Chapter 7

Summary and future scope

7.1 SUMMARY

This chapter contains a brief summary of the research work carried out in this thesis and some possibilities for future work.

In chapter 2, the application of HAM with a non-homogeneous term for a number of single nonlinear ordinary differential equations is described. A relation between the generalized form of HAM [Liao, 2003] and the proposed technique is given. To the best of the author's knowledge, this is the first application of the generalized form of HAM given by Liao to nonlinear problems. We observe that the non-homogeneous term does not affect the region of convergence of the nonlinear problems considered. However, the use of the non-homogeneous term helps in reducing the square residual error. A proof of convergence of the proposed technique provides more weight to our analysis. A faster convergence is obtained at the same order of approximation in comparison to standard HAM solutions. We see faster convergence not only for the solutions but also for the HAM amplitudes found by removing the secular term for the Duffing-oscillator problem in space. We can observe from this analysis that a
little additional effort leads to a significant improvement in the approximate solutions by reducing the square residual error.

The development of limit cycle solutions for the forced Van der Pol Duffing oscillator under the condition that the external frequency is equal to the resultant frequency of the forced nonlinear oscillator is carried out in chapter 3. Instead of choosing a proper value of the convergence control parameter $h$, we have used the optimal value of $h$ by minimizing the square residual error of the problem. On the basis of this analysis and the analysis proposed by Chen and Liu [Chen and Liu, 2009], we conclude that the frequency of the limit cycle (in case of forcing or no forcing) decreases on increasing the value of the damping parameter. We demonstrate that in the case without forcing, we obtain limit cycle solutions as already shown in the literature by Chen and Liu [Chen and Liu, 2009], if we use the proper value of $h$. As the square residual error is minimum and the comparison between numerical and analytical solutions is quite good, we conclude that the optimal homotopy analysis method is a strong tool for solving such nonlinear problems with high accuracy. In the case with forcing we note that the proposed regions of periodic solutions are similar to what have been demonstrated numerically in the literature [Venkatesan and Lakshmanan, 1997]. In addition to the development of the limit cycle solutions, we demonstrate the efficiency of the inclusion of a non-homogeneous term by further reduction of the square residual error for some set of parameter values. We found that the inclusion of the non-homogeneous term helps us in reducing the square residual error for this forced nonlinear oscillator problem also.
Limit cycles and quasi-periodic solutions are obtained for the first time by a modified homotopy approach for the forced Van der Pol and the forced Van der Pol Duffing oscillators in chapter 4. The comparison between the analytical solutions and the numerical solutions is good. The convergence of the analytical solution is demonstrated and the accuracy of the analytical solutions is confirmed by minimizing the square residual error. Therefore the proposed approach seems to be promising for the development of analytical solutions of nonlinear oscillators specially for limit cycles and quasi-periodic solutions. Thus, we have shown that a proper choice of parameter values can lead to limit cycles or quasi-periodic solutions for the forced Van der Pol and the forced Van der Pol Duffing oscillators. The advantage of the proposed technique over numerical techniques is that we can determine a priori the parameter values that may lead to limit cycles or quasi-periodic solutions. As is well known a priori determination of this numerically is quite difficult. A literature survey reveals that such a technique has not been developed previously for obtaining limit cycle solutions of period one and two and quasi-periodic solutions. In addition to this for this case also we obtain a lower square residual error when the non-homogeneous term is introduced in the frame of HAM for a number of cases. The analyses of chapters 2,3 and 4 confirms the applicability, efficiency and usefulness of the non-homogeneous term in the frame of HAM for a number of single nonlinear ordinary differential equations.

In chapter 5, HAM with a non-homogeneous term is applied successfully to a system of coupled nonlinear differential equations for the first time by studying the fully developed and laminar MHD mixed convective flow in a vertical porous space by taking into account viscous dissipation, thermal-diffusion, diffusion-thermo effects and a temperature-dependent heat source. The governing equations are reduced to
nondimensional form and are solved by HAM through the minimization of the average square residual error and then the efficiency of including a non-homogeneous term is demonstrated. The convergence and accuracy of HAM solutions has been discussed graphically. The velocity, temperature, and concentration field have been evaluated for various values of the parameters which appear in the equations. The velocity and flow rate for a micropolar fluid is found to be less than those of Newtonian fluids. Increasing the parameters $\alpha, Pr, Br$ and $D_u$ leads to an increase in the fluid temperature. The given increase of the viscous dissipation parameter and heat source parameter lead to an increase of the Nusselt number at the wall $y = 1$ but the trend is reversed at the other wall. Similarly, the reverse effect is observed for the Sherwood number at the walls. The results of the hydrodynamic case for a non-porous space in the absence of the heat source parameter can be obtained as a limiting case of our analysis by taking $M, \alpha \rightarrow 0$, and $D_a \rightarrow \infty$. The results of Cheng [Cheng, 2006] can be recovered as a limiting case of our analysis by taking $M, \alpha, Br, D_u, Pr, Sr, Sc, S, I \rightarrow 0$ and $D_a \rightarrow \infty$. With suitable choice of the parameters and taking the boundary conditions similar to [Grosan and Pop, 2007] for $\theta$, we note that the HAM solutions show very good agreement with the solutions given by Grosan and Pop [Grosan and Pop, 2007]. Further reduction of the average square residual error for a system of coupled nonlinear ordinary differential equations is obtained when we use HAM with a non-homogeneous term. A good comparison between the standard HAM solution and the solution obtained by the inclusion of the non-homogeneous term is observed. There is no qualitative change in the solutions obtained by the inclusion of a non-homogeneous term but the average square residual error is reduced in comparison to standard HAM solutions.
A mathematical model is analyzed to study the interaction of thermal radiation, thermal-diffusion and diffusion thermo effects on unsteady viscous flow over a contracting cylinder for extending the scope and applicability of HAM with a non-homogeneous technique in chapter 6. The governing equations in cylindrical coordinates are introduced and transformed into a system of nonlinear ordinary differential equations using similarity transformations. Standard HAM and HAM with a non-homogeneous term techniques are employed to obtain the analytical solutions for the system of coupled nonlinear ordinary differential equations. The effectiveness of the inclusion of a non-homogeneous term is shown by further significant reduction of the average square residual error. The results obtained by standard HAM and HAM with a non-homogeneous term are in good agreement with the numerical solutions. The dimensionless temperature distribution decreases for a given increase in radiation parameter while it increases with an increase in Dufour number. The dimensionless concentration distribution decreases for a given increase in the Dufour number while it increases with an increase in Soret number. Further, the Nusselt number increases for a given increase in $R_d, Pr, Sr$ and $Sc$ while it decreases with an increase in $D_u$. The Sherwood number decreases for a given increase in $R_d, Pr, Sr$ and $Sc$ while it increases with an increase in $D_u$. In continuation of the results described in chapters 5 and 6, we conclude that the non-homogeneous term helps us in reducing the average square residual error significantly for this system of coupled nonlinear ordinary differential equations also.
In brief, we have applied the HAM with a non-homogeneous term technique to single nonlinear ordinary differential equations and to systems of coupled nonlinear ordinary differential equations and found that the errors are reduced at the same order of approximation i.e., the convergence rate increases and better solutions are obtained in comparison to the standard HAM procedure.

7.2 FUTURE SCOPE

We need to test the solutions of more nonlinear problems arising from different branches of science and engineering by the proposed HAM with a non-homogeneous term technique, because we have seen significant reduction in the (average) square residual error for a number of nonlinear ordinary differential equations. If we include further terms in the non-homogeneous part, we may achieve greater accuracy of the solutions in fewer iterations. It would be helpful to have some theorems which can guide us in this regard. The non-perturbative approach for the forced Van der Pol and the forced Van der Pol Duffing oscillators can be extended to other nonlinear systems for obtaining their limit cycle and quasi-periodic solutions. Many more nonlinear phenomena like, chaos and fractals, remain to be considered and analytical techniques need to be developed for these problems.