CHAPTER 7

SPATIAL INSTABILITY OF SHEAR FLOW
IN A POROUS MEDIUM

7.1. Introduction.

In this chapter the spatial stability of density stratified flows through a porous medium has been discussed. Flow of density stratified through a porous medium and the stability problem there of have been of significant importance in literature. Contribution to the problem of stability in a porous medium are well summerized in the books by Scheidegger (1960) and Yih (1980).

Undoubtedly the stability of the fluid interfaces moving in a porous medium is of significant importance for the ground water hydrology, petroleum production engineering, civil engineering etc. Extensive studies have been conducted on the stability of the interface between two fluids of different densities and viscosities through porous media when there is movement or displacement perpendicular to the interface as well as parallel to the interface. Several authors have endeavoured to study the thermal instability of a fluid saturated porous layer initiated by Horton and Roger (1945) and Lapwood (1948). An excellent review of the literature is provided by Joseph (1976). Further contributions in this direction are

In all these analysis, the fluid flow has been assumed to be governed by Darcy's law. A general argument has been advanced since the experimental findings of Darcy (1956) that the inclusion of inertia is not interesting for the physics of flow through porous medium. But there are situations in engineering and Geophysics in which a departure from Darcy's law and the inertia effects not included in Darcy's model may become significant.

Jaimala and Aggarwal (1991) investigated the stability of a density stratified fluid with horizontal through a porous medium. They obtained semi-circle type bounds on the complex wave velocity of unstable modes (if exist) under certain conditions. They discussed the temporal stability of the system. In this chapter the same problem has been taken for the discussion in view of spatial stability.

7.2. Formulation of the Problem.

A fluid saturated porous medium boundd by two infinite parallel plates situated at a distance d apart is considered. The x-axis is taken along the main flow in the lower plane and y-axis is taken perpendicular to the planes. The solid incompressible substrate has a constant porosity \( \phi \) and a constant permeability \( K \). A considertion of theoritical possibilities of the structure of porous media makes one realize that a general correlation
between porosity and permeability do not exist. Porosity is the ratio of the volume of unit pore to that of the unit cell and it will be independent of the radius R of the uniform spheres comprising the assemblage where as the permeability of the array is dependent upon the actual dimensions of the pore opening and is proportional to $R^2$ so that the porosity of an assemblage can not alone provide an accurate indication of its permeability. It is possible for two porous media of the same porosity to have different permeabilities. The following simplifying assumptions have been taken in the present analysis:

(i) The saturated fluid is incompressible and all the physical properties of the fluid except the density are constant.

(ii) The porous medium is everywhere isotropic and homogeneous.

(iii) The medium obeys the Boussinesq approximation which states that the variations of density in the equations of motion can safely be ignored everywhere except in its association with the external force.

The theory of porous layer is based mainly on Darey’s law which is applicable to the steady flows. When inertial effects are negligible. Since we wish to study a flow in which inertial effects are included and substantial derivative of $\bar{u}$ is not zero, we have assumed that under such conditions the drag on the fluid can still be approximated by Darey’s law. Thus
the physical system under consideration obeys the following fundamental equations:

\[ \Delta \cdot \overline{u} = 0 \quad \ldots (7.2.1) \]

\[ \frac{\rho}{\phi} \frac{\partial \overline{u}}{\partial t} + \frac{1}{\phi^2} \rho \overline{u} \cdot \Delta \overline{u} = - \Delta p - \frac{\nu \rho u}{k} - g \rho \overline{\lambda} \quad \ldots (7.2.2) \]

and

\[ \frac{\partial \rho}{\partial t} + \frac{1}{\phi} \overline{u} \cdot \nabla \rho = 0, \quad \ldots (7.2.3) \]

where \( g \) is the acceleration due to gravity \( \overline{\lambda} = (0, 1, 0) \) is the unit vector in the vertically upward direction, \( \overline{u} \) is the seepage velocity, \( \rho \) is the density, \( p \) is the pressure and \( \nu \) is the kinematic viscosity of the saturated fluid. The viscosity is introduced through the Darcy resistance term \( \frac{\nu \rho \overline{u}}{k} \).

The seepage velocity \( \overline{u} \) and the pore average velocity say \( \overline{q} \) are related by the equation

\[ \overline{u} = \rho \overline{q} \]

where the porosity is defined as

\[ \phi = \frac{\text{Volume of the voids}}{\text{total volume}} \quad (0 < \phi < 1) \]

For very fluffy foam materials \( \phi \) is nearly one and in beds of packed spheres \( \phi \) is in the range \( 0.25 - 0.50 \).

Suppose the basic state of the system under discussion is given by
\[ \bar{\mathbf{u}} = (\mathbf{U}(y), 0) \]
\[ \bar{\rho} = \bar{\rho}(y) \]
\[ \text{and } \bar{p} = \bar{p}(y) \]

...(7.2.4)

This basic state satisfies the equation (7.2.1) and (7.2.3) identically. The equation (7.2.2) gives

\[ - \frac{\partial \bar{p}}{\partial x} - \nu \frac{\bar{\rho}}{K} \frac{\partial \bar{U}}{\partial y} = 0 \]

and \[ - \frac{\partial \bar{p}}{\partial y} - g \bar{\rho} = 0 \]

...(7.2.5)

These equations yield

\[ \frac{\partial^2 \bar{p}}{\partial x \partial y} = - \frac{1}{K} \frac{\partial}{\partial y} (\mu \bar{U}) \]

and \[ \frac{\partial^2 \bar{p}}{\partial x \partial y} = 0 \]

...(7.2.6)

From these equations, we have

\[ \frac{\partial}{\partial y} (\mu \bar{U}) = 0 \]

or \[ \frac{\partial}{\partial y} (\nu \bar{\rho} \bar{U}) = 0 \].

...(7.2.7)

For stability analysis we take the perturbed state as

\[ \bar{\mathbf{u}} = (\mathbf{U}(y), 0) + (\mathbf{u}', \mathbf{v}') \]
\[ \bar{\rho} = \bar{\rho}(y) + \rho' \]
\[ \bar{p} = \bar{p}(y) + p' \]

...(7.2.8)
Substituting equations (7.2.8) into the governing equations (7.2.1)
to (7.2.3) linearising them and finally analysing the perturbation quantities
of the form
\[ f'(x, y, t) = f(y) \exp \left\{ i k \left[ x - \frac{c t}{\phi} \right] \right\} \] (2.9) the linearised equation

take the form
\[ i k u + dv = 0 \] ...(7.2.9)

\[ \frac{i \rho}{\rho^2} (U - c)v = -\frac{\rho D_p}{K} - \frac{\mu v}{KK} + \frac{g v D_p}{i (U - c) k^2} \] ...(7.2.10)

and
\[ \frac{i \rho}{\phi^2} (U - c)ku + \frac{\rho v}{\phi^2} DU = -\frac{i k^2 p}{k} - \frac{\mu u}{K}. \] ...(7.2.11)

Now eliminating u and p from equations (7.2.9), (7.2.10) and
(7.2.11) we get

\[ k^2 \rho \left[ U - c - \frac{i v \rho}{K k} \right] = D \left[ \rho \left( U - c - \frac{i v \phi^2}{K k} \right) Dv - \rho v DU \right] \]

\[ - g \frac{\rho D_p}{(U - c) v}. \] ...(7.2.12)

Non-dimensionalizing equation (7.2.12) we get

\[ k^2 \rho \left( U - c - i \frac{R_{D_0}^{-1} k^{-1}}{} \right) v \]

\[ = D \left[ \rho \left( U - c - i \frac{R_{D_0}^{-1} k^{-1}}{} \right) Dv - \rho v DU \right] + \frac{J_0 \rho}{U - c} \]

\[ \] ...(7.2.13)
where \[ R_{D_0}^{-1} = \frac{\nu \phi^2 d}{U_0 K} \]

and \[ J_0 = \frac{g \phi^2 \beta d}{U_0^2} \]

in which \( \beta = -\frac{D \rho}{\rho} \), \( d \) is the characteristic length and \( U_0 \) is the characteristic velocity.

Here \( R_{D_0} \) is the ratio of the inertia force to the Darcy drag force called Darcy–Reynold number in analogy with the ordinary Reynolds number which is the ratio of the inertia force to the viscous force and \( J_0 \) is the ratio of the buoyancy force to the inertia force and is called the Richardson number.

The boundary conditions are
\[ v = 0 \text{ at } y = 0 \text{ and } y = 1. \] ...(7.2.14)

Let \[ W_1 = U - c - i R_{D_0}^{-1} k^{-1} = W - i R_{D_0}^{-1} k^{-1}. \]

Taking the transformation \( v = W_1 F \) in (7.2.13) we get
\[ k^2 \rho W_1^2 F = D [\rho W_1^2 D F] + \frac{J_0 \rho W_1 F}{W}. \] ...(7.2.15)

The boundary conditions are
\[ F = 0 \text{ at } y = 0 \text{ and } y = 1. \] ...(7.2.16)
7.3. Analysis.

For spatial stability \( k = k_r + i k_i \) denotes the complex wave number and \( \omega > 0 \) the real frequency. The complex wave velocity \( c \) is given by

\[
c = \frac{\omega}{k}, \quad \text{...(7.3.1)}
\]

which gives

\[
c_r = \frac{\omega k_r}{|k|^2} \quad \text{and} \quad c_i = -\frac{\omega k_i}{|k|^2}. \quad \text{...(7.3.2)}
\]

The phase velocity of the perturbations is defined as

\[
c_p = \frac{\omega}{k_r}. \quad \text{...(7.3.3)}
\]

The unstable modes will be characterised by \( k_i \neq 0 \).

Multiplying equation (7.2.15) by \( \bar{F} \) the complex conjugate of \( F \) and integrating over the flow domain \((0, 1)\), we get

\[
\int \rho W_1^2 (|DF|^2 + k_r^2 |F|^2) - \int \frac{\rho J_0 W_1 |F|^2}{W} = 0. \quad \text{...(7.3.4)}
\]

Separating the real and imaginary parts of (7.3.4), we get

\[
\int \rho \left[ \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right)^2 - \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right)^2 \right]
\times [ |DF|^2 + (k_r^2 - k_i^2) |F|^2 ]
\]
\[ + 4k_r k_i \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right) \int \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right) |F|^2 \]

\[ - \int \rho J_0 \left[ (U - c_r) \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right) + c_i \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right) \right] |F|^2 = 0 \]

...(7.3.5)

\[ - 2 \int \rho \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right) \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right) \left[ |DF|^2 + (k_r^2 - k_i^2) |F|^2 \right] \]

\[ 2k_r k_i \int \rho \left[ \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right)^2 - \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right)^2 \right] \]

\[ - \int \rho J_0 \left[ \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right) \left( U - c_r - \frac{R_D^{-1} k_i}{|k|^2} \right) - (U - c_r) \left( c_i + \frac{R_D^{-1} k_r}{|k|^2} \right) \right] \frac{F^2}{|W|^2} = 0. \]

...(7.3.6)

Equation (7.3.6) can be written as

\[ \int \rho \left( c_i + R c_r \right) \left( U - c_r + c_i R \right) \left[ |DF|^2 + (k_r^2 - k_i^2) |F|^2 \right] \]

\[ - R k_r k_i \int \left[ \left( U - c_r + c_i R \right)^2 - (c_i + R c_r)^2 \right] |F|^2 \]

\[ + \int \frac{\rho J_0}{2 |U - c|^2} \left[ c_i \left( U - c_r + c_i R \right) - \left( U - c_r \right) \left( c_i + R c_r \right) \right] |F|^2 = 0 \]

where \( R = \frac{R_D^{-1}}{\omega} \)
or \[
\int \rho (U - c_r) |DF|^2 + \frac{R}{c_i} \int \left[ c_r (U - c_r) + c_i^2 \right] |DF|^2
\]
\[+ R^2 c_r \int \rho |DF|^2 + \int \rho \frac{|k|^2}{c_p} U (U - c_p) |F|^2
\]
\[+ \frac{R^{-1}}{c_i |k|^2} \int \rho (U - c_p) (|k|^2 - J_0) |F|^2 = 0. \quad \text{(7.3.7)}\]

From equation (7.3.7) it follows that \(c_i < 0\) if
\[
\begin{aligned}
&U - c_p > 0 \\
\text{and} & |k|^2 > J_0
\end{aligned}
\]
\[\quad \text{...(7.3.8)}\]

Thus we have the following result:

**Theorem.** 7.3.1. Stable modes for \(c_r > 0\) lie in the region given

\[k_r > \frac{\omega}{b}\]

and \(k_r^2 + k_i^2 > J_0\).

**Proof:** From (7.3.8) we have
\[c_p < U < b, \quad b = U_{\max}.
\]

or \[\frac{\omega}{k_r} < b\]
and \( k_r^2 + k_l^2 > J_0 \).

This result does not depend upon \( U' \) and \( (\rho U')' \).

Taking substitution \( F = W_1^{-1/2} G \) in (7.2.15) and dividing the resulting equation by \( W_1^{1/2} \), we get

\[
(\rho W_1 G')' - \rho W_1 k^2 G - \frac{(\rho U')'}{2} G - \frac{\rho U^2}{4 W_1} G + \frac{J_0}{W} G = 0.
\]

...(7.3.9)

Also taking the substitution \( F = W_1^{-1} H \) in (7.2.15) and dividing the resulting equation \( W_1 \), we get

\[
(\rho H')' - (\rho k^2 H) - \frac{(\rho U')'}{W_1} H + \frac{\rho J_0}{W W_1} H = 0.
\]

...(7.3.10)

Multiplying equation (7.3.9) by \( \overline{G} \), the complex conjugate of \( G \) and integrating over the flow domain, we get

\[
\int \rho W_1 \left[ |G'|^2 + k^2 |G|^2 \right] + \int \frac{(\rho U')'}{2} |G|^2

+ \int \left( \frac{\rho U^2}{4 W_1} - \frac{\rho J_0}{W} \right) |G|^2 = 0.
\]

...(7.3.11)

Similarly multiplying equation (7.3.10) by \( \overline{H} \), the complex conjugate of \( H \) and integrating over the flow domain, we get
\[
\int \rho \left( |H'|^2 + k^2 |H|^2 \right) + \int \frac{(\rho U')'}{W_1} |H|^2 - \int \frac{\rho J_0 |H|^2}{W W_1} = 0.
\]

...(7.3.12)

The imaginary part of (7.3.11) is given by

\[- \int \rho \left( c_i + c_r R \right) \left( |G'|^2 + (k_r^2 - k_i^2) |G|^2 \right) + 2k_i k_r \int \rho \left( U - c_r + c_i R \right) |G|^2
\]

\[+ \int \left[ \frac{\rho U'^2 [c_i + c_r R]}{4 |W_1|^2} - \frac{\rho J_0 c_i}{|W|^2} \right] |G|^2 = 0
\]

which can be written as

\[
\int \rho |G'|^2 + \frac{R c_r}{c_i} \int |G'|^2 + \frac{k_r^2}{c_r} \int \rho \left( 2U - c_p \right) |G|^2
\]

\[+ \frac{R c_r}{c_i} \int \left( |k|^2 - \frac{U'^2}{4 |W_1|^2} \right) |G|^2 + \int \left( \frac{J_0}{|W|^2} - \frac{U^2}{4 |W_1|^2} \right) \rho |G|^2 = 0.
\]

...(7.3.13)

Now if the conditions

\[2U - c_p > 0 \]

\[|k|^2 - \frac{U'^2}{4 |W_1|^2} > 0\]
and
\[ \frac{J_0}{|W|^2} - \frac{U'^2}{4|W_1|^2} > 0 \]

hold everywhere in the flow domain, then necessarily \( c_1 \) should be negative which implies the stability of the system. These conditions do not depend upon the sign of \( U' \) and \( (\rho U')' \). In the form of \((k_r, k_i)\) the above inequalities take the form

\[ k_r > \frac{\omega}{2b}, \]

\[ \left( k_r - \frac{\omega}{U} \right)^2 + \left( k_i - \frac{\omega}{U} \right)^2 > \frac{U'^2}{4b^2}, \]

and
\[ \left( k_r - \frac{\omega}{U} \right)^2 + \left( k_i - \frac{\omega R}{U} \frac{4J}{1 - 4J} \right)^2 < \frac{\omega^2 R^2}{U^2} \frac{4J}{(1 - 4J)^2}. \]

where 
\[ J = \frac{U'^2}{J_0} < \frac{1}{4} \]

Thus we have the following result:

**Theorem. 7.3.2.** For \( k_r > 0 \) the stable modes lie in the region given by

\[ k_r > \frac{\omega}{2b}, \]

\[ \left( k_r - \frac{\omega}{U} \right)^2 + \left( k_i - \frac{\omega}{U} \right)^2 > \frac{U'^2}{4b^2}. \]
and 

\[
\left( k_r - \frac{\omega}{U} \right)^2 + \left( k_i - \frac{\omega R}{U} \cdot \frac{4J}{1 - 4J} \right)^2 < \frac{\omega^2 R^2}{U^2} \cdot \frac{4J}{(1 - 4J)^2}
\]

This theorem confirms the fact that the porous media with high Darcy resistance has a stabilizing effect.

The imaginary part of (7.3.12) is given by

\[
2k_r k_i \int \rho |H|^2 + \int \frac{(\rho' U') (c_i + c_r R)}{|W_1|^2} |H|^2
\]

\[
- \int \frac{\rho J_0 [(U - c_r) (2c_i + Rc_r) c_i^2 R_2 |H|^2]}{|W|^2 |W_1|^2} = 0. \quad \text{(7.3.14)}
\]

Equation (7.3.14) can also be written as

\[
- 2c_i \left[ \frac{k_r^2}{c_r} \int \rho |H|^2 - \int \frac{(\rho' U') |H|^2}{2 |W_1|^2} + \int \frac{\rho J_0 (U - c_r)}{|W|^2 |W_1|^2} |H|^2 \right]
\]

\[
+ R \left[ \int \frac{(\rho U') c_r |H|^2}{|W_1|^2} - \int \left( \frac{\rho J_0 [(U - c_r) c_r + c_i^2]}{|W|^2 |W_1|^2} \right) |H|^2 \right] = 0
\]

\[
\quad \text{... \text{(7.3.15)}}
\]

If \((\rho' U')\) is negative and \((U - c_r) > 0\) every where in the flow domain, then equation (7.3.15) holds good if \(c_i\) is negative which implies that \(k_i\) is positive.

But \(U - c_r > 0\) every where in the flow domain

\[\Rightarrow \quad c_r < U \text{ for all } y\]
Thus we have the following theorem.

**Theorem. 7.3.3.** If \((\rho U')\)' is negative throughout the flow domain, then for spatially decaying stable modes with \(k_r > 0\) (\(k_r, k_i\)) must lie outside the circle whose centre is \(\left(\frac{\omega}{2b}, 0\right)\) and radius \(\frac{\omega}{2b}\).

The real part of (7.3.12) is given by

\[
\int \rho \left[ |H'|^2 + (k_r^2 - k_i^2) |H|^2 \right] + \int \frac{(\rho U')'[U - c_r + c_i R]}{|W_1|^2} |H|^2
\]

\[
- \int \rho J_0 \left[ (U - c_r) (U - c_r + c_i R) - c_i (c_i + R c_r) \right] \frac{|W_1|^2 |W|^2}{|W_1|} |H|^2 = 0.
\]

...(7.3.16)

Multiplying equation (7.3.14) by \(\frac{k_r^2 - k_i^2}{-2k_r k_i}\) and adding it to the equation (7.3.16), we get
\[
\int \rho |H'|^2 + \int \frac{(\rho \ U')'}{2 |W_1|^2} \left[ 2U - c_p - R_{D_0}^{-1} \right] |H|^2 \\
+ \int \frac{\rho J_0 (c_p - U)}{|W|^2 |W_1|^2} \left( U - \frac{R_{D_0}}{k_i} \right) |H|^2 = 0. \tag{7.3.17}
\]

Now if \((\rho \ U')'\) is positive everywhere and the phase velocity satisfies the condition

\[b < c_p < 2a\]

then necessarily \(k_i\) should be positive. This implies the stability of the system. Thus we have the following result:

**Theorem. 7.3.4.** If \((\rho \ U')'\) is positive everywhere in the flow domain and the phase velocity lies between the maximum and twice the minimum of the flow velocity then the flow is necessarily stable.

Further if \((\rho \ U')' > 0\) everywhere in the flow domain and \(c_p > 2a\) then for unstable modes from equation (7.3.17) we must necessarily have

\[2U - c_p - \frac{R_{D_0}^{-1}}{|k_i|} < 0\] somewhere in the flow domain.

or

\[|k_i| > \frac{R_{D_0}^{-1}}{c_p - 2a}\]
some where in the flow domain.

If \((\rho U')' < 0\) every where in the flow domain and \(c_p > 2b\) then for unstable modes from equation (7.3.17) we must necessarily have

\[
c_p - 2U - \frac{R_D^{-1}}{|k_i|} < 0
\]

or

\[
|k_i| < \frac{R_D^{-1}}{c_p - 2b}
\]

some where in the flow domain.

7.4. Concluding Remarks.

Here the spatial instability of shear flow in a porous medium has been discussed. Eigen values spectrum has been obtained for \(k_r > 0\). It has been shown that porous media with high Darcy resistance has a stabilizing effect.
Fig. 7.1. The region of stability

I. \[ k_r = \frac{\omega}{b} \]

II. \[ k_r^2 + k_t^2 = J_0 \]

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Fig. 7.2. The region of stability when \((\rho U')' > 0\) every where in the flow domain.

I. \[ k_r = \frac{\omega}{b} \]

II. \[ k_r = \frac{\omega}{2a} \]