Chapter 1

Introduction

1.1 HISTORY

Analytical solutions can be used to validate numerical schemes and/or software for developing approximate solutions of a nonlinear problem. From numerical solutions alone it is difficult to establish the validity and accuracy of the solutions obtained. Convergence and accuracy are major points to be considered while developing numerical solutions. Except in a few cases, it is not possible to guarantee the convergence and accuracy of a numerical scheme for a nonlinear problem. One way to validate numerical solutions would be to compare the numerical solution and the analytical solution for the same problem or for some limiting cases where analytical solutions are available. Hence it is of considerable value to generate analytical solutions of nonlinear problems whenever possible.

The perturbation technique was developed by Pierre-Simon Laplace in the eighteenth century and it has been used for analyzing numerous nonlinear differential equations analytically [Cole, 1968; Van Dyke, 1975; Nayfeh, 1981; Nayfeh, 1985; Grasman, 1987; Hinch, 1991, Kevorkian and Cole, 1995; Nayfeh, 2000]. This
technique is based on series solutions in terms of the powers of a perturbation quantity whose presence is essential in the problem considered. Difficulties arise, when the problem does not contain a perturbation quantity. To overcome this problem some nonperturbation techniques have been developed in the literature. We find mainly two types of nonperturbation approaches, one, Lyapunov’s small artificial parameter approach developed by Lyapunov [Lyapunov, 1992] and, two, the $\delta$-expansion method given by Karmishin et al. [Karmishin et al, 1990]. It is difficult to find the appropriate place to position these artificial perturbation quantities even in a simple nonlinear problem, whose exact solution is known, as discussed in [Liao, 2003]. Apart from the issue of the presence or artificial presence of the perturbation quantity in the perturbation and nonperturbation techniques, respectively, there is also no way to control the rate and region of convergence of the approximate analytical series solutions obtained by both these techniques. The Adomian decomposition method (ADM) is much more general than the perturbation and the nonperturbation techniques, since it is independent of the presence of a perturbation quantity as developed by Adomian [Adomian, 1976]. The approximate series solutions obtained by Adomian’s decomposition method are often polynomials and as is well known the convergence region for polynomial series is small. Polynomial series solutions may not often be a good approximation to different types of nonlinear problems, for example, in an oscillation problem we need oscillatory solutions instead of polynomials. We do not have the freedom to express the solution in terms of different base functions in any of the techniques mentioned above. Adomian’s decomposition method also does not provide us any way to control the rate and region of convergence of the approximate analytical series solutions thus obtained. The homotopy analysis method (HAM) proposed by Liao in 1992 can be applied to problems irrespective of the presence of a perturbation quantity. HAM provides us
great freedom to express our solution expression by means of different base functions and provides a convenient way to control the rate and region of convergence of the solution series. Unlike the perturbation or non-perturbation methods mentioned above, the homotopy analysis method [Liao, 2003, 2004a; Liao and Tan 2007; Liao, 2009; Liao, 2012] does not require the existence of a small parameter in terms of which a perturbation solution is developed and is thus valid for both weakly and strongly nonlinear problems. The homotopy analysis method can be applied to ordinary differential equations, partial differential equations, integro-differential equations, delay-differential equation, integral equations etc.

1.2 THE HOMOTOPY ANALYSIS METHOD

First we define a homotopy. Homotopy [Hilton, 1953] is a basic concept in topology [Sen, 1983] and it can be defined as follows:

**Definition 1.2.1.** For two continuous functions \( f(x) \) and \( g(x) \) defined on topological spaces, \( f(x), g(x) : X \rightarrow Y \), a continuous function \( H(x, q) : X \times [0, 1] \rightarrow Y \) is called a homotopy if it satisfies the following:

\[
H(x, 0) = f(x), \quad H(x, 1) = g(x) \quad \forall x \in X
\]

Functions \( f(x) \) and \( g(x) \) deform continuously through the homotopy \( H(x, q) \) as \( q \) varies from 0 to 1. Such a \( f \) and \( g \) are called homotopic to each other. In the context of HAM, \( f(x) \) is assumed to be the initial guess of the solution and \( g(x) \) the approximate analytical solution (which we need to obtain) for any given nonlinear problem.
The homotopy approach has been used for solving nonlinear algebraic equations by the homotopy continuation approach [Alexander and Yorke, 1978]. The homotopy analysis method is a tool to obtain approximate analytical solutions of nonlinear problems. In recent years, HAM has generated a lot of interest due to its applicability and efficiency and this technique has been successfully applied to a number of nonlinear problems, as is clear from the literature [Liao, 2004b; Abbasbandy, 2006; Hang et al., 2006; Sajid et al., 2008; Lopez et al., 2009; Odibat et al., 2010; Srinivas and Muthuraj, 2010a; Turkyilmazoglu, 2010b; Turkyilmazoglu, 2011; Turkyilmazoglu, 2012; Srinivas et al., 2013]. The advantages of using HAM are as follows:

- it can be applied to nonlinear problems whether or not they contain a perturbation parameter
- the solution can be expressed in terms of different base functions as per the requirement of the problem considered
- the rate and region of convergence of the approximation can be controlled with the help of the convergence control parameter present in the frame of HAM.

1.2.1 APPLICATION OF HAM TO DIFFERENTIAL EQUATIONS

To determine the basic features of the homotopy analysis method for a general differential equation proposed by Liao [Liao, 2003], let us consider the following system:

\[ N[u(t)] = 0 \]  

(1.1)
where \( N \) is a nonlinear operator, \( t \) denotes the independent variable and \( u(t) \) is an unknown function. For simplicity we ignore all the boundary and/or initial conditions, which can be treated in a similar manner. We define the zero-order deformation equation as follows:

\[
(1 - q)L[\phi(t; q) - u_0(t)] = q h H(t) N[\phi(t; q)]
\]

(1.2)

where, \( q \in [0, 1] \) is the embedding parameter, \( h \) is a non-zero auxiliary parameter, \( H(t) \) is a non-zero auxiliary function, \( u_0(t) \) is the initial guess of \( u(t) \), \( L \) is an auxiliary linear operator and \( \phi(t; q) \) is an unknown function. It is important, that one has a lot of freedom in choosing the auxiliary linear operator, auxiliary parameter and auxiliary function in the framework of the homotopy analysis method. Obviously, when \( q = 0 \) and \( q = 1 \), we have

\[
\phi(t; 0) = u_0(t), \quad \phi(t; 1) = u(t)
\]

(1.3)

respectively. Thus as \( q \) increases from 0 to 1, \( \phi(t; q) \) varies from the initial guess \( u_0(t) \) to the exact solution \( u(t) \). Here the idea of homotopy discussed in section (1.2) is used. In topology this is called a deformation and \( L[\phi(t; 0) - u_0(t)] \) and \( N[\phi(t; 1)] \) are called homotopic. By expanding \( \phi(t; q) \) in a Taylor’s series with respect to \( q \), we obtain

\[
\phi(t; q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)q^m
\]

(1.4)

where

\[
u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m} |_{q = 0}
\]

(1.5)

As suggested and already applied to a number of problems, if the initial guess, auxiliary linear operator, auxiliary parameter, and auxiliary function are properly chosen, such that the series (1.4) converges at \( q = 1 \), then we have

\[
u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)
\]

(1.6)
which must be one of the solutions of the nonlinear equation as proved by Liao [Liao, 2003]. According to (1.5) the governing equation can be found from the zero order deformation Eq. (1.2). For the sake of brevity, define the vector

$$
\vec{u}(t) = \{u_0(t), u_1(t), \ldots, u_n(t)\}
$$

(1.7)

Now our aim is to obtain $u_1(t), u_2(t), \ldots, u_n(t)$ to approximate the solution (1.6). We use the standard way of obtaining these components and differentiate Eq. (1.2) $m$-times with respect to $q$ and then set $q = 0$ and finally divide by $m!$, we, then, have the so-called $m^{th}$ order deformation equation

$$
L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)R_m(u_{m-1})
$$

(1.8)

where

$$
R_m(u_{m-1}) = \frac{1}{(m - 1)!} \frac{\partial^{m-1}N[\phi(t; q)]}{\partial q^{m-1}}|_{q=0}
$$

(1.9)

and

$$
\chi_m = \begin{cases} 
0; & \text{for } m = 1 \\
1; & \text{for } m \neq 1 
\end{cases}
$$

(1.10)

It is easy to solve the linear differential Eqs. (1.8) for each $m = 1, 2, \ldots$ and we, then, obtain the required $u_1(t), u_2(t), \ldots$ to approximate the solution of the given differential equation. Basically, the homotopy analysis method transforms the given nonlinear differential equation (it could be a system of nonlinear equations also) to an infinite number of linear differential equations. Therefore it is easier to approximate the solution instead of solving the original problem. Throughout the analysis we keep $H(t) = 1$ for simplicity, unless otherwise stated.
The convergence region and the rate of approximation of the solution series are usually controlled by a proper choice of the auxiliary parameter (or convergence control parameter) $h$. Once we define the auxiliary linear operator and the initial guess and after solving the high order deformation Eqs. (1.8) for each $m = 1, 2, \ldots$, the solution series (1.6) contains the only unknown $h$. The proper value of $h$ can be obtained via the $h$-curves [Liao, 2003]. For $h$-curves, we usually calculate the derivatives of (1.6) at different order and after setting $t = 0$ (initial value of the independent variable of the problem) we plot these derivatives vs. $h$. We call these curves $h$-curves and find the region of these curves where they are flat. This flat region in HAM is called the region of convergence. This implies that whatever the value of the convergence control parameter in this region we use, the series converges. This idea has been applied to a number of nonlinear problems. Further, Liao has also given another criterion for the choice of the convergence control parameter, which leads to the optimal homotopy analysis method (OHAM) [Liao, 2010]. In the optimal homotopy analysis method we plot the square residual error vs. $h$ and obtain the value of $h$ which leads to the minimum square residual error. This value of the convergence control parameter for which the square residual error is minimum, is called optimal. Some nonlinear problems have been analyzed through the optimal homotopy analysis method [Liao, 2010; Turkyilmazoglu, 2010b; Liao, 2012; Shukal et al., 2013; Muthuraj et al., 2013; Farooq and Zhi-Liang, 2014] and the efficiency of this technique has been demonstrated.
A literature survey reveals that there are some other approaches based on the homotopy analysis method for different classes of equations. A modified homotopy analysis method based on HAM has been proposed by Bataineh et al. [Bataineh et al., 2009a; Bataineh et al., 2009b] and has been applied successfully to boundary value problems. In this modified HAM they expand the non-homogeneous term of the differential equation in terms of the homotopy parameter. They have shown the effectiveness of the Modified HAM by comparing their solutions with those of the standard HAM. There is another approach, called the “Predictor homotopy analysis method” proposed by Abbasbandy et al. [Abbasbandy et al., 2009] for determining multiple solutions of nonlinear problems. This approach is very important from a physical point of view since one can find multiple solutions of nonlinear problems. The application of the “Predictor homotopy analysis method” has been carried out for some nonlinear problems in [Abbasbandy and Shivanian, 2010; Abbasbandy and Shivanian, 2011]. The multistage homotopy analysis method (MSHAM) proposed by Alomari et al. [Alomari et al., 2010] has been used to analyze the hyperchaotic Chen system. Hyperchaotic solutions are obtained by MSHAM. The MSHAM is based on applying HAM to small sub-intervals of the considered domain of interest instead of the whole domain and obtaining the analytical solution defined over each sub-interval. The accuracy of MSHAM is discussed in [Alomari et al., 2010] by comparing with numerical solutions obtained by the RK4 algorithm. The “Spectral homotopy analysis method” has been proposed by Motsa et al. [Motsa et al., 2010a; Motsa et al., 2010b]. They have used the spectral HAM by using the Chebyshev pseudo-spectral technique to solve the linear high-order deformation equations which occur in HAM. A numerical technique is combined with HAM in this method. This is useful for
physical problems whose solutions are special functions like Chebyshev polynomials and so on. These modifications demonstrate the usefulness and effectiveness of HAM for nonlinear problems. In view of the above, in this thesis, we develop some modifications of HAM which lead to certain advantages in accuracy and help in capturing typical nonlinear behaviour. We use the idea of homotopy for solving specifically, the nonlinear problems arising from solid mechanics to fluid dynamics by modifying HAM and sketch the underlying physics.

1.3 OVERVIEW OF THE WORK

To enhance the rate of convergence and accuracy of the analytical solutions obtained by HAM, we present a modification of the standard HAM by introducing a non-homogeneous term in the auxiliary linear operator in chapter 2. We solve some nonlinear problems from the open literature [Liao, 2003] and make a comparison between solutions obtained by both the techniques. We also prove a convergence theorem which confirms that if the solution obtained by the proposed approach converges, it is a solution of the problem. We obtain the approximate analytical solution by minimizing the square residual error of the problem. We check the efficiency of the proposed technique by obtaining solutions of three nonlinear problems including the Duffing oscillator in space. The proposed approach can be considered as a special case of the “Further Generalization of HAM” described by Liao [Liao, 2003]. These results may be considered as the first application of the further generalization of HAM to nonlinear problems.
Most analytical studies of nonlinear oscillators are in the absence of external forcing as is clear from the literature [Buonomo, 1998; Liao, 2003; Liao, 2004a; Chen and Liu, 2007; Chen and Liu, 2009a; Chen and Liu, 2009b; Lopez et al., 2009; Turkyilmazoglu, 2010b; Abbasbandy et al., 2011]. Therefore, it is important to understand the variation of the frequency of the limit cycle solution with respect to the parameters present in the problem when external forcing is applied. In chapter 3, we analyze, for the first time, the effect of external forcing on the Van der Pol Duffing oscillator under the condition that the external frequency is same as the resultant frequency of the nonlinear oscillator. We verify our results by making a comparison between the proposed solutions and the numerical solution. We obtain nonlinear phenomena, namely, a limit cycle along with the frequency of the limit cycle in different parametric regimes. The efficiency of the inclusion of the non-homogeneous term is checked for this forced nonlinear oscillator also.

Chapter 4 describes the development of a non-perturbative analytical approach based on HAM to obtain different kinds of nonlinear phenomena such as obtaining limit cycle solutions of period one and two and quasi-periodic solutions for forced nonlinear oscillators. Nonlinear oscillators show very complex behaviour like limit cycles (periodic), quasi-periodicity and chaos with respect to different parameter values and different initial conditions. Determining the structure of the phase space of such systems is difficult through numerical techniques alone. Hence it is advantageous to develop approximate analytical techniques which can provide at least a rough guide to the structure of the phase space. Solutions of some nonlinear oscillators have been obtained analytically by a number of authors by using different approaches like the perturbation technique, the homotopy perturbation method, harmonic balance method, the homotopy analysis method (HAM) etc [Buonomo,
The asymptotic perturbation method based on the harmonic balance method and the perturbation method [Maccari, 2000], the method of multiple scales and the asymptotic method based on the harmonic balance method [Rebusco, 2008] and the method of multiple scales [Meccari, 2012] have been applied to analyze the periodic and quasi-periodic behaviour of nonlinear oscillators. It is clear that in all the above three methods we need a perturbation parameter to develop the approximate analytical solution which is a disadvantage of these techniques. An analytical technique free from this restriction would hence be valuable. To the best of our knowledge in the literature only one attempt has been made by Qaisi [Qaisi, 1996] to develop analytical solutions for limit cycles for the case of forcing without damping. He has obtained the solution expressions by using the power series method. We apply the technique proposed in this chapter to the forced Van der Pol and the forced Van der Pol Duffing oscillators. We verify our results by comparing the obtained solutions with the numerical solutions and discuss an advantage of using HAM for these problems in comparison to numerical techniques. In this case also we check the efficiency of the inclusion of the non-homogeneous term through further reduction of the square residual error.

Chapters 2, 3 and 4 detail certain advantages of the proposed approach in the case of a single nonlinear ordinary differential equation either by enhancing the rate of convergence or by obtaining more accurate results at the same order of approximation in comparison to the standard HAM. To extend the application of the proposed approach (HAM with a non-homogeneous term) to a system of nonlinear ordinary
differential equations for the first time, we analyze the dynamics of a micropolar fluid in chapter 5. We examine the effects of thermal-diffusion and diffusion-thermo on fully MHD flow of a micropolar fluid through a porous space in a vertical channel with asymmetric wall temperatures and concentrations. We discuss the convergence and the accuracy of the solution expression by evaluating the square residual error. The effect of important parameters on the heat and mass transfer characteristics are presented graphically. Further reduction of the square residual error by introducing a non-homogeneous term is discussed for a set of parameter values present in the problem.

To check the applicability, efficiency and to provide more weight to the proposed approach for a system of coupled nonlinear ordinary differential equations, we also study the Dufour and Soret effects on unsteady viscous flow over a contracting cylinder in the presence of thermal radiation in chapter 6. The coupled nonlinear partial differential equations that govern the flow are transformed into a system of coupled nonlinear ordinary differential equations by using a suitable similarity transformation. The effectiveness of including a non homogeneous term in HAM to a system of coupled nonlinear boundary value problems is checked. The standard HAM and HAM with a non-homogeneous term techniques are both employed to obtain analytical solutions for this system. The convergence of the obtained series solutions is analyzed with both techniques. A comparison between analytical and numerical solutions is presented for validation. The effects of various parameters on the flow variables have been discussed.

The last chapter 7 of the thesis sums up the contributions of the thesis, presents concluding remarks and discusses possibilities of future work.