Chapter 1

INTRODUCTION

In this chapter, we give a brief survey of some of the recent researches carried out for developing multiple hypergeometric series, basic bilateral series, mock theta functions and their transformations during the last decade. No attempt has been made to give an exhaustive account of the entire multiple hypergeometric series, basic bilateral series, mock theta functions and its transformation theory but only works relevant to the present context have been described.

1.1. Ordinary Hypergeometric Series

Ordinary hypergeometric functions have been gaining more and more importance during the recent past because of their important applications in number theory, combinatorial analysis, theory of partitions, vector spaces etc. The series:

\[
\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}
\]

is called the Gauss Series or the ordinary hypergeometric series. It is represented by the symbol \( {}_2F_1[a,b;c;z] \).

The variable is \( z \) and \( a, b \) and \( c \) are called the parameters of the function. If either of the quantities \( a \) or \( b \) is a negative integer, the series has only a finite number of terms and becomes a polynomial.
In 1812, Gauss defined the modern infinite series (1.1.1) and introduced the notation $F(a,b;c;z)$ for it. He also proved a famous summation theorem

$$2F_1[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (1.1.2)$$

The series

$$1 + \frac{a_1a_2...a_A}{b_1b_2...b_B} z + \frac{a_1(a_1+1)a_2(a_2+1)...a_A(a_A+1)}{b_1(b_1+1)b_2(b_2+1)...b_B(b_B+1)} \frac{z^2}{2!} + ...$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_A)_n}{(b_1)_n(b_2)_n...(b_B)_n} \frac{z^n}{n!} \quad (1.1.3)$$

is called general ordinary hypergeometric series with $A$ parameters in the numerator and $B$ parameters in the denominator. We shall follow the usual Ordinary Hypergeometric notations given below:

$$(a)_n = a(a+1)(a+2)(a+3)...(a+n-1), \quad (a)_0 = 1$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$(a)_n = \frac{(-1)^n}{(1-a)_n}, \quad (a+n)_n = \frac{(a)_zn}{(a)_n} = \frac{\Gamma(a-n)}{\Gamma(a)}$$

$$(a-n)_n = (-1)^n(1-a)_n$$

$$A_{\ F_B}[a_1,a_2,a_3,...,a_A,b_1,b_2,b_3,...,b_B,z] = A_{\ F_B}[(a);(b);z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{((b)_B)_n n!} \quad (1.1.4)$$

In equation (1.1.4), it is assumed that it has $A$ numerator parameters $a_1,a_2,a_3,...,a_A$, $B$ denominator parameters $b_1,b_2,b_3,...,b_B$, and one variable $z$. Any of these quantities may be real or complex but the $b$ parameters must not be negative, as in that case the series is not defined.
If any of the $a$ parameters is a negative integer, the function reduces to a polynomial.

In general, series $\sum A F_B$ is called Saalschutzian if the sum of the numerator parameters exceeds the sum of the denominator parameters by one, that is if

$$b_1 + b_2 + ... + b_B = a_1 + a_2 + ... + a_A + 1$$

(1.1.5)

If $A = B + 1$ and $1 + a_1 = b_1 + a_2 = b_2 + a_3 = ... = b_B + a_{B+1}$

(1.1.6)

then the series is said to be nearly poised, and if all but one of the above pairs of parameters have the same sum, then the series (1.1.4), is called a nearly poised series of the first kind i.e. if

$$1 + a_1 \neq b_1 + a_2 = b_2 + a_3 = ... = b_B + a_{B+1}$$

(1.1.7)

and it is called a nearly poised series of the second kind if

$$1 + a_1 = b_1 + a_2 = b_2 + a_3 = ... = b_B + a_{B+1}^.$$ (1.1.8)

The order of summation of a terminating nearly poised series can be reversed so that the resulting series is either of the first kind or of the second kind.

One of the important identity of ordinary hypergeometric function is given by Euler [71] as:

$$\sum F_1[a,b;c;z] = (1-z)^{c-a-b} F_1[c-a,c-b;c;z].$$

(1.1.9)
Famous summation theorem by Gauss [71] was given as:

\[ _2 F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \] (1.1.10)

Another important summation theorem

\[ _3 F_2[a, b, -n; c, 1 + a + b - c - n; 1] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \] (1.1.11)

is known as Saalschütz’s theorem [71]. It gives the sum of the series

\[ _3 F_2[a, b, c; d, e; 1] = \Gamma \left[ \begin{array}{c} d, 1 + a - e, 1 + b - e, 1 + c - e \\ 1 - e, d - a, d - b, d - c \end{array} \right] \] (1.1.12)

provided that one of the numerator parameters is a negative integer \(-n\), and that

\[ d + e = 1 + a + b + c. \] (1.1.13)

When \( n \to \infty \), in equation (1.1.12), it reduces to equation (1.1.10).

Also, if \( \text{Re}(b) < 1 \), then
\[ _2 F_1[a, b; 1 + a - b; -1] = \Gamma \left[ \begin{array}{c} 1 + a - b, 1 + \frac{1}{2} a \\ 1 + \frac{1}{2} a - b, a + b \end{array} \right]. \] (1.1.14)

which is known as Kumer’s Series.

Dixon theorem [71] gives the sum of a well-poised \(_3 F_2\) series as

\[ _3 F_2[a, b, c; 1 + a - b, 1 + a - c; 1] = \Gamma \left[ \begin{array}{c} 1 + \frac{1}{2} a, 1 + \frac{1}{2} a - b - c, 1 + a - b, 1 + a - c \\ 1 + a, 1 + a - b - c, 1 + \frac{1}{2} a - b, 1 + \frac{1}{2} a - c \end{array} \right]. \] (1.1.15)

provided \( \text{Re} \left( \frac{1}{2} a - b - c \right) > -1 \).

In particular if \( c = -n \), the result reduces to

\[ _3 F_2[a, b, -n; 1 + a - b, 1 + a + n; 1] = \frac{(1 + a)_n (1 + \frac{1}{2} a - b)_n}{(1 + a)_n (1 + a - b)_n}. \] (1.1.16)

Finally, when \( c \to \infty \), Dixon theorem [71] reduces to Kummer theorem [71].

Another summation theorem due to Watson [71] is given by
\[ {}_3F_2 \left[ a,b,c; \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b; 2c; 1 \right] \]

\[ = \Gamma \left[ \frac{1}{2} + c - \frac{1}{2}b, \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b, 2c; \frac{1}{2}a, \frac{1}{2}a - \frac{1}{2}a + \frac{1}{2}b + c, 1 + c - \frac{1}{2}a, \frac{1}{2} + 2c - \frac{1}{2}a - \frac{1}{2}b \right] \]

\[ = \Gamma \left[ \frac{1}{2} + c, \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b, 2c; \frac{1}{2}a - \frac{1}{2}a + \frac{1}{2}b + c \right] \]

\[ \frac{1}{2} + \frac{1}{2}a - \frac{1}{2}b + c, a, \frac{1}{2} + \frac{1}{2}a - \frac{1}{2}b + c, \frac{1}{2} + 2c - a, \frac{1}{2}b - \frac{1}{2}c \] (1.1.17)

provided that the series is convergent, i.e. real part of \( Rl \left( \frac{1}{2} - \frac{1}{2}a - \frac{1}{2}b + c \right) > 0 \).

When \( c \to \infty \), the result reduces to Gauss theorem.

Dougall theorem \([71]\) gives the sum of well poised \( {}_7F_6 \) series in which the sum of the denominator parameters exceeds that the numerator parameters by 2 as

\[ {}_7F_6 \left[ a, \frac{1}{2}a, b, c, d, e, f; \frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f; 1 \right] \]

\[ = \Gamma \left[ 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - f, 1 + a - b - c - d, 1 + a - b - c - f, 1 + a - b - d - f, 1 + c - d - f; \right] \]

\[ 1 + a, 1 + a - b - c, 1 + a - b - d, 1 + a - c - d, 1 + a - b - f, 1 + a - c - f, 1 + a - d - f, 1 + a - b - c - d - f \] (1.1.18)

provided that the series terminates and that

\[ 1 + 2a = b + c + d + e + f \] (1.1.19)
1.2. Multiple Hypergeometric Series

In 1880, Appell [82] generalized Gauss functions by considering the product of two Gauss functions, i.e.

\[ _2F_1(a, b; c; x) _2F_1(a', b'; c'; y') = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m (a')_n (b')_n}{(c)_m (c')_n} \frac{x^m y^n}{m! n!}. \tag{1.2.1} \]

The double series led to five distinct possibilities of new functions. One such possibility, however gives us the double series

\[ \sum_{m,n=0}^{\infty} \frac{(a + m)_n (b + m)_n}{(c + m)_n} \frac{x^m y^n}{m! n!}. \tag{1.2.2} \]

which is simply the Gaussian series,

\[ _2F_1[a, b; c; x + y]. \tag{1.2.3} \]

The remaining four possibilities led to the four Appell functions [82] of two variables.

Appell function [82] of the first kind is given as

\[ F_1[a, b, b'; c, x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (b)_m (b')_n}{(c)_m} \frac{x^m y^n}{m! n!} \]

\[ = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} _2F_1[a + m, b'; c + m; y] \frac{x^m}{m!}. \tag{1.2.4} \]
where for convergence, \( \max\{|x|,|y|\} < 1 \).

This exists for all real or complex values of \( a, b, b', c, x \) and \( y \) except \( c \) a negative integer.

Similarly,

\[
F_2[a;b,b';c,c';x,y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n x^m y^n}{(c)_m (c')_n m! n!} = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \binom{2}{a + m, b'} \frac{x^m}{m!}
\]  

(1.2.5)

provided, \(|x| + |y| < 1\)

\[
F_3[a,a';b,b';c,x,y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_n (b')_n x^m y^n}{(c)_{m+n} m! n!} = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \binom{2}{a', b'} \frac{x^m}{m!}
\]  

(1.2.6)

where for convergence, \( \max\{|x|,|y|\} < 1 \).

\[
F_4[a;b,c,c';x,y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} x^m y^n}{(c)_m (c')_n m! n!} = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \binom{2}{a + m, b + m; c'} \frac{x^m}{m!}
\]  

(1.2.7)

where for convergence, \( \sqrt{|x|} + \sqrt{|y|} < 1 \)
and the denominator parameters $e$ and $e'$ are neither zero nor negative integers.

All four Appell functions reduce to ordinary Gauss Series $\mathbf{2}_1 F_1 [x]$ when $y = 0$.

In 1893, Lauricella [73] further generalized the four Appell functions $F_1, F_2, F_3, F_4$ to functions of $n$ variables and defined his functions as

$$F_A[a, b_1, b_2, ..., b_n, c_1, c_2, ..., c_n, x_1, x_2, ..., x_n] = \sum_{m_1=0}^{\infty} ... \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_m (b_2)_m \cdots (b_n)_m}{(c_1)_m (c_2)_m \cdots (c_n)_m (1)_m (1)_m \cdots (1)_m} x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \quad (1.2.8)$$

where for convergence,

$$|x_1| + |x_2| + ... + |x_n| < 1 \quad (1.2.9)$$

$$F_B[a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c, x_1, x_2, ..., x_n] = \sum_{m_1=0}^{\infty} ... \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1} (a_2)_{m_2} \cdots (a_n)_{m_n} (b_1)_m (b_2)_m \cdots (b_n)_m}{(c)_m (1)_m \cdots (1)_m} x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \quad (1.2.10)$$

provided,

$$|x_1| < 1, |x_2| < 1, ..., |x_n| < 1 \quad (1.2.11)$$

$$F_C[a, b; c_1, c_2, ..., c_n, x_1, x_2, ..., x_n] = \sum_{m_1=0}^{\infty} ... \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b)_{m_1+\cdots+m_n}}{(c_1)_m (c_2)_m \cdots (c_n)_m (1)_m (1)_m \cdots (1)_m} x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \quad (1.2.12)$$

provided,
\begin{align*}
| x_1^{1/2} | + | x_2^{1/2} | + \ldots + | x_n^{1/2} | < 1
\end{align*}

(1.2.13)

and

\[
F_B[a,b_1,b_2,\ldots,b_n;c,x_1,x_2,\ldots,x_n] = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2} \ldots (b_n)_{m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{(c)_{m_1+m_2} \ldots (1)_{m_n} (1)_{m_2} \ldots (1)_{m_1}}
\]

(1.2.14)

where for convergence,

\[
| x_1 | < 1, | x_2 | < 1, \ldots, | x_n | < 1
\]

(1.2.15)

When \( n = 2 \), Lauricella functions correspond to the Appell hypergeometric series of two variables.

\[
F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad F_D^{(2)} = F_1
\]

(1.2.16)

When \( n = 1 \), all four functions reduce to the Gauss Hypergeometric function

\[
F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1(a,b;c;x)
\]

(1.2.17)

The four Appell functions were unified and generalized by Kampe de Feriet [80] in 1921.

He defined a general hypergeometric function of two variables as

\[
F_{p,q,k}^{l,m,n} \left[ \begin{array}{c} (a)_{p};(b)_{q};(c)_{k}; \quad x,y \\ (\alpha)_{l};(\beta)_{m};(\gamma)_{n}; \end{array} \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_{r} \prod_{j=1}^{k} (c_j)_{z}}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_{r} \prod_{j=1}^{n} (\gamma_j)_{z}}
\]

(1.2.18)

where for the convergence,
\[ p + q < l + m + 1, \ p + k < l + n + 1, \ |x| < \infty, \ |y| < \infty \] \hfill (1.2.19)

or

\[ p + q = l + m + 1, \ p + k = l + n + 1 \] \hfill (1.2.20)

\[
\begin{cases}
| x |^{l/(p-l)} + | y |^{l/(p-l)} < 1 & \text{if } p > l \\
\max\{|x|, |y|\} < 1 & \text{if } p \leq l
\end{cases}
\] \hfill (1.2.21)

In 1969, Srivastava and Daoust [80] further generalized Kampe de Feriet function (1.2.18) which is referred as generalized Lauricella function of several variables:

\[
F^{A;B^{(1)},...,B^{(n)}}_{D;E^{(1)},...,E^{(n)}} \left[ \left( a_A \right) : \left( \theta^{(1)},...,\theta^{(n)} \right) ; \left( b^{(1)}_{E^{(1)}}, \Phi^{(1)} \right) ;...,\left( b^{(n)}_{E^{(n)}}, \Phi^{(n)} \right) ; z_1,...,z_n \right] \\
\left( (d_D) : \left( \Psi^{(1)},...,\Psi^{(n)} \right) ; \left( e^{(1)}_{E^{(1)}}, \delta^{(1)} \right) ;...,\left( e^{(n)}_{E^{(n)}}, \delta^{(n)} \right) \right) \\
= \sum_{m_1,...,m_n} \Xi(m_1,...,m_n) \frac{z_1^{m_1}}{(m_1)!} \cdots \frac{z_n^{m_n}}{(m_n)!} \tag{1.2.22}
\]

where for convenience,

\[
\Xi(m_1,...,m_2) = \frac{\prod_{j=1}^{A} (a_j)^{m_j \theta_j^{(1)} + \cdots + m_j \theta_j^{(n)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})^{m_j \Phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})^{m_j \Phi_j^{(n)}} \prod_{j=1}^{D} (d_j)^{m_j \Psi_j^{(1)} + \cdots + m_j \Psi_j^{(n)}} \prod_{j=1}^{E^{(1)}} (e_j^{(1)})^{m_j \delta_j^{(1)}} \cdots \prod_{j=1}^{E^{(n)}} (e_j^{(n)})^{m_j \delta_j^{(n)}}}{\prod_{j=1}^{B^{(1)}} (b_j^{(1)})^{m_j \Phi_j^{(1)}} \cdots \prod_{j=1}^{E^{(n)}} (e_j^{(n)})^{m_j \delta_j^{(n)}}} \tag{1.2.23}
\]

The coefficients $\theta_j^{(k)}$, $j = 1,...,A$, $\Phi_j^{(k)}$, $j = 1,...,B^{(k)}$, $\Psi_j^{(k)}$, $j = 1,...,D$, $\delta_j^{(k)}$, $j = 1,...,E^{(k)}$ for all $k \in \{1,...,n\}$ are zero and real constants (positive, negative) and $(b_j^{(k)})_{E^{(k)}}$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j = 1,...,B^{(k)}$ for all $k \in \{1,...,n\}$ with similar interpretations for others.
1.3. Basic Hypergeometric Series

Heine [71] defined the basic analog of the Gauss function as the infinite series

\[ 1 + \frac{(1-q^a)(1-q^b)z}{(1-q^c)(1-q)} + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})z^2}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)} + \ldots \]  

(1.3.1)

where \(|q| < 1\), so that as \(q \to 1\), this series converges to \( \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n \), the Gauss series.

We shall follow the usual basic hypergeometric notations given below:

Let \(|q| < 1\), then we write

\[ (\alpha)_n = (\alpha, q)_n = (1 - \alpha)(1 - \alpha q)...(1 - \alpha q^{n-1}), \quad n \geq 1 \]  

(1.3.2)

\[ (\alpha)_0 = 1; \quad (\alpha)_\infty = (\alpha; q)_\infty = \prod_{n=0}^{\infty} (1 - \alpha q^n) \]  

(1.3.3)

\[ (\alpha)_{-n} = (\alpha; q)_{-n} = \frac{(-\alpha)^{-n} q^{\frac{1}{2}(n+1)n}}{(\frac{q}{\alpha})_n} \]  

(1.3.4)

Heine series is the basic hypergeometric function which is written as:

\[ {}_2\Phi_1[\alpha, b; c; q; z] = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n \]  

(1.3.5)

where \(|q| < 1\) and \(|z| < 1\). Here \(\alpha, b\) and \(c\) are the parameters, \(z\) is the variable and \(q\) is called the base of the series.

The general basic hypergeometric series is:
\[ \Phi_s(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n} [(-1)^n q^{\binom{n}{s}}]^{1+s-r} z^n \] (1.3.6)

with \( \binom{n}{2} = \frac{n(n-1)}{2} \), where \( q \neq 0 \) and \( r > s + 1 \).

Equation (1.3.6), is sometimes denoted concisely through the system \( s \Phi_r((a_r); (b_s); z) \).

In equation (1.3.6), it is assumed that the parameters \( b_1, b_2, \ldots, b_s \) are such that the denominator factors in the terms of the series are non-zero. When \( r = s + 1 \), the equation (1.3.6), converges for \(|z| < 1\), \(|q| < 1\) and is called Saalschützian i.e. when

\[ qb_1 b_2 \ldots b_s = a_1 a_2 \ldots a_{s+1} \] (1.3.7)

and well poised when

\[ a_1 q = b_1 a_2 = \ldots = b_s a_{s+1} \] (1.3.8)

The series \( \Phi_{s+1} \) is called ‘nearly-poised’ if all but one of the pairs of parameters in (1.3.8), has the same product. If \( a_1 q \) is not a member of equation (1.3.8), then the series is called a ‘nearly-poised’ series of the ‘first’ kind, if \( b_s a_{s+1} \) is not a member of equation (1.3.8), it is called a ‘nearly-poised’ series of the ‘second’ kind.

One of the most important summation formula for basic hypergeometric series is given by the \( q \) analogue of the binomial theorem, namely

\[ \Phi_0(a; -; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty} \] (1.3.9)
for $|z| < 1$, $|q| < 1$.

Heine also proved that for $|c/ab| < 1$,

$$2\Phi_1(a, b; c; c/ab) = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}$$

Equation (1.3.10), is a $q$-analogue of Gauss’s summation formula

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - 1)\Gamma(c - b)}, \text{ Re}(c - a - b) > 0$$

Jacobi’s well known triple product identity

$$(z^{1/2}, q^{1/2})/(z, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{z^n}{2}}$$

can be easily derived by using Heine summation formula equation (1.3.10).

The basic analog of Dougall’s theorem is Jackson’s theorem [71] given as

$$\phi(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^\frac{z^n}{2}\phi$$

where $N$ is a positive integer and
\[ a^2 q^{N+1} = b c d e. \] \hfill (1.3.14)

The basic analog of Dixon’s theorem is

\[
\phi_3 \left[ a, -q \sqrt{a}, b, c; \frac{-\sqrt{a}}{b}, \frac{q \sqrt{a}}{c} \right] = \prod \left[ a q, q \sqrt{a}, b, \frac{q \sqrt{a}}{b}; \frac{a q}{c}, q, \frac{q \sqrt{a}}{b} \right] \hfill (1.3.15)
\]

In Jackson theorem [71] if we write \( \frac{a q}{d} \) in place of \( d \) and substitute \( \frac{a d q^n}{b c} \) for \( e \) and then let \( a \to \infty \), we get

\[
\phi_2 \left[ b, c, q^{-n}; \frac{b c q^{-n}}{d} \right] = \frac{(d q)^n_n}{(b c q)^n_n} \hfill (1.3.16)
\]

which is the basic analogue of Saalschutzian theorem.

In equation (1.3.16), as \( n \to \infty \), we obtain

\[
\phi_1 \left[ b, c; \frac{d}{d'} q \right] = \prod \left[ \frac{d}{c b}; d', \frac{d}{b c} \right] \hfill (1.3.17)
\]

which is a basic analogue of Gauss’s theorem.

Also if \( c = q^{-N} \), where \( N \) is an integer in (1.3.16), then

\[
\phi_1 \left[ b, q^{-N}; \frac{d q^N}{d'} q \right] = \frac{(d q)^N_n}{(d q)^N_n} \hfill (1.3.18)
\]
which is a basic analogue of Vandermonde theorem [71].

1.4. Basic Bilateral Series

Basic hypergeometric series can be extended by extending to infinity in both directions. The basic bilateral series is written as

\[ A \Psi_B(a_1, a_2, \ldots, a_A, b_1, b_2, \ldots, b_B; q, z) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_A)_n z^n}{(b_1, b_2, \ldots, b_B)_n} \]  

(1.4.1)

for \(|q| < 1\), for all values, real or complex, of the parameters \(a_1, a_2, \ldots, a_A, b_1, b_2, \ldots, b_B\) and for all \(|z| \leq 1\). The above series can be rewritten in contracted form as

\[ A \Psi_B((a), (b); q, z) = \sum_{n=-\infty}^{\infty} \frac{((a); q)_n z^n}{((b); q)_n} \]  

(1.4.2)

In Chapter 6, we have proved the following result which is a bilateral extension of Satya Prakash Singh and Amit Kumar Singh’s result [69].
1.5. Mock Theta Functions

Ramanujan’s last mathematical creation was his mock theta functions which he discovered during the last years of his life. The first detailed description of these functions was given by Watson [90] in his celebrated Presidential Address delivered at the meeting of the London Mathematical Society in November, 1935.
Ramanujan’s [63] general definition of a mock theta function is a function of $f(q)$ defined by a $q$-series which is convergent when $|q|<1$ and satisfies the following two conditions,

(a) For every root $\xi$ of unity, there exists a $\theta$-function $\theta(q)$ such that difference between $f(q)$ and $\theta(q)$ is bounded as $q \to \xi$ radially.

(b) There is no single theta function which works for all $\xi$, i.e. for every $\theta$-function $\theta(q)$ there is some root of unity $\xi$ for which $f(q)$ minus the theta function $\theta(q)$ is unbounded as $q \to \xi$ radially.

Ramanujan [63] gave a list of seventeen mock theta functions and labeled them as third, fifth and seventh orders.

Ramanujan’s “lost notebook” also contained several mock theta functions of orders 6 and 10, which, however, were not explicitly identified as mock theta functions by Ramanujan. Their properties have now been investigated in detail by Andrews and Hickerson [9] in 1991, and also by Choi [33, 35] in 1999.

Unfortunately, while known identities make it clear that mock theta functions of “order” $n$ are related to the number $n$, no formal definition for the order of a mock theta function is known. As a result, the term “order” is regarded merely as a convenient label when applied to mock theta functions by Andrews and Hickerson [9].
McIntosh [55] defined the second order mock theta functions as:

\[ A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} (-q; q^2)_n}{(q; q^2)_{(n+1)}^2} \]  \hspace{1cm} (1.5.1)

\[ B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} (-q^2; q^2)_n}{(q; q^2)_{(n+1)}^2} \]  \hspace{1cm} (1.5.2)

\[ \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q; q^2)_n}{(-q^2; q^2)_n^2} \]  \hspace{1cm} (1.5.3)

The mock theta functions of order 3 are given as

\[ f(q) = \sum_{n=0}^{\infty} \frac{q^n}{(1+q)^2(1+q^2)^2 \ldots (1+q^n)^2} \]  \hspace{1cm} (1.5.4)

\[ \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4) \ldots (1+q^{2n})} \]  \hspace{1cm} (1.5.5)

\[ \psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2) \ldots (1-q^{2n-1})} \]  \hspace{1cm} (1.5.6)

\[ \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4) \ldots (1-q^n+q^{2n})} \]  \hspace{1cm} (1.5.7)

\[ \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2 \ldots (1-q^{2n+1})^2} \]  \hspace{1cm} (1.5.8)

\[ \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3) \ldots (1+q^{2n+1})} \]  \hspace{1cm} (1.5.9)

\[ \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6) \ldots (1+q^{2n+1}+q^{4n+2})} \]  \hspace{1cm} (1.5.10)
Ramanujan [63] gave 10 mock theta functions of order five:

\[
f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}
\]  
(1.5.11)

\[
F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}
\]  
(1.5.12)

\[
1 + 2\psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{\frac{n(n+1)}{2}}
\]  
(1.5.13)

\[
\phi_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2}
\]  
(1.5.14)

\[
f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n}
\]  
(1.5.15)

\[
F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}}
\]  
(1.5.16)

\[
\psi_1(q) = \sum_{n=0}^{\infty} (-q)_n q^{\frac{n(n+1)}{2}}
\]  
(1.5.17)

\[
\phi_1(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{(n+1)^2}
\]  
(1.5.18)

\[
\chi_0(q) = 2F_0(q) - \phi_0(-q)
\]  
(1.5.19)

\[
\chi_1(q) = 2F_1(q) + q^{-1}\phi_1(-q)
\]  
(1.5.20)
Ramanujan [63] also gave seven mock theta functions of order six:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q)_{2n}}$$  \hfill (1.5.21)

$$\psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q; q^2)_n}{(-q)_{2n+1}}$$  \hfill (1.5.22)

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q)_n}{(q; q^2)_{n+1}}$$  \hfill (1.5.23)

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+2)}{2}} (-q)_n}{(q; q^2)_{n+1}}$$  \hfill (1.5.24)

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q; q^2)_n}{(-q)_n}$$  \hfill (1.5.25)

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n}{(-q)_n}$$  \hfill (1.5.26)

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (q)_n}{(q^3; q^3)_n}$$  \hfill (1.5.27)

Ramanujan [63, p. 355] also gave three mock theta functions of order seven:

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n}$$  \hfill (1.5.28)

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^n)_n}$$  \hfill (1.5.29)

$$F_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1})_{n+1}}$$  \hfill (1.5.30)
Gordon and McIntosh [46] found eight mock theta functions of order 8,

\[ S_0(q) = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(-q^2; q^3)_n} \]  
\[ S_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)} \frac{(-q; q^2)_n}{(-q^2; q^3)_n} \]  
\[ T_0(q) = \sum_{n=0}^{\infty} q^{n(n+1)(n+2)} \frac{(-q; q^2)_n}{(-q^2; q^3)_{n+1}} \]  
\[ T_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)} \frac{(-q^2; q^3)_n}{(-q; q^2)_{n+1}} \]  
\[ U_0(q) = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(-q^4; q^5)_n} \]  
\[ U_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)^2} \frac{(-q; q^2)_n}{(-q^2; q^4)_{n+1}} \]  
\[ V_0(q) = -1 + 2 \sum_{n=0}^{\infty} q^{2n^2} \frac{(-q^2; q^4)_n}{(q^2 q^3)_{2n+1}} \]  
\[ V_1(q) = \sum_{n=0}^{\infty} q^{2n^2+2n+1} \frac{(-q^4; q^4)_n}{(q^2 q^3)_{2n+2}} \]  

The four tenth order mock theta functions as defined by Ramanujan [64] are given by:

\[ \Phi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{q^{n+1/2}}{(q; q^2)_{n+1}} \]  
\[ \Psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)(n+2)/2} \frac{q^{n+1}}{(q; q^2)_{n+1}} \]  
\[ X(q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2} \frac{(-1)^n q^n}{(-q; q)_{2n}} \]
\[
\chi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} 
\]

(1.5.42)

In chapter 2, investigation of general multiple series identities is done which extend and generalize the theorems of Bailey [19], and Pathan [57]. The theorems given in the chapter is extremely useful as it provides connections with various classes of well-known hypergeometric functions and even new representations of these functions. Some applications of this theorem are also given in the chapter.

In chapter 3, certain multiple series identities are studied which extend and generalize the theorems of Bailey [19] and Pathan [57]. Special cases of the theorems proved in chapter yields various new transformations and reduction formulae involving quadruple hypergeometric function and Srivastava’s quadruple hypergeometric functions and triple hypergeometric functions.

In chapter 4, double series representations of Mock theta functions of order eight have been established in a very compact form. Also the relationships among the partial Mock theta function and Mock theta functions of order eight have been developed.

In chapter 5, certain new identities of tenth order mock theta function using Bailey transform have been established and relationships among the partial mock theta functions and Mock theta functions of order ten have been obtained. Also, in this chapter few general functions which reduce to Mock theta functions of order ten have been defined and these functions are proven to be $F_q$ functions [86].

In chapter 6, bilateral extensions of few results are carried out. The advantage of writing down the bilateral series is to introduce one more parameter and thus to obtain an entire infinite family. Moreover special cases of the extensions gave many new transformation formulae.