Chapter 2

Homotopy analysis method with a non-homogeneous term in the auxiliary linear operator

2.1 OBJECTIVE

As the order of the HAM approximation increases the complexity (length) of the analytical solution increases. Hence, it is important to develop techniques which either increase the rate of convergence or provide better accuracy at the same order of approximation in comparison to HAM (normal HAM or standard HAM). Therefore, our objective in this chapter is to find a simple way based on the homotopy analysis method of enhancing the rate of convergence of the analytical solution obtained by HAM without affecting the region of convergence and the given accuracy. To obtain this we introduce the idea of the inclusion of a non-homogeneous term in the frame of HAM. We use this non-homogeneous term in the auxiliary linear operator present in the frame of HAM. We apply this technique to three nonlinear problems including the Duffing oscillator in space, present in the literature [Liao, 2003] to verify the effectiveness and usefulness of the proposed approach. For the first problem we prove a convergence theorem which confirms that the proposed technique yields
the convergent solution. To provide more weight to our proposed technique we link this technique to a special case of “Further Generalization of HAM” described by Liao [Liao, 2003]. The idea of “Further Generalization of HAM” has been proposed but there is no application of this generalization in the literature. In other words, for the first time we present the application of the proposed idea and demonstrate its advantages in a set of nonlinear problems.

2.2 THE NON-HOMOGENEOUS AUXILIARY LINEAR OPERATOR

In order to apply HAM, a prior knowledge of some of the properties of the solution is desirable either through a knowledge of the physics or through numerical solutions, so that an appropriate choice of base functions can be made. After choosing the base functions we choose the auxiliary linear operator, auxiliary function and auxiliary parameter so that the solution of each deformation equation exists and the solution of each deformation equation can be expressed in terms of the base functions. Suppose (1.1) is the equation that we choose to solve. Our technique modifies Liao’s technique [Liao, 2003] by taking the auxiliary linear operator in the following form:

\[ L[\phi(t; q)] + G(t; q) \]

where, \( L[\phi(t; q)] \) is the usual auxiliary linear operator following Liao and \( G(t; q) \) is a function of \( t \) and \( q \), which is zero at \( q = 0 \) and \( q = 1 \). For illustrating this general idea throughout the thesis we choose a simple form of \( G(t; q) \), namely

\[ G(t; q) = q(1 - q)F(t) \]
The form of the zero order deformation equation taken in this work is:

\[(1 - q)L[\phi(t; q) - u_0(t)] + q(1 - q)F(t) = qhH(t)N[\phi(t; q)] \quad (2.1)\]

At \( F(t) = 0 \) the above equation reduces to the original form (1.2) of the zero order deformation equation as taken by Liao [Liao, 2003]. After this change, the whole procedure of finding either the higher order deformation equations or the approximate solution is the same as mentioned previously. Apart from the auxiliary parameter we also choose \( F(t) \) so that solution converges quickly and the process yields a good solution for a lower order HAM approximation. Applying this procedure may also reduce the computational cost for a given accuracy. At \( q = 0 \) and \( q = 1 \), equation (2.1) reduces to the form (1.3); this is the basic requirement of HAM. The procedure of finding the high order deformation equations is similar to Liao’s. Only the form of the 1\(^{st}\) and 2\(^{nd}\) order deformation equations change, i.e. from the 3\(^{rd}\) order onwards, the form of the higher order deformation equations is exactly the same as Liao’s. The position of \( F(t) \) will be fixed and will be common to all problems in this thesis. Clearly, at \( F(t) = 0 \) we obtain the solution of Liao’s scheme. Therefore, it is easy to recover the solutions of the standard HAM by replacing \( F(t) \) by 0. This helps us in comparing the results from both techniques.

2.2.1 RELATION TO FURTHER GENERALIZATION OF HAM

The approach of using a non-homogeneous term can also be considered as a special case of ”Further Generalization” of HAM given by Liao [Liao, 2003] in his book. Liao suggested a zeroth-order generalized deformation equation in the following form:

\[[1 - B(q)]L[\phi(t; q) - u_0(t)] = A(q)hH(t)N[\phi(t; q)] + h_2H_2(t)\Pi[\phi(t; q)] \quad (2.2)\]
where \( A(0) = B(0) = 0 \) and \( A(1) = B(1) = 1 \) are convergent power series of \( q \) in \([0,1]\) and

\[
\Pi[\phi(t; 0)] = \Pi[\phi(t; 1)] = 0 \tag{2.3}
\]

such as

\[
\Pi[\phi(t; q)] = A(q)[1 - B(q)]F[\phi(t; q)] \tag{2.4}
\]

\[
\Pi[\phi(t; q)] = [1 - A(q)][\phi(t, q)]^{1+q} - \phi(t, q) \tag{2.5}
\]

The proposed approach is a special case of Liao’s generalized form with

\[ A(q) = B(q) = q, H_2(t) = 1, h_2 = -1, F = F(t). \]

This generalization can be developed in some other forms also. This opens up another area of research in the development of analytical solutions for nonlinear problems.

2.2.2 CRITERIA FOR THE CHOICE OF THE NON-HOMOGENEOUS TERM

After a suitable choice of the base functions, it is easy to approximate the solution expressions of nonlinear problems. The base functions help us in choosing the initial guess, the auxiliary linear operator and the auxiliary function. It is important to note that we have great freedom in choosing all these functions in HAM.

Let \( \{c_k(t) : k = 1, 2, 3, \ldots \} \) denote the complete set of base function for the solution of \( u(t) \). Then, we can express

\[ F(t) = \sum_{k=1}^{s} c_k e_k(t) \tag{2.6} \]

where, \( s \) is a positive integer and “\( c_k \)” are constants. Then, it is clear from the \( m^{th} \) order HAM approximation that it contains \( (s + 1) \) convergence-control parameters.
namely: $h, c_1, c_2, \cdots, c_s$, whose optimal value can be determined by the minimization of the square residual error of the governing equation as shown by Liao in the so-called optimal HAM. Suppose $\Delta_m$ is the square residual error of equation (1.1) at the $m^{th}$ order HAM approximation, then

$$\Delta_m = \int_a^b (N_0 \sum_{i=0}^{m} u_i(\xi))^2 d\xi$$  \hspace{1cm} (2.7)

where the interval $[a, b]$ belongs to the domain of the problem considered. We can determine the constants by solving the following algebraic equations:

$$\frac{\partial\Delta_m}{\partial h} = 0, \frac{\partial\Delta_m}{\partial c_1} = 0, \frac{\partial\Delta_m}{\partial c_2} = 0 \cdots$$  \hspace{1cm} (2.8)

The length of the expression for the square residual error of the nonlinear problems solved by HAM increases on increasing the order of approximation. At the same time it is difficult to obtain more than one unknown from Eq. (2.7) because ultimately, we end up with a nonlinear algebraic equation containing more than one unknown. It is still not trivial to solve nonlinear algebraic equations. Therefore, for simplicity we choose to evaluate only one convergence control parameter. In this chapter we know the proper value of $h$ because the considered problems have been solved already by Liao. What remains to be determined is the value of one convergence control parameter among $c_1, c_2$ or $c_s$ by minimizing the square residual error. We find the convergence control parameter excluding $h$ by plotting $\Delta_m$ vs. the corresponding convergence control parameter. For example if we introduce $c_1$ through a non-homogeneous term then our equation (2.7) contains only two convergence control parameters $h$ and $c_1$ and because we already know the value of $h$ we only need to find out $c_1$ in such a way that the square residual error is minimum. This can be done by plotting $\Delta_m$ vs. $c_1$. For the sake of simplicity we introduce only one $c_s$ in all the problems considered in this chapter and obtain its value by using equation (2.7).
The square residual error (Eq. (2.7)) is a function of two variables, namely, $h$ and one $c_i$ for arbitrary $i$ depending on the problem. The square residual error may be minimized by finding the optimal values of $h$ and $c_i$ simultaneously from Eq. (2.8). Alternatively, we can first find the optimal $h$ and then find the value of $c_i$ for arbitrary $i$ which further reduces the square residual error. This process may be iterated further. In this thesis we use the latter approach and stop at the first iteration, since our aim is to show that including the non-homogeneous term improves accuracy. Further iteration is likely to reduce the error further, but for our purpose, one iteration is sufficient to prove the point.

In order to demonstrate the efficiency of the proposed technique we present an analysis of some nonlinear problems and prove a convergence theorem.

2.3 Example (1)

The following non-linear differential equation is taken from Liao’s book [Liao, 2003] as it has been already solved by Liao. It’s approximate analytical solution is expressed by different sets of base functions such as, polynomial, exponential, rational functions and the combination of trigonometric and exponential functions. The nonlinear differential equation is:

$$u'(t) + u^2(t) = 1, \quad u(0) = 0, t \geq 0 \quad (2.9)$$

The exact solution of equation (2.9) is $tanh(t)$. Following Liao, we choose exponential base functions, with the same initial guess of $u(t)$ i.e.

$$u_0(t) = 1 - \exp(-t) \quad (2.10)$$
Choose the non-homogeneous auxiliary linear operator as follows:

\[ L[\phi(t; q)] + q(1 - q)F(t), \]

where

\[ L[\phi(t; q)] = \frac{\partial[\phi(t; q)]}{\partial t} + \phi(t; q) \] \hspace{1cm} (2.11)

From equation (2.9) we define the nonlinear operator as:

\[ N[\phi(t; q)] = \phi'(t; q) + \phi^2(t; q) - 1 \] \hspace{1cm} (2.12)

The high-order deformation equation is

\[ L[u_m(t) - \chi_m u_{m-1}(t)] + g(t) = hH(t)R_m(u_{m-1}) \] \hspace{1cm} (2.13)

where

\[ g(t) = \begin{cases} F(t); & \text{for } m = 1 \\ \chi_m F(t); & \text{for } m = 2 \\ 0; & \text{for } m \geq 3 \end{cases} \] \hspace{1cm} (2.14)

Therefore the \( m^{th} \) order deformation equations are

\[ L[u_1(t)] = hH(t)R_1(t) - F(t), \quad m = 1, \] \hspace{1cm} (2.15)

\[ L[u_2(t) - u_1(t)] = hH(t)R_2(t) + F(t), \quad m = 2, \] \hspace{1cm} (2.16)

\[ L[u_m(t) - u_{m-1}(t)] = hH(t)R_m(t), \quad m \geq 3, \] \hspace{1cm} (2.17)

subject to the initial conditions:

\[ u_m(0) = 0, \quad m \geq 1, \] \hspace{1cm} (2.18)

where

\[ R_m(u_{m-1}) = u'_m + \sum_{n=0}^{m-1} u_n(t)u_{m-n-1}(t) - [1 - \chi_m] \] \hspace{1cm} (2.19)
The $m^{th}$ order approximate solution $u(t)$ is given by:

$$u(t) = \sum_{m=0}^{\infty} u_m(t), \quad (2.20)$$

Before discussing the solution, following Liao, we prove a convergence theorem.

**Theorem 2.3.1.** As long as the series (2.20) converges, where $u_m(t)$ is governed by the high-order deformation Eqs. (2.15), (2.16), (2.17) and (2.18) under the definition (1.10) and (2.19), it must be one exact solution of (2.9).

**Proof.** If the series

$$u(t) = \sum_{m=0}^{\infty} u_m(t), \quad (2.21)$$

converges, we can write

$$S(t) = \sum_{m=0}^{\infty} u_m(t), \quad (2.22)$$

and it holds,

$$\lim_{m \to \infty} u_m(t) = 0 \quad (2.23)$$

From Eqs. (2.15), (2.16), (2.17) and using (1.10) successively we get the following equations for $m = 1, 2, 3, \ldots, n$

$$L[u_1(t)] = hH(t)R_1(t) - F(t), \quad (2.24)$$

$$L[u_2(t) - u_1(t)] = hH(t)R_2(t) + F(t), \quad (2.25)$$

$$\vdots$$

$$L[u_n(t) - u_{n-1}(t)] = hH(t)R_n(t), \quad (2.26)$$

Now adding column-wise we get

$$L[u_1 + (u_2 - u_1) + \ldots + (u_n - u_{n-1})] = ((hH(t)R_1 - F(t))$$

$$+ (hH(t)R_2 + F(t))$$

$$+ \ldots + h(t)H(t)R_n$$

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i.e.,

\[ L[u_n] = hH(t) \sum_{m=1}^{n} R_m \]  \hspace{1cm} (2.27)

Taking the limit \( n \to \infty \) and noting that \( h \neq 0 \) and \( H(t) \neq 0 \) and finally using (2.21), we get

\[ \sum_{m=1}^{\infty} R_m(t) = 0 \] \hspace{1cm} (2.28)

From (2.19) we have

\[
\sum_{m=1}^{\infty} R_m(t) = \sum_{m=1}^{\infty} \left[ u'_{m-1}(t) + \sum_{n=0}^{m-1} u_n(t) u_{m-n-1}(t) - (1 - \chi_m) \right]
\]

\[
= \sum_{m=0}^{\infty} u'_m - 1 + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} u_n(t) u_{m-n-1}(t)
\]

\[
= \sum_{m=0}^{\infty} u'_m - 1 + \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} u_n(t) u_{m-n-1}(t)
\]

\[
= \sum_{m=0}^{\infty} u'_m - 1 + \sum_{n=0}^{\infty} u_n(t) \sum_{i=0}^{\infty} u_i(t)
\]

\[ = \dot{S}(t) + S^2(t) - 1. \]

From the above equations we have

\[ \dot{S}(t) + S^2(t) - 1 = 0; \quad t \geq 0 \]

and it also holds

\[ S(0) = 0 \]

Therefore according to the above two equations \( S(t) \) must be the exact solution of (2.9). Thus the theorem is valid even in the case of the non-homogeneous auxiliary linear operator. It is evident from the proof that this is true for any form of \( F(t) \). \( \square \)
2.3.1 Solution

We choose \( F(t) = c_1 \exp(-t) + c_2 \exp(-2t) \) and fix \( c_1 = 0 \) because it’s use would produce terms which do not follow the rule of solution expression. We fix \( h = -1 \) and \( H(t) = \exp(-t) \) for both the schemes. We can determine \( c_2 \) from the Figure (2.1) where \( \Delta_m \) has been plotted vs. \( c_2 \). For \( c_2 = 0.06 \) the h-curves are presented in Figure (2.2) and for Liao’s scheme (where \( c_1 = c_2 = 0 \)) h-curves are presented in Figure (2.3), the corresponding solutions at different orders of HAM approximation and the corresponding series of derivatives from both the schemes are presented in Table (2.1) and Table (2.2). The solution expression for both the schemes of order 10 with \( h = -1 \) are given in Eqs. (2.29) and (2.30):

\[
\begin{align*}
\mathbf{u}_{\text{Liao}}(t) &= 1 - \frac{e^{-21t}}{1024} + \frac{e^{-20t}}{512} + \frac{5e^{-19t}}{512} - \frac{11e^{-18t}}{512} - \frac{45e^{-17t}}{1024} \\
&\quad + \frac{128}{15e^{-15t}} - \frac{32}{11e^{-14t}} - \frac{512}{105e^{-13t}} + \frac{193e^{-12t}}{256} \\
&\quad + \frac{63e^{-11t}}{319e^{-10t}} - \frac{512}{105e^{-9t}} + \frac{256}{7e^{-8t}} \\
&\quad + \frac{256}{32} + \frac{15e^{-7t}}{256} - \frac{512}{121e^{-6t}} - \frac{1024}{45e^{-5t}} \\
&\quad + \frac{1013e^{-4t}}{512} + \frac{5e^{-3t}}{512} - \frac{1023e^{-2t}}{512} - e^{-t} \quad (2.29)
\end{align*}
\]
\[ u_{\text{proposed}}(t) = 1 - \frac{e^{-21t}}{1024} + \frac{83e^{-20t}}{64000} + \frac{71557e^{-19t}}{640000} - \frac{5175155837e^{-18t}}{30800000000} \]
\[ - \frac{e^{-20t}}{12320000000000} + \frac{71557e^{-19t}}{145445517892193e^{-16t}} - \frac{1501500000000000}{12461231893639e^{-14t}} \]
\[ + \frac{2102100000000000}{27565732152591e^{-13t}} + \frac{3753750000000000}{29492523646261e^{-12t}} \]
\[ - \frac{9240000000000}{3815504752991e^{-11t}} - \frac{3850000000000000}{336389835907e^{-10t}} \]
\[ + \frac{105000000000000}{959889392353e^{-9t}} + \frac{262500000000000}{8939444197793e^{-8t}} \]
\[ - \frac{28000000000000}{2448652614491e^{-7t}} - \frac{490000000000000}{10030822156459e^{-6t}} \]
\[ + \frac{115500000000000}{1639774008067e^{-5t}} + \frac{577500000000000}{376214885173e^{-4t}} \]
\[ - \frac{92400000000000}{92400000000000} + \frac{184800000000000}{1235490931e^{-3t}} - \frac{3007101185274421e^{-2t}}{1501500000000000} \]
\[ + \frac{2234375000000}{1422289524479e^{-t}} + \frac{1422289524479e^{-t}}{8408400000000000} \]  

(2.30)

2.4 Example (2)

Consider the nonlinear differential equation called the Duffing oscillator in space [Liao, 2003]

\[ v''(t) + \epsilon(v(t) - v^3(t)) = 0, \]  

(2.31)

with the boundary conditions:

\[ v(0) = v(\pi) = 0. \]  

(2.32)

2.4.1 Solution

Define

\[ A = v\left(\frac{\pi}{2}\right), v(t) = Au(t), \]  

(2.33)
then equation (2.31) becomes

$$u''(t) + \epsilon(u(t) - A^2u^3(t)) = 0, \quad u(0) = u(\pi) = 0. \quad (2.34)$$

It is obvious that

$$u\left(\frac{\pi}{2}\right) = 1.$$

According to Kahn and Zarmi, the exact relation between $\epsilon$ and $A$ is:

$$\epsilon = \frac{2}{\pi \sqrt{1 - \frac{A^2}{2}}} K\left(\frac{A^2}{2 - A^2}\right)^2 \quad (2.35)$$

where $K(\zeta)$ is the complete elliptic integral of the first kind. According to the above exact relation, $\epsilon$ tends to infinity as $|A|$ approaches to 1. We find the amplitude $A$ using the proposed scheme by removing the secular terms and then we compare our results with Liao’s results. We choose the same initial guess $u_0(t) = \sin(t)$ and the auxiliary function $H(t) = 1$ for both the schemes. The set of base functions in this case is

$$\{\sin[(2m+1)t] \ : \ m \geq 0\} \quad (2.36)$$

and the auxiliary linear operator is

$$L[\phi(t; q)] = \frac{\partial^2[\phi(t; q)]}{\partial t^2} + \phi(t; q) \quad (2.37)$$

This is the auxiliary operator in Liao’s scheme. We apply the above mentioned procedure to this problem. We choose $F(t) = c_1 \sin(t) + c_2 \sin(3t) + c_3 \sin(5t)$ and fix $c_1 = 0$ because it’s use would produce secular terms. We fix $h = -\frac{1}{5}$ and $H(t) = 1$ for both the schemes. As a special case we use $c_2 = 0$ and find $c_3$ from Figure(2.4).
The corresponding results are presented in Table (2.3). Following Liao, we take

\[ h = -\frac{1}{(1 + \frac{\epsilon}{3})} \]  

(2.38)

then for each value of \( \epsilon \) we obtain the square residual error in Figure (2.5) and we choose \( c_1 = 0 \) for \( F(t) = c_1 \sin(t) + c_2 \sin(3t) \). Clearly, the square residual error decreases. We find approximately the same \( h \)-curves as found by Liao [Liao, 2003], so that the region of convergence is approximately the same.

### 2.5 Example (3)

For the above example it has been shown in [Liao, 2003] that it admits multiple solutions, when the set of base functions is of the form

\[ \{ \sin[(2m + 1)\kappa t] : m \geq 0, \kappa \geq 1 \} \]  

(2.39)

Following Liao, we determine the multiple solutions including the non-homogeneous term. Without loss of generality, define

\[ A = v\left(\frac{\pi}{2\kappa}\right), v(t) = Au(t) \]  

(2.40)

then equation (2.31) becomes

\[ u''(t) + \epsilon(u(t) - A^2u^3(t)) = 0, \quad u(0) = u(\pi) = 0. \]  

(2.41)

Clearly \( A \) is unknown in the above equation and we obtain this by removing the secular terms like [Liao, 2003] for different values of \( \epsilon \) and \( \kappa \). From equation (2.40), it holds

\[ u\left(\frac{\pi}{2\kappa}\right) = 1 \]  

(2.42)
2.5.1 Solution

As the problem is the same except for the parameter $\kappa$, we can apply the above technique here also. We choose the same initial guess $u_0(t) = \sin(\kappa t)$ and the auxiliary function $H(t) = 1$ as taken by Liao [Liao, 2003]. In this case we choose $F(t) = c_1 \sin(2t) + c_2 \sin(6t)$ and $F(t) = c_1 \sin(3t) + c_2 \sin(9t)$ for $\kappa = 2$ and $\kappa = 3$, respectively. We fix $c_1 = 0, h = -\frac{1}{2}$ and $H(t) = 1$. We find $c_2$ from Figures (2.6) and (2.7) for $\kappa = 2$ and $\kappa = 3$, respectively. We present a comparison between both the schemes in Tables (2.4) and (2.5). The exact analytical result is given by the implicit formula

$$\epsilon = \frac{8\kappa^2}{\pi^2(2 - A^2)} K\left(\frac{A^2}{2 - A^2}\right)$$

(2.43)

where $K(\zeta)$ is the complete elliptic integral of first kind. For $\epsilon = 40, \kappa = 2$ and $c_2 = 2/5$, the solution expression of $u(t)$ of order 5 at $h = -1/2$ for $H(t) = 1$ is given in Equation (2.44). Similarly, for $\epsilon = 90, \kappa = 3$ and $c = 1.2$, the solution expression of $u(t)$ of order 5 at $h = -1/2$ for $H(t) = 1$ is given in Eq. (2.45).

$$u_{(\text{proposed}, \epsilon=40)}(t) = 1.1764 \sin(2t) + 0.21834 \sin(6t)$$

$$+ 0.053345 \sin(10) + 0.013777 \sin(14t)$$

$$+ 0.0025753 \sin(18t) + 0.00020878 \sin(22t)$$

(2.44)

$$u_{(\text{proposed}, \epsilon=90)}(t) = 1.1766 \sin(3t) + 0.21862 \sin(9t)$$

$$+ 0.053344 \sin(15t) + 0.013708 \sin(21t)$$

$$+ 0.002566421 \sin(27t) + 0.000209525 \sin(33t)$$

(2.45)
2.6 RESULTS AND DISCUSSION

It is observed from Figs. (2.2) and (2.3), that the non-homogeneous term does not effect the region of convergence of the problems considered so far. However the use of the non-homogeneous term helps in reducing the square residual error, as is clear from Figs. (2.1), (2.4), (2.5), (2.6) and (2.7). It is clear from Table (2.1), that we obtain adequate accuracy with our proposed scheme for a maximum approximation of order 10. This is a significant improvement over Liao’s scheme with a little additional effort. The first derivative of the approximate solution found by the proposed scheme also agrees well with the first derivative of the exact solution in comparison to Liao’s scheme as shown in Table (2.2). Similarly, we see faster convergence not only for the solutions but also for the HAM amplitudes found by removing the secular term for the Duffing-oscillator problem in space. This is clear from Tables (2.3), (2.4) and (2.5). We can observe from the above analysis that a little additional effort leads to a significant improvement in the approximate solutions by reducing the square residual error.
Table 2.1: Comparison between Liao’s scheme and the proposed scheme with the exact solution when $h = -1, H(t) = \exp(-t), c_2 = 0.06$ and $F(t) = c_2 \exp(-2t)$.

<table>
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<tr>
<th>t</th>
<th>HAM $5^{th}$ order</th>
<th>proposed $5^{th}$ order</th>
<th>HAM $10^{th}$ order</th>
<th>proposed $10^{th}$ order</th>
<th>Exact solution</th>
</tr>
</thead>
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<td>$1/4$</td>
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<td>0.24492</td>
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<td>0.46184</td>
<td>0.46212</td>
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</tr>
<tr>
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<td>0.63414</td>
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<tr>
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</tr>
<tr>
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<td>0.90225</td>
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<tr>
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<tr>
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<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>$100$</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

Table 2.2: Comparison between the corresponding first derivative series for both the schemes with the first derivative of the exact solution at different times.

<table>
<thead>
<tr>
<th>t</th>
<th>derivative of (2.29)</th>
<th>derivative of (2.30)</th>
<th>derivative of $\tanh(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000000</td>
<td>1.000000000</td>
<td>1.000000000</td>
</tr>
<tr>
<td>1</td>
<td>0.419900000</td>
<td>0.419900000</td>
<td>0.420000000</td>
</tr>
<tr>
<td>2</td>
<td>0.070670000</td>
<td>0.070670000</td>
<td>0.070670000</td>
</tr>
<tr>
<td>3</td>
<td>0.009902000</td>
<td>0.009868000</td>
<td>0.009866000</td>
</tr>
<tr>
<td>4</td>
<td>0.001357000</td>
<td>0.001340000</td>
<td>0.001341000</td>
</tr>
<tr>
<td>5</td>
<td>0.000188000</td>
<td>0.000180700</td>
<td>0.000181600</td>
</tr>
<tr>
<td>6</td>
<td>0.000026970</td>
<td>0.000024190</td>
<td>0.000024580</td>
</tr>
</tbody>
</table>
\[
\epsilon = 25, c_2 = -0.3, F(t) = c_2 \sin(5t)
\]

<table>
<thead>
<tr>
<th>order of approximation</th>
<th>(\epsilon = 25) by Liao’s scheme</th>
<th>(\epsilon = 25), (c_2 = -0.3), (F(t) = c_2 \sin(5t)) by proposed scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.01046</td>
<td>0.99642</td>
</tr>
<tr>
<td>10</td>
<td>1.00313</td>
<td>1.00169</td>
</tr>
<tr>
<td>15</td>
<td>1.00117</td>
<td>1.00044</td>
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<td>20</td>
<td>1.00049</td>
<td>1.00014</td>
</tr>
<tr>
<td>25</td>
<td>1.00017</td>
<td>1.00003</td>
</tr>
</tbody>
</table>

Table 2.3: Analytical approximations of \(A\) when \(\epsilon = 25, h = -\frac{1}{5}\) and \(H(t) = 1\).

\[
\epsilon = 40, \kappa = 2, c_2 = \frac{2}{5}, F(t) = c_2 \sin(6t)
\]

<table>
<thead>
<tr>
<th>order of approximation</th>
<th>(\epsilon = 40, \kappa = 2) Liao’s scheme</th>
<th>(\epsilon = 40, \kappa = 2) proposed scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.98070</td>
<td>0.97788</td>
</tr>
<tr>
<td>4</td>
<td>0.99912</td>
<td>1.00064</td>
</tr>
<tr>
<td>6</td>
<td>0.99613</td>
<td>0.99524</td>
</tr>
<tr>
<td>8</td>
<td>0.99656</td>
<td>0.99683</td>
</tr>
<tr>
<td>10</td>
<td>0.99656</td>
<td>0.99632</td>
</tr>
<tr>
<td>12</td>
<td>0.99635</td>
<td>0.99646</td>
</tr>
<tr>
<td>14</td>
<td>0.99649</td>
<td>0.99644</td>
</tr>
<tr>
<td>16</td>
<td>0.99641</td>
<td>0.99643</td>
</tr>
<tr>
<td>18</td>
<td>0.99645</td>
<td>0.99644</td>
</tr>
<tr>
<td>20</td>
<td>0.99643</td>
<td>0.99643</td>
</tr>
<tr>
<td>22</td>
<td>0.99644</td>
<td>0.99644</td>
</tr>
</tbody>
</table>

Table 2.4: Comparison of the analytical approximation of \(A\) with Liao’s scheme and the proposed scheme when \(\epsilon = 40, \kappa = 2\) for \(h = -\frac{1}{2}\) and \(H(t) = 1\).
Table 2.5: Comparison of the analytical approximation of $A$ with Liao’s scheme and the proposed scheme when $\epsilon = 90, \kappa = 3$ for $h = -\frac{1}{2}$ and $H(t) = 1$.

<table>
<thead>
<tr>
<th>order of approximation</th>
<th>$\epsilon = 90, \kappa = 3$</th>
<th>$\epsilon = 90, \kappa = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Liao’s scheme</td>
<td>proposed scheme</td>
</tr>
<tr>
<td>2</td>
<td>0.98170</td>
<td>0.97792</td>
</tr>
<tr>
<td>4</td>
<td>0.99854</td>
<td>1.0005</td>
</tr>
<tr>
<td>6</td>
<td>0.99639</td>
<td>0.99528</td>
</tr>
<tr>
<td>8</td>
<td>0.99625</td>
<td>0.99682</td>
</tr>
<tr>
<td>10</td>
<td>0.99658</td>
<td>0.99632</td>
</tr>
<tr>
<td>12</td>
<td>0.99635</td>
<td>0.99646</td>
</tr>
<tr>
<td>14</td>
<td>0.99648</td>
<td>0.99643</td>
</tr>
<tr>
<td>16</td>
<td>0.99642</td>
<td>0.99643</td>
</tr>
<tr>
<td>18</td>
<td>0.99644</td>
<td>0.99644</td>
</tr>
<tr>
<td>20</td>
<td>0.99643</td>
<td>0.99643</td>
</tr>
<tr>
<td>22</td>
<td>0.99643</td>
<td>0.99644</td>
</tr>
</tbody>
</table>

Figure 2.1: The square residual error $\Delta_m$ vs. $c_2$ for $h = -1, c_1 = 0$, orders 6, 8, 10 with $H(t) = \exp(-t)$ and $F(t) = c_2 \exp(-2t)$.
Figure 2.2: $h$-curves found by Liao’s scheme; Solid line: $u''(0)$ vs. $h$; Dashed line: $u'''(0)$ vs. $h$, order 10, for $H(t) = \exp(-t)$.

Figure 2.3: $h$-curves found by the proposed scheme with $c_2 = 0.06$ & $F(t) = c_2 \exp(-2t)$; Solid line: $u''(0)$ vs. $h$; Dashed line: $u'''(0)$ vs. $h$, order 10, for $H(t) = \exp(-t)$. 
Figure 2.4: The square residual error $\Delta_m$ vs. $c_3$ for $\epsilon = 25, h = -\frac{1}{5}, c_1 = c_2 = 0$ and $F(t) = c_3 \sin(5t)$, order 11.

Figure 2.5: The square residual error $\Delta_m$ vs. $c_2$ for $\epsilon = 1, 2, 3, 4, 5; h = -\frac{1}{(1+\epsilon)}; c_1 = 0$ and $F(t) = c_2 \sin(3t)$, order 3; Black line: $\epsilon = 1$; Blue line: $\epsilon = 2$; Red line: $\epsilon = 3$; Green line: $\epsilon = 4$; Orange line: $\epsilon = 5$; vertical line corresponds to the residual error for Liao’s scheme for each $\epsilon$. 
Figure 2.6: The square residual error $\Delta_m$ vs. $c_2$ for $h = -\frac{1}{2}, c_1 = 0, \kappa = 2, \epsilon = 40$, order 11, $H(t) = 1$ and $F(t) = c_2 \sin(6t)$.

Figure 2.7: The square residual error $\Delta_m$ vs. $c_2$ for $h = -\frac{1}{2}, c_1 = 0, \kappa = 3, \epsilon = 90$, order 11, $H(t) = 1$ and $F(t) = c_2 \sin(9t)$. 