Chapter 3

Cubic B-Spline Collocation Method

A variety of numerical techniques are available in the literature to solve differential equations. In this work, the technique of cubic B-spline collocation method (CSCM) is followed, in view of its advantage of being a piecewise approximation function. Subsequently, in this chapter, the splines based numerical methods for solving complex models represented by partial differential equations are discussed from literature. For completeness, the basic concepts and definitions of splines are introduced. The symbolic solution scheme with asymptotic convergence analysis is also given.

3.1 Numerical Approach

The analysis of systems of applied sciences, e.g. technology, economy, biology etc., needs a constantly growing use of numerical methods and computer sciences. In fact, once a physical system has been observed and phenomenologically analyzed, it is often appropriate to use mathematical models suitable to describe the evolution of system in
time and space. Indeed, the interpretation of system consisting of physical phenomena, which occasionally involves complex interactions, on the basis of numerical methods leads towards the simulation of the process. During recent years, numerical analysis is established as the most powerful tool for solving intricate practical problems. Two of the most popular techniques for solving partial differential equations are the finite element method (FEM) and the finite difference method (FDM). The finite element method involves dividing the domains into a finite number of elements and placing nodes at predetermined locations around the elements boundary. The method finds the solution at each of the nodes very accurately. An advantage of the FEM is that each element can have its own distinct geometry of varying complexities. Another equally popular computational technique to solve mathematical models is the finite difference method. In finite difference method, the solution is derived at a finite number of points by approximating the derivatives at each of them. The accuracy of this method is based on the refinement level of the grid points where the solution is being evaluated.

Over the past few years, another numerical technique has been increasingly used to solve mathematical models in engineering research, namely the Spline Collocation Method (SCM). The SCM has a few distinct advantages over the FEM and FDM. The key advantage over the FDM is that the SCM provides a piecewise-continuous, closed form solution. A major advantage over the FEM is that the SCM procedure is simple and easy to apply to many problems involving differential equations. The application of this technique in combination with orthogonal collocation points, over different fluid flow problems has resulted in great success.
3.2 Concepts and Definitions of Spline Functions

Spline is a draftsman tool which is used to draw a smooth curve passing through the specified points in a plane. It is a flexible metal strip attached with weights which can be adjusted to keep the strip in the required shape. As an analogy, in numerical analysis a function (piecewise polynomial) that describes a smooth curve through pre-assigned points is called ‘spline’.

Firstly, Schoenberg ([88], [89]), introduced spline functions or splines to the mathematical world. Since then, splines have proved to be enormously popular in branches of mathematics such as approximation theory, numerical analysis and statistics. Also, they have become useful tools in fields of applications, especially computer-aided methods in manufacturing, animation and tomography. According to Ahlberg et.al. [2], the mathematical spline is continuous and has both a continuous first derivative and a continuous second derivative. Prenter [77] generalized the statement of Ahlberg et.al. [2] as: the \( n^{th} \) order spline is a function which is continuous and its first \( n - 1 \) order derivatives are also continuous.

According to Prenter [77], the mathematical subfield of numerical analysis, a B-spline, \textit{i.e.} non-zero only on a finite interval, is a spline function that has minimal compact support with respect to a given degree, smoothness and domain partition. The B-splines are better suited to represent functions that are localized both in time and space. In particular, B-splines enable to represent functions with sharp spikes. According to De Boor [31], each spline can be written as linear combination of basis function of given spline as follows:
\[ S(x) = \sum_{i=0}^{N-p-1} \delta_i B_{i,p}(x), \]

\[ B_{i,p}(x) = \frac{x - x_i}{x_i + p - x_i} B_{i,p-1} + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}. \]

This is Cox De Boor recursion formula. The recurrence relation starts with the 1st order B-spline, just boxes, and builds up successively higher orders. For degree \( p = 0 \), the basis function is the step function as under:

\[ B_{i,0}(x) = \begin{cases} 
1 & \text{if } [x_i, x_{i+1}] \\
0 & \text{otherwise}.
\end{cases} \]

**Higher Order B-spline**

By using the Cox De Boor recursion formula and 1st order B-spline function, higher order B-splines can be evaluated. They are named according to the degree '\( p \)'.

- \( p=1 \) Linear
- \( p=2 \) Quadratic
- \( p=3 \) **Cubic**
- \( p=4 \) Quartic
- \( p=5 \) Quintic
- \( p=6 \) Sextic
- \( p=7 \) Septic
- \( p=8 \) Octic
Hereunder review of spline collocation methods employed to solve different types of mathematical problems is given.

### 3.3 Review of Spline Collocation Methods

Two point boundary value problems of hyperbolic, elliptic or parabolic nature are widely solved using splines of different degrees. In recent literature, the spline collocation method is used extensively and referred to as the pseudospectral method. The spline collocation provides approximations to the solution and its derivative with respect to independent variable at all points of the domain of the problem. Due to the simplicity and easy adaptability to the computer codes, the method has gained momentum and is frequently applied to many real world problems. Some prominent papers on splines are discussed below:

De Boor [30] demonstrated both the existence and uniqueness of certain bi-cubic splines of interpolation. Ahlberg *et al.* [1] have presented the orthogonality properties of the spline function. A more abstract approach to spline theory has been made by Atteia [12]. De Boor and Swartz [29] used an orthogonal collocation approach that used the Gaussian points as collocation points. The Gaussian points are the zeroes of the appropriate degree Legendre polynomials over the normalized knot interval. Daniel and Swartz [26] extrapolated collocation for two-point boundary value problems using cubic splines. De Boor [31] has presented a practical approach to splines in the form of piecewise linear
approximation, piecewise cubic interpolation and its error, parabolic spline interpolation and B-splines.

Fairweather and Meade [33] provided a comprehensive survey of spline collocation methods for the numerical solution of differential equations. Robinson and Fairweather [81] have examined use of orthogonal spline collocation (OSC) at Gauss points, for semi-discretization of cubic Schrödinger equation and 2-D parabolic equation of Tappert. Christara [23] has presented parallel solvers for spline collocation equations arising from discretization of elliptic PDEs. The convergence properties of semi iterative and Krylov subspace acceleration methods applied to the system of spline collocation equations have also been studied.

Fernandes [35] has analyzed two schemes namely piecewise Hermite bi-cubic OSC Laplace modified (LM) and OSC alternating direction implicit (ADI) for approximate solution of linear second order hyperbolic problems on rectangles. The schemes were unconditionally stable. The OSC ADI scheme was found less expensive than OSC LM method. Lou et.al. [60] have formulated and analyzed OSC methods for solution of certain Dirichlet biharmonic problems in unit square. They have presented existence, uniqueness and convergence results for OSC methods for the solution of three biharmonic problems based on splitting principle.

Manickam et.al. [61] have presented second-order splitting combined with orthogonal cubic spline collocation (OCSC) method for the Kuramoto-Sivashinsky equation. The continuous OCSC method was analyzed and optimal error was estimated using $L_2$ and $L_\infty$ norms. Li et.al. [55] have formulated a Schrödinger-type system, which is approximated by a Crank Nicolson OSC scheme for a problem governing the transverse
vibrations of a clamped thin square plate. This scheme is shown to be second-order accurate in time and of optimal order accuracy in space in the $H^1$ and $H^2$ norms. They presented the theoretical properties like existence, uniqueness and stability of the scheme.

Bialecki and Fairweather [14] have presented formulation, analysis and implementation of OSC for numerical solution of PDEs in two space variables. They reviewed applications of OSC to elliptic, parabolic, hyperbolic and Schrödinger-type PDEs as well as to parabolic and hyperbolic partial intro-differential equations. Layton [54] has given a numerical method for solving shallow water equations (SWE) in spherical coordinates. The SWE are discretized in time with semi-implicit leapfrog method and in space with cubic spline collocation method on a skipped latitude-longitude grid. Botella [17] described a B-spline collocation method for the solution of the incompressible Navier-Stokes equations, but without accounting for the pressure terms. Wang and Chen [101] have studied mixed convection boundary layer flows of non-Newtonian fluids over the wavy surfaces by coordinate transformation and cubic spline collocation numerical method. Bialecki [15] has developed an efficient algorithm for solving biharmonic Dirichlet problem on a rectangle using a fourth order mixed method based on the piecewise Hermite bicubic OSC discretization.

Johnson [44] has approximated the solution of the steady convection-diffusion problem with constant coefficients for two values of the Peclet number, with the objective of determining relative accuracy and efficiency for quartic Greville method and traditional cubic orthogonal collocation method. Danumjaya and Pani [27] have formulated and analyzed a second-order splitting procedure combined with OCSC method for the extended Fisher-Kolmogorov equation and also discussed the convergence analysis. Aziz
et.al. [13] have presented spline method for the solution of fourth-order parabolic PDEs. Stability analysis of the method has been carried out. Once mesh refinement is possible, much of the work has been done approximating the solution of singularly perturbed fluid flow problems. Kadalbajoo and Aggarwal [46] have presented fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed BVPs. They have used fitted mesh technique to generate piecewise uniform mesh along with B-spline method, which leads to a tridiagonal linear system. The method was shown to have uniform convergence of second order. Ramadan et.al. [79] used seventh order B-splines with collocation points uniformly distributed to solve the nonlinear Burgers’ equation. The Burgers’ equation is the simplest nonlinear model equation for studying diffusive waves in fluid dynamics.

Christara and Ng ([21], [22]) have developed optimal quadratic and cubic spline collocation methods for solving linear second-order two-point BVPs on non-uniform partitions, adaptive grid techniques and grid size and error estimators. The existence and uniqueness of spline collocation approximations were also shown under some conditions. Optimal global and local orders of convergence of spline collocation approximations and derivatives were also derived. They have focused on the discussion to 1-D BVPs but some of the ideas and methods considered are useful for higher-order and multi-dimensional BVPs.

Danumjaya and Nandakumaran [28] have presented OCSC method for the Cahn-Hilliard equation. They have studied the behaviour of the solution and derived optimal order error estimates. They presented computational experiments by using monomial basis functions in spatial direction and RADAU 5 time integrator. Mazzia et.al. [62] have
analyzed the convergence of multiple methods that have the collocation points coinciding with the knot points. Rao and Kumar [80] have presented a B-spline collocation method of higher order for a class of self-adjoint singularly perturbed BVPs by dividing the domain of differential equation into three non-overlapping subdomains.

Akyildiz and Vajravelu [4] have used OCSC method to solve nonlinear parabolic equation arising in magneto-hydrodynamic unsteady Poiseuille flow of the generalized Newtonian fluid. Jator and Sinkala [43] have presented a high order B-spline collocation method for linear BVPs. Wang and Zhang [102] have presented a class of conservative orthogonal spline collocation schemes for solving coupled Klein-Gordon-Schrodinger equations. The approach of Mazzia et al. [62] has been validated by Saka and Dag [85]. They have compared the B-spline results with the computed results of the Burgers’ equation and modified Burgers’ equation.

Mittal and Jain ([64], [65]) have proposed the method based on collocation of modified cubic B-splines over finite elements to approximate the solution of nonlinear Burgers’ equation and second order one dimensional hyperbolic telegraph equation.

### 3.4 Cubic B-Spline Function

Consider the space $S_3(\pi)$ as the set of all functions $s(x) \in C^2[a, b]$ that reduce to cubic polynomial on each element $(x_i, x_{i+1}), 0 \leq i \leq n - 1$, of $[a, b]$. There exists a unique function $s(x)$ in $S_3(\pi)$, for evenly spaced knots $x_i = x_0 + i(b - a)/n$. In addition, four additional knots $x_{-2} < x_{-1} < x_0$ and $x_{n+2} > x_{n+1} > x_n$ are introduced and the
function $B_{i,3}(x)(= B_i(x))$ is defined by Prenter [77] as follows:

\[
B_i(x) = \frac{1}{h^3} \begin{cases} 
(x - x_{i-2})^3, & [x_{i-2}, x_{i-1}] \\
 h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & [x_{i-1}, x_i] \\
h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & [x_i, x_{i+1}] \\
(x_{i+2} - x)^3, & [x_{i+1}, x_{i+2}] \\
0, & \text{otherwise.}
\end{cases}
\]

These functions on a single curve segment, where $x$ varies from 0 to 1, are graphed in Figure 3.1. Over the element $[x_i, x_{i+1}]$, cubic B-spline functions are plotted in Figure 3.2.

Each of the functions $B_i(x)$ is twice continuously differentiable on the entire real line, with

\[
B_i(x_j) = \begin{cases} 
4 & \text{if } j = i \\
1 & \text{if } j = i - 1 \text{ or } i + 1 \\
0 & \text{if } j = i + 1 \text{ or } j = i - 1,
\end{cases}
\]

and $B_i(x) \equiv 0$ for $x \geq x_{i+2}$ and $x \leq x_{i-2}$. Schoenberg [90] defined that the $B$-splines are the unique nonzero splines of smallest compact support with knots at $x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2}$. That is, any cubic spline $s(x)$ with these knots, that vanishes identically outside every interval $(x_{i-1}, x_{i+2})$, must be identically equal to zero. Since each $B_i(x)$ is a piecewise cubic polynomial with knots at $\pi$, each $B_i(x) \in S_3(\pi)$, and is called cubic basic spline (B-spline). To compute $B_i(x)$ and its derivatives at the knots, Table 3.1 is used. Since $B_i(x)$ and its derivatives vanish at all other knots, these values are omitted from the Table 3.1.
Let $\mathcal{S} = \{ B_{-1}, B_0, ..., B_{n+1} \}$. The functions $\mathcal{S}$ are linearly independent on $[a, b]$ and

$B_3(\pi) (= S_3(\pi)) = \text{span } \mathcal{S}$. There by, each cubic spline can be written as linear combination of cubic B-spline function and $B_3(\pi)$ is $(n + 3)$ dimensional.

### 3.5 Selection of Collocation Points

A collocation method involves satisfying a differential equation to some tolerance at a finite number of points, called collocation points. Choice of the collocation points is an important and sensitive part of the solution technique.

Firstly, the collocation method was used by Frazer et.al. [36] in 1937. There after, Bickley [16] used it along with the least squares method and the Galerkin method to solve unsteady heat condition problems. In 1963, Schetz [87] has applied the low-order collocation method to a number of boundary-layer problems. Villadsen and Stewart [100], in their pioneer work, have incorporated the orthogonality property in the collocation method. They have discovered that collocation points chosen as the zeros of orthogonal polynomials give good results due to some attractive features of these polynomials. They have selected the trial functions as the Jacobi polynomials and picked the collocation points as the corresponding zeros of these polynomials. The collocation method has been used over a wide range of problems in recent times.

According to Villadsen and Michelsen [99], for symmetric boundary value problems, the Jacobi polynomials are defined in terms of the hypergeometric functions depending upon the particle geometry, whereas for nonsymmetric boundary value problems, simple Jacobi polynomials are considered for different values of boundary points. Usually, the
zeros of Chebyshev and Legendre polynomials are taken as collocation points and these polynomials are also the particular cases of Jacobi polynomials.

**Zeros of shifted Chebyshev polynomial**

The Chebyshev polynomial can be written in the following form:

\[ T_r(x) = \cos r\theta, \quad \cos \theta = x, \quad -1 \leq x \leq 1, \]

where \( T_r(x) \) have the value unity at \( x = 1 \), and at \( x = -1 \), the value is +1 for even \( r \) and is −1 for odd \( r \). The turning points of \( T_r(x) \) occur at the zeros of \( \sin r\theta/\sin \theta \), i.e. at the \( r-1 \) points as:

\[
\theta_i = \frac{i\pi}{r}, \quad x_i = \cos \left( \frac{i\pi}{r} \right), \quad i = 1, 2, \ldots, r-1,
\]

and both turning points and zeros are symmetrically disposed about the origin \( x = 0 \).

Any finite range \( a \leq \xi \leq b \), can be transformed to the basic range \( -1 \leq x \leq 1 \) with the change of variable \( \xi = 0.5(b - a)x + 0.5(b + a) \), and for required range \( 0 \leq \xi \leq 1 \), it will be \( \xi = 0.5(x + 1) \), i.e., \( x = 2\xi - 1 \). It is convenient to have a special notation for this range, which is written as follows:

\[ T^*_r(\xi) = T_r(2\xi - 1), \quad 0 \leq \xi \leq 1. \]

All the properties of \( T^*_r(\xi) \) can be deduced from those of \( T_r(2\xi - 1) \) and in particular, all values of \( T^*_r(\xi) \) can be generated from the recurrence system:
\[ T_{r+1}^{\ast}(\xi) = 2(2\xi - 1)T_{r}^{\ast}(\xi) - T_{r-1}^{\ast}(\xi), \quad \text{with} \quad T_{0}^{\ast}(\xi) = 1, \quad T_{1}^{\ast}(\xi) = 2(2\xi - 1). \]

The zeros of shifted Chebyshev polynomials are used as collocation points in present study for axial direction. It has been employed because they have the tendency to keep the error down to a minimum at the corners ([20], [68]).

**Zeros of shifted Legendre polynomial**

The shifted Legendre polynomials \( P_{r}^{\ast}(\eta) \) defined on \([0, 1]\) are obtained from Legendre polynomials \( P_{r}(x) \) valid on \([-1, 1]\) by the variable transformation \( x = 2\eta - 1 \). using the recurrence relation,

\[
(r + 1)P_{r+1}(x) = (2r + 1)xP_{r}(x) - rP_{r-1}(x), \quad \text{with} \quad P_{0}(x) = 1, \quad P_{1}(x) = x.
\]

By using the variable transformation, relation between Legendre polynomials and shifted Legendre polynomials can be obtained as:

\[
P_{r}^{\ast}(\eta) = P_{r}(2\eta - 1), \quad 0 \leq \eta \leq 1.
\]

All the properties of \( P_{r}^{\ast}(\eta) \) can be deduced from those of \( P_{r}(2\eta - 1) \) and in particular, all values of \( P_{r}^{\ast}(\eta) \) can be generated from the recurrence system:

\[
(r+1)P_{r+1}^{\ast}(\eta) = (2r+1)(2\eta-1)P_{r}^{\ast}(\eta) - rP_{r-1}^{\ast}(\eta), \quad \text{with} \quad P_{0}^{\ast}(\eta) = 1, \quad P_{1}^{\ast}(\eta) = 2\eta - 1.
\]
In radial domain, the zeros of shifted Legendre polynomial have been taken as collocation points, because they have the tendency to give good approximation on the average, and in radial domain the results are required on average. In literature, the researchers ([8], [9], [58]) have also followed the Legendre polynomial to check the behavior of the dependent variable on the average.

### 3.6 Symbolic Solution using CSCM

For collocation method using B-spline curves, the locations of the knots and the values of the De Boor points play a critical role. The location of knots should be chosen so as to provide sufficient resolution for the solution. The values of the De Boor points are found by forcing the differential equations to be satisfied to some tolerance at a finite set of evaluation or collocation points.

Consider a partition of the interval $[0, 1]$ like, $\pi : 0 = x_0 < x_1 < \ldots < x_n = 1$, with piecewise uniform span $h_i (= h) = x_i - x_{i-1}, i = 1, 2, \ldots, n$. The approximate solution $\tilde{c}(x, t)$ of $c(x, t)$ for equation (1.8), in the space $S_3(\pi)$ of cubic B-spline, can be expressed as:

$$\tilde{c} = \sum_{i=-1}^{n+1} \delta_i(t) B_i(x).$$  (3.1)

As each cubic B-spline covers four nodal elements, therefore each element is covered by four cubic B-splines, i.e., $B_{i-1}, B_i, B_{i+1}, B_{i+2}$, for element $[x_i, x_{i+1}]$. Therefore, for the $i^{th}$ element, the representation of the approximate solution $\tilde{c}(x, t)$ is:

$$\tilde{c}(x, t) = \sum_{k=1}^{4} \delta_{k+i-1}(t) B_k(x), \quad i = 1, 2, \ldots, n.$$  (3.2)
In MATLAB [version 7.5.0.342 (R2007b)] code, which is used for the simulation of given diffusion models, the four cubic B-spline functions and their derivatives, for variable \( x \), are declared as:

\[
\begin{align*}
Bx_1 &= @(x) \frac{(1-x)^3}{6}; \\
Bx_2 &= @(x) \frac{(3x^3 - 6x^2 + 4)}{6}; \\
Bx_3 &= @(x) \frac{(-3x^3 + 3x^2 + 3x + 1)}{6}; \\
Bx_4 &= @(x) \frac{x^3}{6}; \\
\end{align*}
\]

\[
\begin{align*}
BPx_1 &= @(x) \frac{-(1-x)^2}{2}; \\
BPx_2 &= @(x) \frac{(3x^2 - 4x)}{2}; \\
BPx_3 &= @(x) \frac{(-3x^2 + 2x + 1)}{2}; \\
BPx_4 &= @(x) \frac{x^2}{2}; \\
\end{align*}
\]

\[
\begin{align*}
BPPx_1 &= @(x) 1-x; \\
BPPx_2 &= @(x) 3x-2; \\
BPPx_3 &= @(x) -3x+1; \\
BPPx_4 &= @(x) x; \\
\end{align*}
\]

In the element \([x_i, x_{i+1}]\), orthogonal collocation points are selected as follows:

\[
u_{ij} = x_i + h_i \theta_j, \quad i = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, m \tag{3.3}
\]
where \(0 \leq \theta_1 < \theta_2 < ... < \theta_m \leq 1\), are the zeros of \(m^{th}\) orthogonal polynomial on each element \([x_{i-1}, x_i]\), \(i = 0, 1, ..., n\), with step size \(h_i\). In the present work, the zeros of shifted Chebyshev and shifted Legendre polynomials are used as collocation points.

For the \(i^{th}\) element, the evaluation of B-spline function and its derivatives at chosen collocation points are expressed in MATLAB code as:

\[
B(i, j) = Bx\{i\}(u(j));
\]
\[
BP(i, j) = BPx\{i\}(u(j));
\]
\[
BPP(i, j) = BPPx\{i\}(u(j));
\]

By using the transformation equation (3.3), the trial function and its derivatives at the collocation points \((u'_j)'s\) are expressed as:

\[
\begin{aligned}
\tilde{c}(u_j, t) &= \sum_{k=1}^{4} \delta_{k+i-1}(t) B_k(u_j) \\
\frac{d\tilde{c}(u_j, t)}{du} &= \frac{1}{h_i} \sum_{k=1}^{4} \delta_{k+i-1}(t) B'_k(u_j) \\
\frac{d^2\tilde{c}(u_j, t)}{du^2} &= \frac{1}{h_i^2} \sum_{k=1}^{4} \delta_{k+i-1}(t) B''_k(u_j)
\end{aligned}
\]

\[i = 0, 1, ...n, \ j = 1, 2, ..., m. \tag{3.4}\]

By using the right hand side of equation (3.4) in equation (1.8), one gets a system of \((n + 1)\) linear differential equations as:

\[
A \frac{d\tilde{\delta}(t)}{dt} = A \tilde{\delta}'(t) = -\frac{1}{h^2} B \tilde{\delta}(t). \tag{3.5}
\]

Above equation contains \((n+1)\) unknowns \(\delta_0, \delta_1, ..., \delta_n\), with \(\tilde{\delta}'(t) = \frac{d\tilde{\delta}(t)}{dt} = (\delta_0, \delta_1, ..., \delta_n)^T\) and \(\tilde{\delta}(t) = (\delta_0, \delta_1, ..., \delta_n)^T\). Here \(A\) and \(B\) are the matrices of order \((n+1)\) and elements.
of these matrices represent the coefficients of the trial functions. These matrices are diagonally dominant and highly sparse in nature.

These sparse matrices save a significant amount of computer memory during data storage and at the same time speed up the processing of data to be evaluated. For \( n \) elements, a system of \((n + 3)\) equations is obtained from equations (1.8), (1.10) and (1.11). Of these, two unknowns are found using discretized boundary conditions and the rest \((n + 1)\) are obtained from the equation (3.5). The banded matrix system thus obtained is solved by MATLAB ode15s solver, using discretized form of equation (1.12) as an initial approximation. The MATLAB code for ode15s solver is given below. Its detail can be taken from MATLAB Help menu.

```matlab
options=odeset('Mass',Mass,'Jacobian',A,'BDF','on','
'normcontrol','on','Abstol',1e-9,'Reltol',1e-6);
sol=ode15s(@FB,[0 T],yo,options);
```

### 3.7 Asymptotic Convergence

The convergence analysis of parabolic differential equations of second order for fitted mesh finite difference method is given in Miller \textit{et.al.} [63]. Onah [71] has discussed the convergence of Galerkin finite element method and orthogonal spline collocation method. The asymptotic convergence of CSCM is derived using more comprehensive equation by extending the work of Onah [71] with general boundary and initial conditions. To check the theoretical asymptotic convergence behavior of the solution technique CSCM, a linear
convection diffusion time dependent parabolic partial differential problem (1.8)-(1.12), given in Chapter 1, is considered by taking $a_4 = 0$.

The operator $H$ defined by $\frac{\partial^2}{\partial x^2}$ in the spatial and time domain is positive definite in $L_2(0, 1)$, the space of all real valued Lebesgue measurable functions square integrable on $(0, 1)$, for all $t > 0$. Hence, there exist a continuous function $v(x, t) \in D(H)$, the space of all the functions twice continuously differentiable on the interval $[0, 1]$, such that for every constant $\lambda (> 0)$:

$$\lambda \langle v, v \rangle \leq \langle Hv, v \rangle, \quad \text{for } x \in (0, 1) \text{ and } t > 0, \quad (3.6)$$

and

$$\langle Hv, v \rangle \leq \langle v, Hv \rangle, \quad \forall \ x \text{ and } t. \quad (3.7)$$

The fundamental solution of system (1.8)-(1.12) is readily available in Carslaw and Jaeger [19]. The solution can be written in the form:

$$c(x, t) = \int_0^1 \exp(-\gamma t)V(x) \, dx, \quad (3.8)$$

where $V(x)$ is a function of $x$ and $\gamma$ is dependent on $x$, and the parameters $a_i$’s and $b_i$’s, which are used in equations (1.8)-(1.11). Therefore by Cauchy-Schwarz inequality, following has been obtained:

$$\| c(x, t) \| \leq \exp(-\sigma t) \| V \|, \quad t > 0, \quad (3.9)$$

where $\| V \| \leq 1$ and $\sigma > 0$. The equation (3.9) shows that the analytic solution of the
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parabolic problem (1.8)-(1.12) is asymptotically convergent, i.e., \( c(x, t) \to 0 \) as \( t \to \infty \).

Hereunder, an analysis is carried out to show the asymptotic convergence of system (1.8)-(1.12) through CSCM. Writing equation (3.5) as:

\[
N^* \frac{d\vec{\delta}(t)}{dt} = -I\vec{\delta}(t), \quad t > 0,
\]

(3.10)

with initial condition (1.12), where \( N^* = A^*B^{-1} \) and \( A^* = h^2A \) are nonsingular matrices, and \( I \) is identity matrix of order \( n + 1 \). For arbitrary constants \( \alpha_0 \) and \( \beta_0 \), using inequality (3.6), one gets:

\[
\alpha_0 \| \vec{\delta} \|^2 \leq \langle N^*\vec{\delta}, \vec{\delta} \rangle \leq \| N^* \| \| \vec{\delta} \|^2,
\]

(3.11)

and

\[
\beta_0 \| \vec{\delta} \|^2 \leq \langle -I\vec{\delta}, \vec{\delta} \rangle \leq \| I \| \| \vec{\delta} \|^2,
\]

(3.12)

where \( \| N^* \| \) and \( \| I \| \) are the spectral norms of matrices \( N^* \) and \( I \), respectively. From equation (3.10), inner product of \( N^*\vec{\dot{\delta}}(t) \) and \( \vec{\dot{\delta}}(t) \) can be written as: \( \langle N^*\vec{\dot{\delta}}, \vec{\dot{\delta}} \rangle = \langle I\vec{\dot{\delta}}, \vec{\dot{\delta}} \rangle \).

Using the concept that \( \langle N^*\vec{\dot{\delta}}, \vec{\dot{\delta}} \rangle = \frac{1}{2} \frac{d}{dt} \langle N^*\vec{\delta}, \vec{\delta} \rangle \), in equations (3.11) and (3.12), one obtains:

\[
\frac{d}{dt} \langle N^*\vec{\delta}, \vec{\delta} \rangle \leq -\frac{2\beta_0}{\| N^* \|} \langle N^*\vec{\delta}, \vec{\delta} \rangle,
\]

(3.13)

after integration (3.13), one gets:

\[
\langle N^*\vec{\delta}, \vec{\delta} \rangle \leq \| N^* \| \| \vec{c}_0 \|^2 \exp \left( -\frac{2\beta_0 t}{\| N^* \|} \right).
\]

(3.14)
Using the property of Euclidean norm, each $\delta_i$ satisfies the inequity:

$$
\delta_i \leq \| c_0 \| \exp \left( \frac{-2\beta_0 t}{\| N^* \|} \right),
$$

(3.15)

where $c_0$ is a constant (taken from equation 1.12).

Using $\| N^* \| = \| A^*B^{-1} \| = h^2 \| AB^{-1} \|$ and properties of exponential function, equation (3.15) can be written as:

$$
\delta_i \leq p^* h^2,
$$

(3.16)

where $p^*$ is constant.

Hence, it is clear that CSCM has asymptotic convergence of $O(h^2)$, which converges more rapidly than order $h^2$ as $h \to 0$.

### 3.8 Summary

From literature survey, it is clear that orthogonal collocation method with spline functions, is widely applicable for the solution of complex process systems. A new numerical approach of cubic B-spline collocation method, is proposed for simulation of time-dependent diffusion models for fluid flow. The state variables are approximated by cubic B-spline functions in space direction. These approximated functions are evaluated at collocation points, which are the zeros of orthogonal polynomial. The asymptotic convergence analysis of the CSCM technique is also discussed and the method is found to be of order $h^2$. 
The potential of the cubic B-spline collocation method will be investigated for solving a variety of linear or nonlinear diffusion models, with high numerical accuracy. The effectiveness of this method will be compared with existing methods through test problems, in next chapters.
**Figure 3.1:** Cubic B-spline functions for $x \in [0, 1]$.
Figure 3.2: Cubic B-spline functions for $x \in [x_i, x_{i+1}]$. 
Table 3.1: Values of $B_i(x)$, $B'_i(x)$ and $B''_i(x)$ at knots.

<table>
<thead>
<tr>
<th></th>
<th>$x_{i-2}$</th>
<th>$x_{i-1}$</th>
<th>$x_i$</th>
<th>$x_{i+1}$</th>
<th>$x_{i+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_i(x)$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$B'_i(x)$</td>
<td>$\frac{3}{h}$</td>
<td>0</td>
<td>$-\frac{3}{h}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B''_i(x)$</td>
<td>$\frac{6}{h^2}$</td>
<td>$-\frac{12}{h^2}$</td>
<td>$\frac{6}{h^2}$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>