions \( J_0 > \frac{1}{4} \) and \( U'' < 0 \) hold everywhere in the flow domain and the phase velocity \( c_p \) is smaller than the minimum of flow velocity then \( k_r \) and \( k_i \) the inside the circle.
\[
\left(k_r - \frac{\omega}{Uy_0}\right)^2 \left(k_1 + \frac{\omega t}{Uy_0 (4J_0 - 1)}\right) = \frac{\omega^2 t^2}{U^2 y_0^2} \left[ \frac{1}{4J_0 - 1} + \frac{1}{(4J_0 - 1)^2} \right]
\]

4.4 Analysis.

(a) Case of Fine Dust.

For fine dust case, \( \tau < < 1 \) so that \( A = 1 \), and

\[
\bar{U} - c = (U - c) (1 + f), \quad \bar{U}' = U' (1 + f)
\]

Therefore equation (4.2.2) becomes

\[
\phi'' - k^2 \phi - \frac{U''}{U - c} \phi + \frac{\beta g \phi}{(U - c)^2 (1 + f)} = 0 \quad \ldots (4.4.1)
\]

Substituting \( \phi = (U - c)^n \psi \) in the equation (4.4.1) and multiplying the resulting equation by \((U - c)^{n-1}\), we get

\[
\left[(U - c)^{2n} \psi'\right]' - k^2 (U - c)^{2n} \psi + (n - 1) U'' (U - c)^{2n-1} \psi
\]

\[
+ \left[ n(n - 1) + \frac{\beta g}{U'} \right] \frac{U'^2 (U - c)^{2n-2} \psi}{(1 + f)} = 0 \quad \ldots (4.4.2)
\]

The associated boundary conditions are

\[
\psi (y) = 0 \text{ at } y = 0, \ h \quad \ldots (4.4.3)
\]

Multiplying equation (4.4.2) by \( \overline{\psi} \), the complex conjugate of \( \psi \) and integrating over the flow domain, we get
\[
\int (U - c)^{2n} \left[ |\psi'|^2 + k^2 |\psi|^2 \right] - (n - 1) \int (U - c)^{2n - 1} U'' |\psi|^2 \\
- \int \left[ n(n - 1) + \frac{\beta g}{(1 + f) U'^2} \right] U^2 [U - c]^{2n - 2} |\psi|^2 = 0.
\]

...(4.4.4)

Putting \( n = 0, \frac{1}{2} \) and 1 and \( \psi = H, G \) and \( F \) respectively, we get

\[
\int \left[ |H'|^2 + k^2 |H|^2 \right] + \int \frac{U''^2 |H|^2}{(U - c)} - \int \frac{\beta g |H|^2}{(1 + f)(U - c)} = 0
\]

...(4.4.5)

\[
\int U - c \left[ |G'|^2 + k^2 |G|^2 \right] + \int \frac{U''^2}{2} |G|^2 \\
+ \frac{\left( \frac{U^2}{4} - \beta g \right)(1 + f)|G|^2}{(U - c)} = 0
\]

...(4.4.6)

and \( \int (U - c)^2 \left[ |F'|^2 + k^2 |F|^2 \right] - \int \frac{\beta g |F|^2}{(1 + f)} = 0
\)

...(4.4.7)

Separating the real and imaginary parts of (4.4.5), (4.4.6) and (4.4.7), we get

\[
\int \left[ |H'|^2 + \left( k_r^2 - k_i^2 \right) |H|^2 \right] + \int \frac{U''(U - c)|H|^2}{|U - c|^2} - \int \frac{\beta g [U - c]^2 - c_f^2}{(1 + f)|U - c|^4} |H|^2 = 0.
\]

...(4.4.8)
\[ k_f^2 \int |H|^2 - c_r \int \frac{U'' |H|^2}{2 |U - c|^2} + c_r \int \frac{(U - c_r) (\beta g) |H|^2}{(1 + f) |U - c|^2} = 0, \]

...(4.4.9)

\[ \int (U - c_r) \left[ |G'|^2 + (k_f^2 - k_f^2) |G|^2 \right] - 2k_f^2 c_r \int |G|^2 \]

\[ + \int \frac{U''}{2} |G|^2 + \int \frac{\left( \frac{1}{4} - \frac{J}{1 + f} \right) (U - c_r) U'^2 |G|^2}{|U - c|^2} = 0. \]

...(4.4.10)

\[ \int |G'|^2 + \int \left( \frac{2U}{c_p} - 1 \right) |k|^2 |G|^2 \]

\[ - \int \frac{\left( \frac{1}{4} - \frac{J}{1 + f} \right) U'^2 |G|^2}{|U - c|^2} = 0. \]

...(4.4.11)

\[ \int \left[ (U - c_r)^2 - c_f^2 \right] \left[ |F'|^2 + (k_f^2 - k_f^2) |F|^2 \right] \]

\[- 4k_f^2 c_r \int (U - c_r) |F|^2 - \int \frac{\beta g}{1 + f} |F|^2 = 0, \]

...(4.4.12)

\[ \int (U - c_r) \left[ |F'|^2 + (k_f^2 - k_f^2) |F|^2 \right] \]

\[ + \frac{k_f^2}{c_r} \int \left[ (U - c_r)^2 - c_f^2 \right] |F|^2 = 0 \]

...(4.4.13)
where \( J = \frac{\beta g}{U^2} \)

4.5. Spectrum of Eigen Values.

We consider the case when \( k_r \) is negative throughout. For \( k_r < 0 \) both \( c_p \) and \( c_r \) are negative and \( c_p < c_r \).

Adding \( \left( \frac{k_r^2 - k_p^2}{k_r^2} \right) c_r \) times the equation (4.4.13) to (4.4.12), we get

\[
\int \left[ (U - c_p)^2 - c_r^2 \right] |F'|^2 + \frac{c_r}{k_r^2} (k_t^2 - k_r^2) \int (U - c_r) |F|^2
\]

\[
- \frac{c_r}{k_t^2} \int (U - c_r) (k_t^2 - k_r^2)^2 |F|^2 - 4k_r^2 \int (U - c_r) |F|^2
\]

\[
- \int \frac{\beta g}{1 + f} |F|^2 = 0
\]

Collecting the terms of \(|F'|^2\) and \(|F|^2\) separately in the above equation, we get

\[
\int \left[ (U - c_r)^2 - c_r^2 + \frac{c_r^2 U}{c_r} - c_r^2 - c_r (U - c_r) \right] |F'|^2
\]

\[
- \frac{c_r}{k_t^2} \int (U - c_r) |k|^2 |F|^2 - \int \frac{\beta g}{1 + f} |F|^2 = 0.
\]
This equation can also be written as

\[ \int \left[ (U - 2cr) \left( U - cr + \frac{c^2_t}{cr} \right) \right] |F'|^2 \]

\[ - \frac{\omega^2}{cr} \int \left[ U - cr + \frac{\beta g cr}{(1 + f) \omega^2} \right] |F|^2 = 0 \quad \text{...(4.5.1)} \]

Now

\[ U - cr + \frac{c^2_t}{cr} = U - 2cr + c_p, \quad \therefore \quad c_p = cr + \frac{c^2_t}{cr} \]

Therefore the equation (4.5.1) reduces to

\[ \int \left[ (U - 2cr) (U - 2cr + c_p) \right] |F'|^2 - \frac{\omega^2}{cr} \int U |F|^2 \]

\[ + \int \left( \frac{\omega^2 - \beta g}{1 + f} \right) |F|^2 = 0. \quad \text{...(4.5.2)} \]

If \( \omega^2 - \frac{\beta g}{1 + f} > 0 \) throughout the flow domain then the necessary condition that the equation (4.5.2) is satisfied, is

\[ U - 2cr + c_p < 0 \quad \text{...(4.5.3)} \]

at least at one point in the flow domain. It gives

\[ c_p < -a + 2cr < -a. \quad \text{...(4.5.4)} \]
Since \( c_p = \frac{|c|^2}{c_r} \), expression (4.5.4) can be written as
\[
\frac{|c|^2}{c_r} < -a + 2c_r
\]
or
\[
|c|^2 > -a c_r + 2c_r \tag{\cdot\cdot\cdot c_r < 0}
\]
or
\[
\left( c_r - \frac{a}{2} \right)^2 - c_r^2 < \left( \frac{a}{2} \right)^2
\]
or
\[
ak_r^3 - \omega k_i^2 + ak_r k_i^2 + \omega k_i^2 > 0. \tag{4.5.5}
\]
Also from (4.5.4), we have
\[
c_p < 2c_r
\]
or
\[
\frac{|c|^2}{c_r} < 2c_r
\]
or
\[
c_i^2 > c_r^2
\]
which can be expressed
\[
k_i^2 > k_r^2. \tag{4.5.6}
\]
Further from (4.5.4), we have
\[
k_r > -\frac{\omega}{a}. \tag{4.5.7}
\]
Also equation (4.4.13) can be written as
\[
\int \left[ |F'|^2 + (k_r^2 - k_i^2) |F|^2 \right] = \int \frac{U}{c_r} \left[ |F'|^2 + (k_r^2 - k_i^2) \right] |F|^2
\]
\[ + \frac{k_r^2}{c_r^2} \int \left[ (U - c_r)^2 - c_i^2 \right] |F|^2 = 0. \quad \ldots(4.5.8) \]

Adding \( c_i^2 \) times of this equation to the equation (4.4.12), we get

\[ \int \left[ (U - c_r)^2 - \frac{U c_i^2}{c_r} \right] \left[ |F'|^2 + (k_r^2 - k_i^2) |F|^2 \right] \]
\[ - k_i \int \left[ (U - c_r)^2 - c_i^2 + 4c_r(U - c_r) \right] |F|^2 \]
\[ - \int \frac{\beta g}{1 + f} |F|^2 = 0 \]

which can be written as

\[ \int \left[ (U - c_r)^2 - \frac{U c_i^2}{c_r} \right] \left[ |F'|^2 + k_r^2 |F|^2 \right] \]
\[ - k_i^2 \int \left[ (U + c_r) \left( 2U - 2c_r - \frac{c_i^2}{c_r} \right) + \frac{\beta g}{(1 + f) k_r^2} \right] |F|^2 = 0. \]

\ldots(4.5.9) \]

Equation (4.5.9) will be satisfied if

\[ k_i^2 (U + c_r) \left[ 2U - 2c_r - \frac{c_i^2}{c_r} \right] + \frac{\beta g}{1 + f} > 0 \]
at least at one point within the flow domain. If it implies

\[(2b - \omega^2) k_r^3 + (2b k_r - \omega) k_i^2 + \left( \frac{\beta g}{1 + f} \right)_{\text{max.}} k_r < 0 \quad \ldots(4.5.10)\]

Further from equation (4.4.9), we see that for \(k_i \neq 0\).

\[
\frac{U''}{2} - \frac{(U - c_r) \beta g}{(1 + f) |U - c|^2} = 0
\]

at least at one point in the flow domain

or

\[
U'' |U - c|^2 - 2(U - c_r) \frac{\beta g}{1 + f} < 0. \quad \ldots(4.5.11)
\]

For \(U'' > 0\) the above expression implies that

\[(a - c_r)^2 + c_r^2 - 2 (b - c_r) \left( \frac{\beta g}{(1 + f) U''} \right)_{\text{max.}} < 0\]

\[
\Rightarrow \left[ a - \left( \frac{\beta g}{(1 + f) U''} \right)_{\text{max.}} \right] - c_r \right]^2 + c_r^2
\]

\[
< \left[ a - \left( \frac{\beta g}{(1 + f) U''} \right)_{\text{max.}} \right]^2 + 2 \left( \frac{\beta g}{(1 + f) U''} \right)_{\text{max.}} \ldots(4.5.12)
\]

For \(U'' < 0\) the expression (4.5.11) implies that

\[
|U - c|^2 + \frac{2 (U - c_r) \beta g}{(1 + f) |U''|^2} > 0
\]
or \[
\left[ a - c_r + \left( \frac{\beta g}{(1 + f) |U''|} \right) \right]^2 + c_f^2
\]

\[
> \left[ \frac{\beta g}{(1 + f) |U''|} \right]^2 . \quad \text{(4.5.13)}
\]

Further adding \[
\frac{k_i^2 - k_r^2}{k_r^2} \left( \frac{c_i^2 - c_r^2}{c_r^2} \right) \text{ times of equation (4.4.9)}
\]
to the equation (4.4.8), we get

\[
\int |H'|^2 - \int \frac{(c_p - 2U) U''}{2 |U - c|^2} |H|^2
\]
\[
+ \int \frac{\rho \beta g U (c_p - U)}{(1 + f) |U - c|^4} |H|^2 = 0. \quad \text{(4.5.14)}
\]

Now for \( U'' > 0 \), equation (4.5.14) will hold good if

\[
\frac{U''}{2} - \frac{\beta g U}{(1 + f) |U - c|^2} < 0 \quad \text{(4.5.15)}
\]
at least at point in the flow domain.

It implies

\[
c_f^2 < 2b \left( \frac{\beta g}{(1 - f)U''} \right)_{\text{max.}} \quad \text{(4.5.16)}
\]

This gives an upper bound of unstable modes.

Also relation (4.4.11) holds good if
\[
\left( \frac{2U}{c_p} - 1 \right) |k|^2 - \frac{\left( \frac{1}{4} - \frac{J}{1 + f} \right) U^2}{|U - c|^2} < 0
\]

at least at one point in the flow domain.

It implies

\[
\left( \frac{2b}{c_p} - 1 \right) |k|^2 < \frac{\left( \frac{1}{4} - \frac{J_0}{1 + f} \right) U^2_{min}}{b^2 - 2b c_r + |c|^2}
\]

where \( J_0 = \min J \).

For \( \frac{J_0}{1 + f} > \frac{1}{4} \) it implies that

\[
\left( \frac{2b k_r}{\omega} \right) - 1 |k|^2 < \frac{\left( \frac{1}{4} - \frac{J_0}{1 + f} \right) |k|^2}{b^2 |k|^2 - 2b \omega k_r + \omega^2}
\]

\[
\Rightarrow \quad k_r^2 + k_r k_f^2 - \frac{5 \omega k_f^2}{2b} + \frac{2 \omega^2 k_r}{b^2} - \frac{\omega^3}{2b^3}
\]

\[
- \left( \frac{1}{4} - \frac{J_0}{1 + f} \right) \omega U^2_{min} < 0. \quad \ldots (4.5.17)
\]

In the above discussion, we see that \( \beta > 0 \) has stabilizing effect while the mass concentration \( f \) has a destabilizing effect.
(b) Case of Coarse Dust.

In case of coarse dust, equation (4.2.2) reduces to

\[
\phi'' - k_2 \phi - \frac{U''}{\left(U - c + \frac{s}{i k}\right)} \phi + \frac{\beta g \phi}{(U - c) \left(U - c + \frac{s}{i k}\right)} = 0
\]

...(4.5.18)

where \( s = \frac{f}{r} \) is dust parameter.

If we substitute

\[
\phi = (U - c)^n \psi
\]

in the equation (4.5.18) then the transformed equation is

\[
(U - c)^n \psi'' + 2n (U - c)^{n-1} U' \psi' + n (U - c)^{n-1} U'' \psi
\]

\[
+ n (n - 1) (U - c)^{n-2} U' \psi - k^2 (U - c)^n \psi - \frac{U''}{\left(U - c + \frac{s}{i k}\right)}
\]

\[
(U - c)^n \psi + \frac{\beta g}{\left(U - c + \frac{s}{i k}\right)} (U - c)^{n-1} \psi = 0.
\]  ...(4.5.19)

The associated boundary conditions are

\[
\psi(y) = 0 \text{ at } y = 0, h.
\]  ...(4.5.20)

Multiplying equation (4.5.19) by \((U - c)^n \bar{\psi}\) and integrating over the flow domain

\[
\int (U - c)^{2n} (|\psi'|^2 + k^2 |\psi|^2)
\]
\[- \int U'' \left( n - \frac{U - c}{(U - c + \frac{s}{i k})} \right) (U - c)^{2n-1} |\psi|^2 \]

\[- \int (U - c)^{2n-2} \left[ n(n-1) U'^2 + \frac{\beta g (U - c)}{U - c + \frac{s}{i k}} \right] |\psi|^2 = 0 \]

...(4.5.21)

Putting \( n = 0, \frac{1}{2} \) and 1 and \( \psi = H, G \) and \( F \) respectively, we get

\[ \int \left[ |H'|^2 + k^2 |H|^2 \right] + \int \frac{U''}{U - c_1} |H|^2 \]

\[- \int \frac{\beta g |H|^2}{(U - c)(U - c_1)} = 0, \]

...(4.5.2)

\[ \int (U - c) \left[ |G'|^2 + k^2 |G|^2 \right] - \int U' \left( \frac{1}{2} + \frac{U - c}{U - c_1} \right) |G|^2 \]

\[ + \int \left[ \frac{U'^2}{4(U - c)} - \frac{\beta g}{U - c_1} \right] |G|^2 = 0, \]

...(4.5.23)

\[ \int (U - c)^2 \left[ |F'|^2 + k^2 |F|^2 \right] \]

\[- \int U'' (U - c) \left[ 1 - \frac{U - c}{U - c_1} \right] |F|^2 - \int \frac{\beta g(U - c)}{(U - c_1)} |F|^2 = 0, \]

...(4.5.24)
where \( U - c_1 = \left( U - c + \frac{s}{i k} \right) \)

\[ = U - c (1 + i t) \] \ ...(4.5.25)

\[ t = \frac{s}{\omega} \]

Separating real and imaginary parts (4.5.22) we get

\[ \int \left[ |H'|^2 + (k_r^2 - k_i^2) |H|^2 \right] + \int \frac{U'' (U - c_r + t c_i)}{|U - c_1|^2} |H|^2 \]

\[ - \int \frac{\beta g [(U - c_r) (U - c_r + t c_i) - c_i (c_i + t c_r)]}{|U - c|^2 |U - c_1|^2} |H|^2 = 0 \] \ ...(4.5.26)

and \( 2k_r k_i \int |H|^2 + \int \frac{U'' (t c_r + c_i)}{|U - c_1|^2} |H|^2 \)

\[ - \int \frac{\beta g (c_i (U - c_r + t c_i) + (U - c_r) (t c_r + c_i))}{|U - c|^2 |U - c_1|^2} |H|^2. \] \ ...(4.5.27)

Equation (4.5.27) can be written as

\[ - 2c_i \left[ \frac{k_r^2}{c_r} \int |H|^2 - \int \frac{U''}{2 |U - c_1|^2} |H|^2 \right. \]

\[ + \int \frac{\beta g (U - c_r)}{|U - c|^2 |U - c_1|^2} |H|^2 \]
\[ t \left[ \int \frac{U'' c_r}{|U - c_1|^2} |H|^2 - \int \frac{\beta g \left( c_i^2 + c_r (U - c_r) \right)}{|U - c|^2 |U - c_1|^2} |H|^2 \right] = 0. \]  

...(4.5.28)

From (4.5.28) we see that for \( k_r < 0 \) i.e. \( c_r < 0 \) and \( U'' < 0 \). \( c_i \) can not be zero.

Multiplying equation (4.5.27) by \( \left( \frac{k_r^2}{2k_r k_i} \right) \) and adding it to the equation (4.5.26), we get

\[ \int |H'|^2 - \int \frac{U'' (c_p - 2U)}{2 |U - c_1|^2} |H|^2 \]

\[ + \int \frac{\beta g U (c_p - U)}{|U - c|^2 |U - c_1|^2} |H|^2 - \frac{s}{k_i} \left[ \int \frac{U''}{2 |U - c_1|^2} |H|^2 \right. \]

\[ + \int \frac{\beta g (c_p - U)}{2 |U - c|^2 |U - c_1|^2} |H|^2 \left. \right] = 0. \]

If the velocity profile is linear then \( U'' = 0 \). Therefore equation (4.5.29) reduces to

\[ \int |H'|^2 + \int \frac{\beta g (c_p - U)}{|U - c|^2 |U - c_1|^2} \left( U - \frac{1}{2} \frac{s}{k_i} \right) |H|^2 = 0. \]

...(4.5.30)

Equation (4.5.30) holds good if

\[ U - \frac{s}{2k_i} > 0 \]
at least at one point in the flow domain. It implies

\[ b - \frac{s}{2k_i} > 0 \]

\[ \Rightarrow k_i > \frac{s}{2b} \text{ for } k_i > 0 \text{ i.e. the stable modes. Thus } \frac{s}{2b} \text{ is the lower bond of stable modes.} \]

Equation (4.5.29) can be written as

\[
\int |H'|^2 + \int \frac{U''}{2|U - c_i|^2} \left[ 2U - c_p - \frac{s}{k_i} \right]|H|^2 \\
+ \int \frac{\beta g (c_p - U) \left( U - \frac{1}{2} \frac{s}{k_i} \right)}{|U - c|^2 |U - c|^2} |H|^2 = 0. \quad \cdots(4.5.31)
\]

If \( U'' \) is positive every where in the flow domain then

\[ U - \frac{1}{2} \frac{s}{k_i} > 0 \]

at least at one point in the flow domain.

It implies \( k_i > \frac{s}{2b} \text{ for } k_i > 0 \text{ i.e. the stable modes. Thus we have the following result:} \)

**Theorem. 4.5.1.** If \( U'' \geq 0 \) every where in the flow domain then

\( \frac{s}{2b} \text{ is the lower bond of stable mode.} \)

Equation (4.5.29) can also be written as
\[
\frac{2k_r^2}{c_r} \int |H|^2 + \int \frac{U'' \left[ \frac{tk_r}{k_i} - 1 \right] |H|^2}{|U - c_i|^2}
\]

\[
\beta g \left[ (U - c_r) \left[ \frac{tk_r}{k_i} - 2 \right] + \omega k_i t \right] |k|^2
\]

\[
= \frac{- \beta g \left[ 2(U - c_r) + \omega k_i t \right] |k|^2}{|U - c_i|^2 |U - c_i|^2}.
\]

For \( k_r < 0, k_i < 0 \) i.e. the unstable modes and \( U'' < 0 \) every where in the flow domain, the equation (4.5.32) will hold good if

\[
U'' + \frac{\beta g \left[ 2(U - c_r) + \omega k_i t \right] |k|^2}{|U - c_i|^2} > 0.
\]

at least at one point in the flow domain.

It implies that

\[
- |U''_{\text{min}}| \left[ b^2 - 2bc_r + |c|^2 \right] + (\beta g)_{\text{min}} \left[ 2(c_i - c_r) + \frac{\omega k_i t}{|k|^2} \right] > 0
\]

or

\[
- |U''_{\text{min}}| \left[ b^2 |k|^2 - 2b \omega k_r + \omega \right]
\]

\[
+ (\beta g)_{\text{min}} \left[ 2(a|k|^2 - \omega k_i) + \omega k_i t \right] > 0
\]

or

\[
[2a(\beta g)_{\text{min}} - |U''_{\text{min}}| b^2] |k|^2
\]

\[
+ |2 \left[ b |U''_{\text{min}}| (\beta g)_{\text{min}} \right] \omega k_r
\]

\[
+ \omega k_i t (\beta g)_{\text{min}} - \omega |U''_{\text{min}}| > 0. \quad \text{(4.5.32)}
\]

Thus we have the following result:
Theorem. 4.5.2. For $k_r < 0$, $k_i < 0$ and $U'' < 0$ every where in the flow domain then $(k_r, k_i)$ lies outside the region given by

$$[2a (\beta g)_{\text{min}} - |U''_{\text{min}}| b^2] |k|^2 + 2 [b |U''_{\text{min}}| - (\beta g)_{\text{min}}] \omega k_r + \omega k_i t (\beta g)_{\text{min}} - \omega^2 |U''_{\text{min}}| = 0$$

For linear velocity profile i.e. for $U'' = 0$ every where in the flow domain equation (4.5.32) reduces to

$$\frac{2k_r^2}{c_r} \int |H|^2 - \frac{\beta g \left[ (U - c_r) \left( \frac{t k_r}{k_i} - 2 \right) + \frac{\omega k_i t}{|k|^2} \right] |H|^2}{|U - c|^2 |U - c|^2} = 0.$$  

...(4.5.34)

Equation (4.5.34) will hold good for $k_r < 0$, $k_i < 0$ if

$$(U - c_r) \left( \frac{t k_r}{k_i} - 2 \right) + \frac{\omega k_i t}{|k|^2} < 0$$

at least at one point in the flow domain.

It implies that

$$(b - c_r) \frac{t k_r}{k_i} - 2(a - c_r) + \omega k_i t/|k|^2 < 0$$

$$\Rightarrow b |k|^2 k_r t - \omega k_r^2 t - 2a k_i |k|^2 + 2\omega k_i k_r$$

$$+ \omega k_i^2 t > 0$$  

...(4.5.35)

Thus we have the following result:
**Theorem. 4.5.3.** For $k_r < 0, k_i < 0$ and linear velocity profile, $(k_r, k_i)$ must lie outside the region given by

\[ b \ |k| \ t - \omega \ k_r \ t - 2a \ k_i \ |k|^2 + 2\omega \ k_i \ k_r + \omega \ k_i^2 \ t = 0 \]

The imaginary part of equation (4.5.23) can be written as

\[
\int \ |G'|^2 + \frac{|k|^2}{c_p} \int \ (2U - c_p) \ |G|^2 \\
+ \int \ \left( \frac{\beta g}{|U - c_i|^2} - \frac{U'^2}{4|U - c|^2} \right) \ |G|^2 \\
+ \frac{t}{c_i} \int \frac{U' \ c_i (U - c_p)}{|U - c_i|^2} \ |G|^2 + \frac{t}{c_i} \int \frac{\beta g \ c_i}{|U - c_i|^2} \ |G|^2 = 0
\]

...(4.5.36)

Equation (4.5.36) implies that if $U'' > 0$, $U'' U - \beta g > 0$ throughout the flow domain then for $c_i > 0$ i.e. unstable modes, it will hold good if

\[
\frac{|k|^2}{c_p} [2U - c_p] + \frac{\beta g}{|U - c_1|^2} - \frac{U'^2}{4|U - c|^2} < 0
\]

at least at one point in the flow domain.

It implies that

\[
\frac{|k|^2}{c_p} (2b - c_p) + \frac{(\beta g)_{\text{min}}}{|U - c_1|^2}_{\text{max.}} - \frac{U'^2_{\text{max}}}{4c_i^2} < 0
\]
or \[ |k|^2 \left( \frac{2b k_r}{\omega} - 1 \right) + \frac{(\beta g)_{\text{min}}}{b^2 \left( \frac{\omega}{k^2} (1 + t)^2 \right)} - \frac{U^2_{\text{max}}}{4\omega^2 k^4} |k|^4 \]

or \[ \left( \frac{2b k_r}{\omega} - 1 \right) + \frac{(\beta g)_{\text{min}}}{b^2 |k|^2 + \omega^2 (1 + t)^2} - \frac{U^2_{\text{max}} |k|^2}{4\omega^2 k^2} < 0. \]

...(4.5.37)

Thus we have the following results:

**Theorem. 4.5.4.** If \( U'' > 0 \) and \( U'' U - \beta g > 0 \) throughout the flow domain then for \( k_r < 0 \) the unstable modes lie in the region given by

\[ \left( \frac{2b k_r}{\omega} - 1 \right) + \frac{(\beta g)_{\text{min}}}{b^2 |k|^2 + \omega^2 (1 + t)^2} - \frac{U^2_{\text{max}} |k|^2}{4\omega k^2} < 0 \]

Also if \( U'' < 0, U'' U - \beta g < 0 \) throughout the flow domain then for \( c_i < 0 \) i.e. the stable modes equation (4.5.36) will hold good if

\[ \frac{|k|^2}{c_p} \left[ 2U - c_p \right] + \frac{\beta g}{|U - c_i|^2} - \frac{U^2}{4|U - c_i|^2} < 0 \]

at least at one point in the flow domain.

It implies that

\[ \frac{|k|^2}{c_p} (2b - c_p) + \frac{(\beta g)_{\text{min}}}{(|U - c_i|^2)_{\text{min}}} - \frac{U^2_{\text{max}}}{4c_i^2} < 0 \]
or \( \left( \frac{2b}{\omega} \frac{k_r}{\omega} - 1 \right) + \frac{(\beta g)_{\text{min}}}{b^2 |k|^2 + \omega (1 + t^2)} - \frac{U_{\text{max}}^2 |k|^2}{4\omega^2 k_r^2} < 0 \).

...(4.5.38)

Thus we have the following result:

**Theorem.** 4.5.5. If \( U'' < 0 \) and \( U'' U - \beta g < 0 \) throughout the flow domain then the stable modes for \( k_r < 0 \) lie in the region given by

\[
\left( \frac{2b k_r}{\omega} - 1 \right) + \frac{(\beta g)_{\text{min}}}{b^2 |k|^2 + \omega^2 (1 + t^2)} - \frac{U_{\text{max}}^2 |k|^2}{4\omega^2 k_r^2} < 0
\]

The imaginary part of (4.5.25) is given by

\[
\int (U - c_r) |F'|^2 + \frac{|k|^2}{c_p} \int U (U - c_p) |F|^2 + \frac{t^3}{c_l}
\]

\[
\int \frac{|c|^2 c_r U''}{|U - c_l|^2} |F|^2 + \frac{t^2}{2} \int \frac{c_r U'' (2U - c_p)}{|U - c_l|^2} |F|^2 + \frac{t}{2 c_l}
\]

\[
\int \left[ -U'' c_r + \beta g (U - c_p) c_r \right] |F|^2.
\]

...(4.5.39)

From equation (4.5.38) we see that if
\[ \frac{|k|^2}{c_p} U + \frac{c_r t^2 U''}{2 |U - c_i|^2} \geq 0 \]
\[ \text{(4.5.39)} \]

and \[ 2t^2 |c|^2 U' - U' + \beta_g (U - c_p) < 0 \]

throughout the flow domain then we must necessarily have \( c_i < 0 \)

which implies the stability of the system.

Above relations imply that:

**Theorem 4.5.6.** The stable modes lie in the region given by

\[ |k|^2 b - \frac{\omega^2 t^2 |U''_{\text{max}}|}{b^2 |k|^2 + \omega^2 |1 + t^2|} < 0 \]

and

\[ -2t^2 k_r |k|^2 |U''_{\text{max}}| + k_r |U''_{\text{min}}| \omega^2 \]

\[ + (\beta g)_{\text{max}} \omega^2 (b k_r - \omega) > 0, \]

**4.6 Concluding Remarks.**

In this chapter, I have discussed the spatial stability of stratified parallel shear dusty flow and obtained some stability conditions for \( k_r < 0 \). It has been confirmed that stratification and coarse dust have stabilizing effect while the fine dust has destabilizing effect.