Chapter 7

Waves in an orthorhombic porous piezo-thermoelastic laminated structure immersed in a fluid

7.1 Introduction

A layered medium consists of two or more material components attached at their interfaces in some fashion. Time-harmonic wave propagation in a layered media is of interest in many fields of research, such as in the analysis of laminated composite structures, composite smart structures, geophysics and submarine acoustics. For such studies, a layered media may be modelled in various ways and the choice of the pertinent model is a matter of closeness to the physical reality. The mathematical approach to wave propagation is strictly related to the model of layered medium. The response of the material system is given by the reflected waves and the transmitted waves, in the two half-spaces, and the field in the layers (or the layer). In discretely layered media, the field within each layer is decomposed into up and downgoing waves.

The transfer matrix method was proposed by Thompson (1950) and Haskell (1953). This method was used by Bufler (1971) and Bahar (1972) to study the isothermal elasticity problems in multilayered medium. Dynamic distribution of displacements and thermal stresses in multilayered media, in generalized thermoelasticity, has been studied by Verma et al. (1999). Verma and Hasebe (2001) studied the wave propagation in
plates of general anisotropic solids in generalized theory of thermoelasticity. The interaction of free harmonic waves with multilayered media in generalized thermoelasticity by utilizing the combination of the linear transformation and transfer matrix method approach was studied by Verma (2011).

Nayfeh and Chimenti (1989) studied dispersion curves of elastic waves one layered anisotropic media, i.e. composite lamina. Nayfeh and chimenti (1991) developed a transfer matrix technique to obtain the dispersion curves of elastic wave propagating in multilayered anisotropic media. Caviglia and Morro (2000) studied the propagation of waves in multilayered anisotropic solids. The phenomena of scattering of shear horizontal waves in layered piezoelectric composites immersed in an elastic fluid in terms of a recursive system of equations involving the piezoelectric impedance was discussed by Mesquida et al. (2001). Using octet formalism via hybrid matrix of the structure, the transmission of acoustic waves through multilayered piezoelectric materials was examined by Lam et al. (2009). The reflection and transmission of plane waves from a fluid-saturated porous piezoelectric solid interface was studied by Vashishth and Gupta (2011b). After then, a theoretical model, based on the transfer matrix method, was elaborated by Vashishth and Gupta (2012), for describing reflection and transmission of plane elastic waves through a porous piezoelectric laminated plate, immersed in a fluid. The reflection-transmission of acoustic/ultrasonic waves at an interface between liquid layer and porous piezoelectric layered half space was also studied by Vashishth and Gupta (2013b). Vashishth and Dahiya (2013) studied shear waves in a piezoceramic layered structure and Vashishth et al. (2015) study Generalized Bleustein Gulyaev type of waves in a layered porous piezoelectric material.

In this Chapter, a study of reflection and transmission of plane elastic waves in porous piezo-thermoelastic laminated plate immersed in a fluid has been done. A theoretical model is derived which is based on transfer matrix method. The layered structure is considered to be consisting of ‘n’ number of layers of porous piezo-thermoelastic materials immersed in a fluid. The formulation of the model is sketched in Section 7.2. The formal solutions for the mechanical displacements, electric potential, mechanical stresses, electrical displacements, temperature changes and temperature gradient are
obtained in Section 7.3. In Section 7.4, the transfer matrix technique is used to study the layered materials. The closed form expressions of the elements of transfer matrix are derived. The analytical expressions for the reflection coefficient and transmission coefficient are obtained in the Section 7.5. The effects of frequency, angle of incidence, number of layers, layer’s thickness and porosity on the reflection coefficient, reflection loss and transmission loss are investigated numerically for different configurations in the Section 7.6.

7.2 Formulation of the problem

We consider a laminated plate consisting of \( n \) porous piezo-thermoelastic layers having ‘crystal symmetry 2mm’ and of thickness \( h_j (j = 1, 2, ..., n) \). Layers are bonded at their interfaces (Figure 7.1). Let \( h_j (j = 1, 2, ..., n) \) be thickness of \( j^{th} \) layer and \( h \) be the total thickness of the plate. The laminated plate is sandwiched between two fluid half spaces and plate occupies the region \( 0 \leq x_3 \leq h \). An elastic wave from the upper fluid half space (UFHS), making an angle \( \theta_I \) with the \( x_3 \)-axis, is assumed to strike the interface \( x_3 = 0 \) which results in one reflected wave in the UFHS and one transmitted wave in the lower fluid half space (LFHS) after getting multiple refracted through \( n \) porous piezo-thermoelastic intervening layers.

7.3 Transfer Matrix

As proved in Chapter 6 that 14 waves can propagate in orthorhombic PPTE material having 2mm type of symmetry. Out of these 7 waves are upward moving and 7 waves are moving downward.

Let \( k_{3i}^{(j)} \) \((i = 1, 2, ..., 14; j = 1, 2, ..., n) \) correspond to the roots of the equation (6.11) in the \( j^{th} \) layer. Let \( k_{3i}^{(j)} \) \((i = 1, 2, ..., 7; j = 1, 2, ..., n) \) represent vertical components of wave vectors corresponding to the waves moving downward. Real and imaginary parts of these \( k_{3i}^{(j)} \) are positive. Similarly, \( k_{3i}^{(j)} \) \((i = 8, 9, ..., 14; j = 1, 2, ..., n) \) correspond to those roots of the equation (6.11) whose real and imaginary parts are negative and
Figure 7.1: Geometry of the problem.

represent upward going waves. \( k_{31}^{(j)} \), \( k_{32}^{(j)} \), \( k_{33}^{(j)} \) correspond to propagating \( qP_1 \), \( qS_1 \), \( qP_2 \) waves. \( k_{34}^{(j)} \) and \( k_{35}^{(j)} \) correspond to the thermal waves \( qT_1 \) and \( qT_2 \) propagating in biphase porous medium. \( k_{36}^{(j)} \) and \( k_{37}^{(j)} \) correspond to non-propagating electric potential wave modes in the \( j^{th} \) layer.

These fourteen roots \( k_{3i}^{(j)} \) of equation (6.11) are related among themselves as

\[
\begin{align*}
    k_{3i}^{(j)} & = -k_{31}^{(j)}, k_{32}^{(j)} = -k_{33}^{(j)}, k_{34}^{(j)} = -k_{35}^{(j)}, k_{36}^{(j)} = -k_{37}^{(j)}, \\
    k_{3i}^{(j)} & = -k_{31}^{(j)}, k_{33}^{(j)} = -k_{34}^{(j)}, k_{36}^{(j)} = -k_{37}^{(j)}. 
\end{align*}
\]  

(7.1)

Repeating the same steps, as detailed out in Chapter 6, the formal solutions for the mechanical displacements, electric potentials, temperature variations, mechanical stresses, electrical displacements and temperature gradients in the \( j^{th} \) layer can be
The transpose of matrix. The matrices \( X^{(j)} \) from equations (6.14) and (6.16) for the \( j \)-th layer, where
\[
(u_1^{s(j)}, u_3^{s(j)}, u_1^{f(j)}, u_3^{f(j)}, \phi^{s(j)}, \phi^{f(j)}, \theta^{s(j)}, \theta^{f(j)}) = \sum_{i=1}^{14} (1, U_{3i}^{s(j)}, U_{1i}^{f(j)}, U_{3i}^{f(j)}, \Phi_i^{s(j)},\Phi_i^{f(j)}),
\]
\[
\Phi_i^{f(j)}, \Theta_i^{s(j)}, \Theta_i^{f(j)} U_{1i}^{s(j)} \exp(\iota(k_1 x_1 + k_{3i} x_3 - \omega t)).
\]

Further, equations (7.2) and (7.3) can be written as
\[
\begin{align*}
\sigma_{31}^{(j)}, \sigma_{33}^{(j)}, \sigma^{(j)}, D_3^{s(j)}, D_3^{f(j)}, \theta_{s3}^{(j)}, \theta_{s3}^{(j)} &= \\
\sum_{i=1}^{14} (g_{1i}^{(j)}, g_{2i}^{(j)}, g_{3i}^{(j)}, g_{4i}^{(j)}, g_{5i}^{(j)}, g_{6i}^{(j)}, g_{7i}^{(j)}) U_{1i}^{s(j)} \exp(\iota(k_1 x_1 + k_{3i} x_3 - \omega t)).
\end{align*}
\]

where, the coefficients \( U_{3i}^{s(j)}, U_{1i}^{f(j)}, U_{3i}^{f(j)}, \Phi_i^{s(j)}, \Phi_i^{f(j)}, \Theta_i^{s(j)}, \Theta_i^{f(j)} \)

\( (i = 1, 2, \ldots, 14; j = 1, \ldots, n) \) and \( g_{1i}^{(j)}, g_{2i}^{(j)}, g_{3i}^{(j)}, g_{4i}^{(j)}, g_{5i}^{(j)}, g_{6i}^{(j)}, g_{7i}^{(j)} \) can be obtained from equations (6.14) and (6.16) for the \( j \)-th layer.

Further, equations (7.2) and (7.3) can be written as
\[
V^{(j)} = X^{(j)} W^{(j)} S^{(j)},
\]

where
\[
\begin{align*}
V^{(j)} &= (u_1^{s(j)}, u_3^{s(j)}, u_1^{f(j)}, u_3^{f(j)}, \sigma_{33}^{(j)}, \sigma^{(j)}, D_3^{s(j)}, D_3^{f(j)}, \sigma_{13}^{(j)}, \\
&\quad \phi^{f(j)}, \theta^{s(j)}, \theta^{f(j)}, \theta_{s3}^{(j)}, \theta_{s3}^{(j)}, \theta_{s3}^{(j)}, \theta_{s3}^{(j)}),
\end{align*}
\]

\( V^{(j)} \) is the field vector corresponding to the \( j \)-th layer. Here, superscript \( T \) denotes the transpose of matrix. The matrices \( X^{(j)}, W^{(j)} \) and \( S^{(j)} \) are described as
\[
X_{11}^{(j)} = 1, X_{12}^{(j)} = 1, X_{13}^{(j)} = 1, X_{14}^{(j)} = 1, X_{15}^{(j)} = 1, X_{16}^{(j)} = 1, X_{17}^{(j)} = 1,
X_{18}^{(j)} = 1, X_{19}^{(j)} = 1, X_{1_{11}}^{(j)} = 1, X_{1_{12}}^{(j)} = 1, X_{1_{13}}^{(j)} = 1, X_{1_{14}}^{(j)} = 1,
X_{21}^{(j)} = U_{s_{11}}^{r(j)}, X_{22}^{(j)} = -U_{s_{11}}^{s(j)}, X_{23}^{(j)} = U_{s_{12}}^{s(j)}, X_{24}^{(j)} = -U_{s_{12}}^{s(j)}, X_{25}^{(j)} = U_{s_{11}}^{s(j)},
X_{26}^{(j)} = -U_{s_{11}}^{s(j)}, X_{27}^{(j)} = U_{s_{3}}^{s(j)}, X_{28}^{(j)} = -U_{s_{3}}^{s(j)}, X_{29}^{(j)} = U_{s_{3}}^{s(j)}, X_{2_{10}}^{(j)} = -U_{s_{3}}^{s(j)},
X_{2_{11}}^{(j)} = U_{s_{13}}^{s(j)}, X_{2_{12}}^{(j)} = -U_{s_{13}}^{s(j)}, X_{2_{13}}^{(j)} = U_{s_{14}}^{s(j)}, X_{2_{14}}^{(j)} = -U_{s_{14}}^{s(j)},
X_{31}^{(j)} = U_{f_{1}}^{r(j)}, X_{32}^{(j)} = -U_{f_{1}}^{f(j)}, X_{33}^{(j)} = U_{f_{1}}^{f(j)}, X_{34}^{(j)} = -U_{f_{1}}^{f(j)}, X_{35}^{(j)} = U_{f_{1}}^{f(j)},
X_{36}^{(j)} = U_{f_{1}}^{f(j)}, X_{37}^{(j)} = U_{f_{1}}^{f(j)}, X_{38}^{(j)} = -U_{f_{1}}^{f(j)}, X_{39}^{(j)} = U_{f_{1}}^{f(j)}, X_{3_{10}}^{(j)} = -U_{f_{1}}^{f(j)},
X_{3_{11}}^{(j)} = U_{f_{1}}^{f(j)}, X_{3_{12}}^{(j)} = -U_{f_{1}}^{f(j)}, X_{3_{13}}^{(j)} = U_{f_{1}}^{f(j)}, X_{3_{14}}^{(j)} = -U_{f_{1}}^{f(j)},
X_{41}^{(j)} = g_{21}, X_{42}^{(j)} = g_{21}, X_{43}^{(j)} = g_{22}, X_{44}^{(j)} = g_{22}, X_{45}^{(j)} = g_{23}, X_{46}^{(j)} = g_{23}, X_{47}^{(j)} = g_{24},
X_{48}^{(j)} = g_{24}, X_{49}^{(j)} = g_{25}, X_{4_{10}}^{(j)} = g_{25}, X_{4_{11}}^{(j)} = g_{26}, X_{4_{12}}^{(j)} = g_{26}, X_{4_{13}}^{(j)} = g_{27}, X_{4_{14}}^{(j)} = g_{27},
X_{51}^{(j)} = \Phi_{1}^{s(j)}, X_{52}^{(j)} = -\Phi_{1}^{s(j)}, X_{53}^{(j)} = \Phi_{2}^{s(j)}, X_{54}^{(j)} = -\Phi_{2}^{s(j)}, X_{55}^{(j)} = \Phi_{3}^{s(j)},
X_{56}^{(j)} = -\Phi_{3}^{s(j)}, X_{5_{7}}^{(j)} = \Phi_{4}^{s(j)}, X_{5_{8}}^{(j)} = -\Phi_{4}^{s(j)}, X_{5_{9}}^{(j)} = \Phi_{5}^{s(j)}, X_{5_{10}}^{(j)} = -\Phi_{5}^{s(j)},
X_{5_{11}}^{(j)} = \Phi_{6}^{s(j)}, X_{5_{12}}^{(j)} = -\Phi_{6}^{s(j)}, X_{5_{13}}^{(j)} = \Phi_{7}^{s(j)}, X_{5_{14}}^{(j)} = -\Phi_{7}^{s(j)},
X_{61}^{(j)} = g_{41}, X_{6_{2}}^{(j)} = g_{41}, X_{6_{3}}^{(j)} = g_{42}, X_{6_{4}}^{(j)} = g_{42}, X_{6_{5}}^{(j)} = g_{43}, X_{6_{6}}^{(j)} = g_{43}, X_{6_{7}}^{(j)} = g_{44},
X_{6_{8}}^{(j)} = g_{44}, X_{6_{9}}^{(j)} = g_{45}, X_{6_{10}}^{(j)} = g_{45}, X_{6_{11}}^{(j)} = g_{46}, X_{6_{12}}^{(j)} = g_{46}, X_{6_{13}}^{(j)} = g_{47}, X_{6_{14}}^{(j)} = g_{47},
X_{71}^{(j)} = g_{51}, X_{7_{2}}^{(j)} = g_{51}, X_{7_{3}}^{(j)} = g_{52}, X_{7_{4}}^{(j)} = g_{52}, X_{7_{5}}^{(j)} = g_{53}, X_{7_{6}}^{(j)} = g_{53}, X_{7_{7}}^{(j)} = g_{54},
X_{7_{8}}^{(j)} = g_{54}, X_{7_{9}}^{(j)} = g_{55}, X_{7_{10}}^{(j)} = g_{55}, X_{7_{11}}^{(j)} = g_{56}, X_{7_{12}}^{(j)} = g_{56}, X_{7_{13}}^{(j)} = g_{57}, X_{7_{14}}^{(j)} = g_{57},
X_{81}^{(j)} = g_{61}, X_{8_{2}}^{(j)} = g_{61}, X_{8_{3}}^{(j)} = g_{62}, X_{8_{4}}^{(j)} = g_{62}, X_{8_{5}}^{(j)} = g_{63}, X_{8_{6}}^{(j)} = g_{63}, X_{8_{7}}^{(j)} = g_{64},
X_{8_{8}}^{(j)} = g_{64}, X_{8_{9}}^{(j)} = g_{65}, X_{8_{10}}^{(j)} = g_{65}, X_{8_{11}}^{(j)} = g_{66}, X_{8_{12}}^{(j)} = g_{66}, X_{8_{13}}^{(j)} = g_{67}, X_{8_{14}}^{(j)} = g_{67},
X_{91}^{(j)} = \Phi_{1}^{r(j)}, X_{9_{2}}^{(j)} = -\Phi_{1}^{r(j)}, X_{9_{3}}^{(j)} = \Phi_{2}^{r(j)}, X_{9_{4}}^{(j)} = -\Phi_{2}^{r(j)}, X_{9_{5}}^{(j)} = \Phi_{3}^{r(j)},
X_{9_{6}}^{(j)} = -\Phi_{3}^{r(j)}, X_{9_{7}}^{(j)} = \Phi_{4}^{r(j)}, X_{9_{8}}^{(j)} = -\Phi_{4}^{r(j)}, X_{9_{9}}^{(j)} = \Phi_{5}^{r(j)}, X_{9_{10}}^{(j)} = -\Phi_{5}^{r(j)},
X_{9_{11}}^{(j)} = \Phi_{6}^{r(j)}, X_{9_{12}}^{(j)} = -\Phi_{6}^{r(j)}, X_{9_{13}}^{(j)} = \Phi_{7}^{r(j)}, X_{9_{14}}^{(j)} = -\Phi_{7}^{r(j)},
X_{101}^{(j)} = g_{51}, X_{10_{2}}^{(j)} = g_{51}, X_{10_{3}}^{(j)} = g_{52}, X_{10_{4}}^{(j)} = g_{52}, X_{10_{5}}^{(j)} = g_{53},
X_{10_{6}}^{(j)} = g_{53}, X_{10_{7}}^{(j)} = g_{54}, X_{10_{8}}^{(j)} = g_{54}, X_{10_{9}}^{(j)} = g_{55}, X_{10_{10}}^{(j)} = g_{55},
X_{10_{11}}^{(j)} = g_{56}, X_{10_{12}}^{(j)} = g_{56}, X_{10_{13}}^{(j)} = g_{57}, X_{10_{14}}^{(j)} = g_{57},
All the non diagonal elements of matrix $W^{(j)}$ are zero and its diagonal elements are given by

$$
W_{11}^{(j)} = E_{1}^{(j)}, W_{12}^{(j)} = E_{2}^{(j)}, W_{13}^{(j)} = E_{3}^{(j)}, W_{14}^{(j)} = E_{4}^{(j)},
$$
$$
W_{22}^{(j)} = E_{5}^{(j)}, W_{23}^{(j)} = E_{6}^{(j)}, W_{24}^{(j)} = E_{7}^{(j)},
$$
$$
W_{33}^{(j)} = E_{8}^{(j)}, W_{34}^{(j)} = E_{9}^{(j)}, W_{44}^{(j)} = E_{10}^{(j)},
$$
$$
W_{55}^{(j)} = E_{11}^{(j)}, W_{56}^{(j)} = E_{12}^{(j)}, W_{57}^{(j)} = E_{13}^{(j)},
$$
$$
W_{66}^{(j)} = E_{14}^{(j)}, W_{67}^{(j)} = E_{15}^{(j)}, W_{68}^{(j)} = E_{16}^{(j)},
$$
$$
W_{77}^{(j)} = E_{17}^{(j)}, W_{78}^{(j)} = E_{18}^{(j)}, W_{79}^{(j)} = E_{19}^{(j)},
$$
$$
W_{88}^{(j)} = E_{20}^{(j)}, W_{89}^{(j)} = E_{21}^{(j)}, W_{810}^{(j)} = E_{22}^{(j)},
$$
$$
W_{99}^{(j)} = E_{23}^{(j)}, W_{910}^{(j)} = E_{24}^{(j)}, W_{911}^{(j)} = E_{25}^{(j)}, W_{912}^{(j)} = E_{26}^{(j)},
$$
$$
W_{1010}^{(j)} = E_{27}^{(j)}, W_{1011}^{(j)} = E_{28}^{(j)}, W_{1012}^{(j)} = E_{29}^{(j)}, W_{1013}^{(j)} = E_{30}^{(j)},
$$
$$
W_{1111}^{(j)} = E_{31}^{(j)}, W_{1112}^{(j)} = E_{32}^{(j)}, W_{1113}^{(j)} = E_{33}^{(j)}, W_{1114}^{(j)} = E_{34}^{(j)}.
$$

where

$$
E_{i}^{(j)} = \exp (i \omega k_{3i}^{(j)} x_3), \quad (i = 1, 2, ..., 14).
$$

(7.6)

$S^{(j)}$ is the amplitudes vector corresponding to $j^{th}$ layer and is given by
\[ \mathbf{S}^{(j)} = [U_{11}^{s'(j)} U_{14}^{s'(j)} U_{12}^{s'(j)} U_{13}^{s'(j)} U_{11}^{s'(j)} U_{12}^{s'(j)} U_{10}^{s'(j)} U_{16}^{s'(j)} U_{11}^{s'(j)} U_{15}^{s'(j)} U_{19}^{s'(j)} U_{17}^{s'(j)} U_{18}^{s'(j)} ] \] (7.7)

The mechanical displacements, electric potentials, temperature variations, mechanical stresses, electrical displacements and temperature gradients at the top of the \( j \)th layer are related with the amplitudes corresponding to \( j \)th layer as

\[ \mathbf{V}^-_j = \mathbf{X}^{(j)} \mathbf{W}^-_j \mathbf{S}^{(j)}, \] (7.8)

where

\[ \mathbf{V}^-_j = [\mathbf{V}^{(j)}]_{x_3=h_1+h_2+...+h_{j-1}}, \quad (2 \leq j \leq n) \quad \text{and} \quad \mathbf{V}^-_1 = [\mathbf{V}^{(1)}]_{x_3=0}, \]
\[ \mathbf{W}^-_j = [\mathbf{W}^{(j)}]_{x_3=h_1+h_2+...+h_{j-1}}, \quad (2 \leq j \leq n) \quad \text{and} \quad \mathbf{W}^-_1 = [\mathbf{W}^{(1)}]_{x_3=0}. \] (7.9)

The mechanical displacements, electric potentials, temperature variations, mechanical stresses, electrical displacements and temperature gradients, at the bottom of the \( j \)th layer are related with the amplitudes corresponding to \( j \)th layer as

\[ \mathbf{V}^+_j = \mathbf{X}^{(j)} \mathbf{W}^+_j \mathbf{S}^{(j)}, \] (7.10)

where

\[ \mathbf{V}^+_j = [\mathbf{V}^{(j)}]_{x_3=h_1+h_2+...+h_j}, \quad (1 \leq j \leq n), \]
\[ \mathbf{W}^+_j = [\mathbf{W}^{(j)}]_{x_3=h_1+h_2+...+h_j}, \quad (1 \leq j \leq n). \] (7.11)

Eliminating the common amplitude vector \( \mathbf{S}^{(j)} \) from the equations (7.8) and (7.10), we obtain

\[ \mathbf{V}^-_j = \mathbf{X}^{(j)} \mathbf{W}^-_j (\mathbf{W}^+_j)^{-1} (\mathbf{X}^{(j)})^{-1} \mathbf{V}^+_j. \] (7.12)
The matrix \((X^{(j)})^{-1}\) is obtained as

\[
(X^{(j)})^{-1} = \begin{pmatrix}
(X_1^{(j)}) & (X_2^{(j)}) \\
(X_3^{(j)}) & (X_4^{(j)})
\end{pmatrix}
\] (7.13)

where

\[
X_1^{(j)} = (H_1^{(j)})^{-1} - (H_1^{(j)})^{-1}H_2^{(j)}X_3^{(j)},
\]
\[
X_2^{(j)} = -(H_1^{(j)})^{-1}H_2^{(j)}X_4^{(j)},
\]
\[
X_3^{(j)} = -(X_4^{(j)})(H_3^{(j)})((H_1^{(j)})^{-1},
\]
\[
X_4^{(j)} = (H_1^{(j)} - H_3^{(j)}(H_1^{(j)})^{-1}(H_2^{(j)}))^{-1},
\] (7.14)

where

\[
(H_1)^{(j)}_{rs} = X_{rs}, \quad r, s = 1, \ldots, 10,
\]
\[
(H_2)^{(j)}_{rs} = X_{r,s+10}, \quad r = 1, \ldots, 10; s = 1, \ldots, 4,
\]
\[
(H_3)^{(j)}_{rs} = X_{r+10,s}, \quad r = 1, \ldots, 4; s = 1, \ldots, 10,
\]
\[
(H_4)^{(j)}_{rs} = X_{r+10,s+10}, \quad r = 1, \ldots, 4; s = 1, \ldots, 4
\]

The matrix \((H_1^{(j)})^{-1}\) is obtained as

\[
(H_1^{(j)})^{-1} = \begin{pmatrix}
y_{41} & z_{41} & z_{42} & y_{42} & z_{43} & y_{43} & y_{21} & z_{21} & z_{22} & y_{22} \\
y_{41} & -z_{41} & -z_{42} & y_{42} & -z_{43} & y_{43} & y_{21} & -z_{21} & -z_{22} & y_{22} \\
y_{44} & z_{44} & z_{45} & y_{45} & z_{46} & y_{46} & y_{23} & z_{23} & z_{24} & y_{24} \\
y_{44} & -z_{44} & -z_{45} & y_{45} & -z_{46} & y_{46} & y_{23} & -z_{23} & -z_{24} & y_{24} \\
y_{47} & z_{47} & z_{48} & y_{48} & z_{49} & y_{49} & y_{25} & z_{25} & z_{26} & y_{26} \\
y_{47} & -z_{47} & -z_{48} & y_{48} & -z_{49} & y_{49} & y_{25} & -z_{25} & -z_{26} & y_{26} \\
y_{31} & z_{31} & z_{32} & y_{32} & z_{33} & y_{33} & y_{11} & z_{11} & z_{12} & y_{12} \\
y_{31} & -z_{31} & -z_{32} & y_{32} & -z_{33} & y_{33} & y_{11} & -z_{11} & -z_{12} & y_{12} \\
y_{34} & z_{34} & z_{35} & y_{35} & z_{36} & y_{36} & y_{13} & z_{13} & z_{14} & y_{14} \\
y_{34} & -z_{34} & -z_{35} & y_{35} & -z_{36} & y_{36} & y_{13} & -z_{13} & -z_{14} & y_{14}
\end{pmatrix}
\] (7.15)

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where

\[
\begin{align*}
    z_{11} &= s_{14}/(s_{11} s_{14} - s_{12} s_{13}), \quad z_{12} = -s_{12}/(s_{11} s_{14} - s_{12} s_{13}), \\
    z_{13} &= -s_{13}/(s_{11} s_{14} - s_{12} s_{13}), \quad z_{14} = s_{11}/(s_{11} s_{14} - s_{12} s_{13})
\end{align*}
\]

\[
\begin{align*}
    z_{21} &= -(\alpha_{21} (z_{11} U_{34}^{s(j)} + z_{13} U_{35}^{s(j)}) + \alpha_{12} (z_{11} U_{34}^{f(j)} + z_{13} U_{35}^{f(j)}) + \alpha_{13} (z_{11} \Phi_{4}^{s(j)} + z_{13} \Phi_{5}^{s(j)})), \\
    z_{22} &= -(\alpha_{21} (z_{12} U_{34}^{s(j)} + z_{14} U_{35}^{s(j)}) + \alpha_{12} (z_{12} U_{34}^{f(j)} + z_{14} U_{35}^{f(j)}) + \alpha_{13} (z_{12} \Phi_{4}^{s(j)} + z_{14} \Phi_{5}^{s(j)})), \\
    z_{23} &= -(\alpha_{31} (z_{11} U_{34}^{s(j)} + z_{13} U_{35}^{s(j)}) + \alpha_{22} (z_{11} U_{34}^{f(j)} + z_{13} U_{35}^{f(j)}) + \alpha_{33} (z_{11} \Phi_{4}^{s(j)} + z_{13} \Phi_{5}^{s(j)})), \\
    z_{24} &= -(\alpha_{31} (z_{12} U_{34}^{s(j)} + z_{14} U_{35}^{s(j)}) + \alpha_{22} (z_{12} U_{34}^{f(j)} + z_{14} U_{35}^{f(j)}) + \alpha_{33} (z_{12} \Phi_{4}^{s(j)} + z_{14} \Phi_{5}^{s(j)})), \\
    z_{25} &= -(\alpha_{41} (z_{11} U_{34}^{s(j)} + z_{13} U_{35}^{s(j)}) + \alpha_{42} (z_{11} U_{34}^{f(j)} + z_{13} U_{35}^{f(j)}) + \alpha_{43} (z_{11} \Phi_{4}^{s(j)} + z_{13} \Phi_{5}^{s(j)})), \\
    z_{26} &= -(\alpha_{41} (z_{12} U_{34}^{s(j)} + z_{14} U_{35}^{s(j)}) + \alpha_{42} (z_{12} U_{34}^{f(j)} + z_{14} U_{35}^{f(j)}) + \alpha_{43} (z_{12} \Phi_{4}^{s(j)} + z_{14} \Phi_{5}^{s(j)})), \\
    z_{31} &= -(z_{11} (\alpha_{21} g_{11}^{(j)} + \alpha_{31} g_{12}^{(j)} + \alpha_{41} g_{13}^{(j)}) + z_{12} (\alpha_{21} \Phi_{1}^{f(j)} + \alpha_{31} \Phi_{2}^{f(j)} + \alpha_{41} \Phi_{3}^{f(j)})), \\
    z_{32} &= -(z_{11} (\alpha_{12} g_{11}^{(j)} + \alpha_{22} g_{12}^{(j)} + \alpha_{12} g_{13}^{(j)}) + z_{12} (\alpha_{12} \Phi_{1}^{f(j)} + \alpha_{22} \Phi_{2}^{f(j)} + \alpha_{12} \Phi_{3}^{f(j)})), \\
    z_{33} &= -(z_{11} (\alpha_{13} g_{11}^{(j)} + \alpha_{33} g_{12}^{(j)} + \alpha_{43} g_{13}^{(j)}) + z_{12} (\alpha_{13} \Phi_{1}^{f(j)} + \alpha_{33} \Phi_{2}^{f(j)} + \alpha_{43} \Phi_{3}^{f(j)})), \\
    z_{34} &= -(z_{11} (\alpha_{21} g_{11}^{(j)} + \alpha_{31} g_{12}^{(j)} + \alpha_{41} g_{13}^{(j)}) + z_{12} (\alpha_{21} \Phi_{1}^{f(j)} + \alpha_{31} \Phi_{2}^{f(j)} + \alpha_{41} \Phi_{3}^{f(j)})), \\
    z_{35} &= -(z_{13} (\alpha_{12} g_{11}^{(j)} + \alpha_{22} g_{12}^{(j)} + \alpha_{12} g_{13}^{(j)}) + z_{14} (\alpha_{12} \Phi_{1}^{f(j)} + \alpha_{22} \Phi_{2}^{f(j)} + \alpha_{12} \Phi_{3}^{f(j)})), \\
    z_{36} &= -(z_{13} (\alpha_{13} g_{11}^{(j)} + \alpha_{33} g_{12}^{(j)} + \alpha_{43} g_{13}^{(j)}) + z_{14} (\alpha_{13} \Phi_{1}^{f(j)} + \alpha_{33} \Phi_{2}^{f(j)} + \alpha_{43} \Phi_{3}^{f(j)})), \\
    z_{41} &= \alpha_{21} - 2 (\alpha_{21} (z_{31} U_{34}^{s(j)} + z_{34} U_{35}^{s(j)}) + \alpha_{12} (z_{31} U_{34}^{f(j)} + z_{34} U_{35}^{f(j)}) + \alpha_{13} (z_{31} \Phi_{4}^{s(j)} + z_{34} \Phi_{5}^{s(j)})), \\
    z_{42} &= \alpha_{12} - 2 (\alpha_{21} (z_{32} U_{34}^{s(j)} + z_{35} U_{35}^{s(j)}) + \alpha_{12} (z_{32} U_{34}^{f(j)} + z_{35} U_{35}^{f(j)}) + \alpha_{13} (z_{32} \Phi_{4}^{s(j)} + z_{35} \Phi_{5}^{s(j)})), \\
    z_{43} &= \alpha_{13} - 2 (\alpha_{21} (z_{33} U_{34}^{s(j)} + z_{36} U_{35}^{s(j)}) + \alpha_{12} (z_{33} U_{34}^{f(j)} + z_{36} U_{35}^{f(j)}) + \alpha_{13} (z_{33} \Phi_{4}^{s(j)} + z_{36} \Phi_{5}^{s(j)})), \\
    z_{44} &= \alpha_{31} - 2 (\alpha_{31} (z_{31} U_{34}^{s(j)} + z_{34} U_{35}^{s(j)}) + \alpha_{22} (z_{31} U_{34}^{f(j)} + z_{34} U_{35}^{f(j)}) + \alpha_{33} (z_{31} \Phi_{4}^{s(j)} + z_{34} \Phi_{5}^{s(j)})), \\
    z_{45} &= \alpha_{22} - 2 (\alpha_{31} (z_{32} U_{34}^{s(j)} + z_{35} U_{35}^{s(j)}) + \alpha_{22} (z_{32} U_{34}^{f(j)} + z_{35} U_{35}^{f(j)}) + \alpha_{33} (z_{32} \Phi_{4}^{s(j)} + z_{35} \Phi_{5}^{s(j)})), \\
    z_{46} &= \alpha_{33} - 2 (\alpha_{31} (z_{33} U_{34}^{s(j)} + z_{36} U_{35}^{s(j)}) + \alpha_{22} (z_{33} U_{34}^{f(j)} + z_{36} U_{35}^{f(j)}) + \alpha_{33} (z_{33} \Phi_{4}^{s(j)} + z_{36} \Phi_{5}^{s(j)})), \\
    z_{47} &= \alpha_{41} - 2 (\alpha_{41} (z_{31} U_{34}^{s(j)} + z_{34} U_{35}^{s(j)}) + \alpha_{42} (z_{31} U_{34}^{f(j)} + z_{34} U_{35}^{f(j)}) + \alpha_{43} (z_{31} \Phi_{4}^{s(j)} + z_{34} \Phi_{5}^{s(j)})), \\
    z_{48} &= \alpha_{42} - 2 (\alpha_{41} (z_{32} U_{34}^{s(j)} + z_{35} U_{35}^{s(j)}) + \alpha_{42} (z_{32} U_{34}^{f(j)} + z_{35} U_{35}^{f(j)}) + \alpha_{43} (z_{32} \Phi_{4}^{s(j)} + z_{35} \Phi_{5}^{s(j)})), \\
    z_{49} &= \alpha_{43} - 2 (\alpha_{41} (z_{33} U_{34}^{s(j)} + U_{34}^{s(j)} + z_{36} U_{35}^{f(j)} + z_{34} U_{35}^{f(j)} + \alpha_{43} (z_{33} \Phi_{4}^{s(j)} + z_{36} \Phi_{5}^{s(j)})), \\
    y_{11} &= t_{14}/(t_{11} t_{14}' - t_{12} t_{13}' ), \\
    y_{12} &= -t_{12}/(t_{11} t_{14}' - t_{12} t_{13}' ), \\
    y_{13} &= -t_{13}/(t_{11} t_{14}' - t_{12} t_{13}' ), \\
    y_{14} &= t_{11}/(t_{11} t_{14}' - t_{12} t_{13}' ),
\end{align*}
\]
\[\begin{align*}
y_{21} &= -\gamma_{21} (y_{11} + y_{13}) + \gamma_{12} (y_{11} g_{24}^{(j)} + y_{13} g_{25}^{(j)}) + \gamma_{13} (y_{11} g_{44}^{(j)} + y_{13} g_{45}^{(j)}), \\
y_{22} &= -\gamma_{21} (y_{14} + y_{12}) + \gamma_{12} (y_{12} g_{24}^{(j)} + y_{14} g_{25}^{(j)}) + \gamma_{13} (y_{12} g_{44}^{(j)} + y_{14} g_{45}^{(j)}), \\
y_{23} &= -\gamma_{31} (y_{11} + y_{13}) + \gamma_{22} (y_{11} g_{24}^{(j)} + y_{13} g_{25}^{(j)}) + \gamma_{33} (y_{11} g_{44}^{(j)} + y_{13} g_{45}^{(j)}), \\
y_{24} &= -\gamma_{31} (y_{14} + y_{12}) + \gamma_{22} (y_{12} g_{24}^{(j)} + y_{14} g_{25}^{(j)}) + \gamma_{33} (y_{12} g_{44}^{(j)} + y_{14} g_{45}^{(j)}), \\
y_{25} &= -\gamma_{41} (y_{11} + y_{13}) + \gamma_{42} (y_{11} g_{24}^{(j)} + y_{13} g_{25}^{(j)}) + \gamma_{43} (y_{11} g_{44}^{(j)} + y_{13} g_{45}^{(j)}), \\
y_{26} &= -\gamma_{41} (y_{14} + y_{12}) + \gamma_{42} (y_{12} g_{24}^{(j)} + y_{14} g_{25}^{(j)}) + \gamma_{43} (y_{12} g_{44}^{(j)} + y_{14} g_{45}^{(j)}), \\
y_{31} &= - (y_{11} (\gamma_{12} g_{31}^{(j)} + \gamma_{31} g_{32}^{(j)} + \gamma_{41} g_{33}^{(j)}) + y_{12} (\gamma_{21} g_{51}^{(j)} + \gamma_{31} g_{52}^{(j)} + \gamma_{41} g_{53}^{(j)}), \\
y_{32} &= - (y_{11} (\gamma_{12} g_{31}^{(j)} + \gamma_{22} g_{32}^{(j)} + \gamma_{42} g_{33}^{(j)}) + y_{12} (\gamma_{12} g_{51}^{(j)} + \gamma_{22} g_{52}^{(j)} + \gamma_{42} g_{53}^{(j)}), \\
y_{33} &= - (y_{11} (\gamma_{13} g_{31}^{(j)} + \gamma_{33} g_{32}^{(j)} + \gamma_{43} g_{33}^{(j)}) + y_{12} (\gamma_{13} g_{51}^{(j)} + \gamma_{33} g_{52}^{(j)} + \gamma_{43} g_{53}^{(j)}), \\
y_{34} &= - (y_{13} (\gamma_{21} g_{31}^{(j)} + \gamma_{31} g_{32}^{(j)} + \gamma_{41} g_{33}^{(j)}) + y_{14} (\gamma_{21} g_{51}^{(j)} + \gamma_{31} g_{52}^{(j)} + \gamma_{41} g_{53}^{(j)}), \\
y_{35} &= - (y_{13} (\gamma_{12} g_{31}^{(j)} + \gamma_{22} g_{32}^{(j)} + \gamma_{42} g_{33}^{(j)}) + y_{14} (\gamma_{12} g_{51}^{(j)} + \gamma_{22} g_{52}^{(j)} + \gamma_{42} g_{53}^{(j)}), \\
y_{36} &= - (y_{13} (\gamma_{13} g_{31}^{(j)} + \gamma_{33} g_{32}^{(j)} + \gamma_{43} g_{33}^{(j)}) + y_{14} (\gamma_{13} g_{51}^{(j)} + \gamma_{33} g_{52}^{(j)} + \gamma_{43} g_{53}^{(j)}), \\
y_{41} &= \gamma_{21} - 2 (\gamma_{21} (y_{31} + y_{34}) + \gamma_{12} (y_{31} g_{24}^{(j)} + y_{34} g_{25}^{(j)}) + \gamma_{13} (y_{31} g_{44}^{(j)} + y_{34} g_{45}^{(j)}), \\
y_{42} &= \gamma_{12} - 2 (\gamma_{21} (y_{32} + y_{35}) + \gamma_{12} (y_{32} g_{24}^{(j)} + y_{35} g_{25}^{(j)}) + \gamma_{13} (y_{32} g_{44}^{(j)} + y_{35} g_{45}^{(j)}), \\
y_{43} &= \gamma_{13} - 2 (\gamma_{21} (y_{33} + y_{36}) + \gamma_{12} (y_{33} g_{24}^{(j)} + y_{36} g_{25}^{(j)}) + \gamma_{13} (y_{33} g_{44}^{(j)} + y_{36} g_{45}^{(j)}), \\
y_{44} &= \gamma_{31} - 2 (\gamma_{31} (y_{31} + y_{34}) + \gamma_{22} (y_{31} g_{24}^{(j)} + y_{34} g_{25}^{(j)}) + \gamma_{33} (y_{31} g_{44}^{(j)} + y_{34} g_{45}^{(j)}), \\
y_{45} &= \gamma_{22} - 2 (\gamma_{31} (y_{32} + y_{35}) + \gamma_{22} (y_{32} g_{24}^{(j)} + y_{35} g_{25}^{(j)}) + \gamma_{33} (y_{32} g_{44}^{(j)} + y_{35} g_{45}^{(j)}), \\
y_{46} &= \gamma_{33} - 2 (\gamma_{31} (y_{33} + y_{36}) + \gamma_{22} (y_{33} g_{24}^{(j)} + y_{36} g_{25}^{(j)}) + \gamma_{33} (y_{33} g_{44}^{(j)} + y_{36} g_{45}^{(j)}), \\
y_{47} &= \gamma_{41} - 2 (\gamma_{41} (y_{31} + y_{34}) + \gamma_{42} (y_{31} g_{24}^{(j)} + y_{34} g_{25}^{(j)}) + \gamma_{43} (y_{31} g_{44}^{(j)} + y_{34} g_{45}^{(j)}), \\
y_{48} &= \gamma_{42} - 2 (\gamma_{41} (y_{32} + y_{35}) + \gamma_{42} (y_{32} g_{24}^{(j)} + y_{35} g_{25}^{(j)}) + \gamma_{43} (y_{32} g_{44}^{(j)} + y_{35} g_{45}^{(j)}), \\
y_{49} &= \gamma_{43} - 2 (\gamma_{41} (y_{33} + y_{36}) + \gamma_{42} (y_{33} g_{24}^{(j)} + y_{36} g_{25}^{(j)}) + \gamma_{43} (y_{33} g_{44}^{(j)} + y_{36} g_{45}^{(j)}), \\
\end{align*}\]
\[
\begin{align*}
\gamma_{11} &= (1 + 2 \gamma_{32} g_{21}^{(j)})/2, \\
\gamma_{12} &= 2 \gamma_{32} g_{42}^{(j)} - \gamma_{32} - 2 \gamma_{11} \gamma_{42}, \\
\gamma_{13} &= 2 (-\gamma_{11} g_{43} + \gamma_{32} g_{23}^{(j)}), \\
\gamma_{21} &= \gamma_{11} - 2 \gamma_{11} \gamma_{41} + 2 \gamma_{32} \gamma_{41} g_{23}^{(j)}, \\
\gamma_{22} &= \gamma_{32} + 2 \gamma_{32} \gamma_{42} g_{21}^{(j)} - 2 \gamma_{32} \gamma_{42} g_{23}^{(j)} - 2 \gamma_{32} \gamma_{41} g_{23}^{(j)}, \\
\gamma_{31} &= 2 \gamma_{32} \gamma_{41} g_{21}^{(j)} - \gamma_{32} g_{21}^{(j)} - 2 \gamma_{32} \gamma_{41} g_{23}^{(j)}, \\
\gamma_{32} &= 1/2 (g_{22}^{(j)} - g_{21}^{(j)}), \\
\gamma_{33} &= 2 (\gamma_{32} \gamma_{43} g_{21}^{(j)} - \gamma_{32} \gamma_{43} g_{23}^{(j)}), \\
\gamma_{41} &= -2 \gamma_{43} (\gamma_{11} g_{41}^{(j)} - \gamma_{32} g_{21}^{(j)} g_{42}^{(j)}), \\
\gamma_{42} &= -2 \gamma_{43} (\gamma_{32} g_{42}^{(j)} - \gamma_{32} g_{41}^{(j)}), \\
\end{align*}
\]
\[ \gamma_{43} = 1/2 \left( g_{43}^{(j)} - 2 \gamma_{11} g_{41}^{(j)} + 2 \gamma_{32} g_{21}^{(j)} g_{42}^{(j)} - 2 g_{23}^{(j)} (\gamma_{32} g_{42}^{(j)} - \gamma_{32} g_{41}^{(j)}) \right). \] (7.16)

The equation (7.12) can be written as

\[ V_j^- = T_j V_j^+, \] (7.17)

where

\[ T_j = X^{(j)} W_j^- (W_j^+)^{-1} (X^{(j)})^{-1}. \] (7.18)

\(T_j\) is the transfer matrix relating the mechanical displacements, electric potentials, temperature variations, mechanical stresses, electric displacements and temperature gradients at the top of the \(j^{th}\) layer to the corresponding quantities at the bottom of the \(j^{th}\) layer.

Repeating this process for each layer successively and considering the continuity conditions of mechanical displacements, electric potentials, temperature variations, mechanical stresses, electric displacements and temperature gradients at the each interface in the layered structure, the relation between the field vectors at the top and bottom of the laminated plate is given by

\[ V_1^- = T^* V_n^+, \] (7.19)

where

\[ V_1^- = [V^{(1)}]_{x_3=0}, \]
\[ V_n^+ = [V^{(n)}]_{x_3=h_1+h_2+...+h_n}, \]
\[ T^* = T_1 \ T_2 \ ... \ T_n. \] (7.20)

\(T^*\) is the global transfer matrix relating mechanical displacements, electric potentials, temperature change, mechanical stresses, electric displacements and temperature gradient at the top and the bottom of the laminated plate.

The normal displacement \((u_1^{us}, u_3^{us})\) and normal stress \((\sigma_{33}^{us})\) in the upper fluid half-

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space can be written as

\[
(u_{1}^{us}, u_{3}^{us}) = \sum_{p=1,2}(1, W_{p}^{us}) U_{p}^{us} \exp \left( \iota \left( k_{1}x_{1} + (-1)^{p+1}k_{3}^{us}x_{3} - \omega t \right) \right), \tag{7.21}
\]

\[
\sigma_{33}^{us} = \frac{i\omega^{2}\rho^{us}}{k_{1}} \sum_{p=1,2} U_{p}^{us} \exp \left( \iota \left( k_{1}x_{1} + (-1)^{p+1}k_{3}^{us}x_{3} - \omega t \right) \right). \tag{7.22}
\]

where

\[
W_{1}^{us} = k_{3}^{us}/k_{1}, \quad W_{2}^{us} = -k_{3}^{us}/k_{1},
\]

\[
k_{1} = \omega/c, \quad k_{3}^{us} = k_{1}\sqrt{\frac{\omega^{2}}{(v^{us})^{2}k_{1}^{2}2k_{1}^{2}} - 1}, \tag{7.23}
\]

\[v^{us}\text{ and }\rho^{us}\text{ are the incident wave velocity and density in the UFHS.}

Similarly, in the lower fluid half-space, the normal displacement \((u_{3}^{ls})\) and normal stress \((\sigma_{33}^{ls})\) can be written as

\[
u_{3}^{ls} = (k_{3}^{ls}/k_{1}) U_{ls} \exp \left( \iota \left( k_{1}x_{1} + k_{3}^{ls}x_{3} - \omega t \right) \right), \tag{7.24}\]

\[
\sigma_{33}^{ls} = \frac{i\omega^{2}\rho^{ls}}{k_{1}} \sum_{p=1,2} U_{p}^{ls} \exp \left( \iota \left( k_{1}x_{1} + k_{3}^{ls}x_{3} - \omega t \right) \right), \tag{7.25}
\]

where \(k_{3}^{ls} = k_{1}\sqrt{\frac{\omega^{2}}{(v^{ls})^{2}k_{1}^{2}2k_{1}^{2}} - 1}; v^{ls}\) and \(\rho^{ls}\) are the incident wave velocity and density in the LFHS.

### 7.4 Boundary conditions

The mechanical, electrical (free case) and thermal (isothermal) boundary conditions at the interface \(x_{3} = 0\) are
\[
\sigma_{33}^{(1)} = (1 - f)\sigma_{33}^{us}, \\
\sigma^{(1)} = f\sigma_{33}^{us}, \\
\sigma_{13}^{(1)} = 0, \\
(1 - f)\dot{u}_{3}^{s(1)} + f\dot{u}_{3}^{f(1)} = \dot{u}_{3}^{us}, \\
D_{3}^{s(1)} = 0, \\
D_{3}^{f(1)} = 0, \\
\theta^{s(1)} = 0, \\
\theta^{f(1)} = 0. \\
\tag{7.26}
\]

Similarly, the mechanical, electrical (free case) and thermal (isothermal) boundary conditions at the interface \( x_3 = h \) can be written as

\[
\sigma_{33}^{(n)} = (1 - f)\sigma_{33}^{ls}, \\
\sigma^{(n)} = f\sigma_{33}^{ls}, \\
\sigma_{13}^{(n)} = 0, \\
(1 - f)\dot{u}_{3}^{s(n)} + f\dot{u}_{3}^{f(n)} = \dot{u}_{3}^{ls}, \\
D_{3}^{s(n)} = 0, \\
D_{3}^{f(n)} = 0, \\
\theta^{s(n)} = 0, \\
\theta^{f(n)} = 0. \\
\tag{7.27}
\]
Using the equations (7.21) and (7.22), the equations (7.26) can be written as

\[
\sigma^{(1)}_{33} = (1 - f)x'_1[U^{us}_1 + U^{us}_2],
\]
\[
\sigma^{(1)}_1 = f x'_1[U^{us}_1 + U^{us}_2],
\]
\[
\sigma^{(1)}_{13} = 0,
\]
\[
(1 - f)\dot{u}^{s(1)}_3 + f \dot{u}^{f(1)}_3 = y'_1[U^{us}_1 - U^{us}_2],
\]
\[
D^{s(1)}_3 = 0,
\]
\[
D^{f(1)}_3 = 0,
\]
\[
\theta^{s(1)} = 0,
\]
\[
\theta^{f(1)} = 0,
\]

(7.28)

where \( x'_1 = i\omega^2 \rho^{us}/k_1 \), \( y'_1 = k^{us}_3/k_1 \)

Similarly, substitution of the equations (7.24) and (7.25) into the equation (7.27) gives

\[
\sigma^{(n)}_{33} = (1 - f)x'_2 U^{ls},
\]
\[
\sigma^{(n)}_1 = f x'_2 U^{ls},
\]
\[
\sigma^{(n)}_{13} = 0,
\]
\[
(1 - f)\dot{u}^{s(n)}_3 + f \dot{u}^{f(n)}_3 = y'_2 U^{ls},
\]
\[
D^{s(n)}_3 = 0,
\]
\[
D^{f(n)}_3 = 0,
\]
\[
\theta^{s(n)} = 0,
\]
\[
\theta^{f(n)} = 0,
\]

(7.29)

where \( x'_2 = (i\omega^2 \rho^{ls}/k_1) \exp(ihk^{ls}_3) \), \( y'_2 = (k^{ls}_3/k_1) \exp(ihk^{ls}_3) \).

Making use of the equations (7.19) in the equation (7.28) and following the convention that \( u^{s(n)}_1, u^{s(n)}_3, u^{f(n)}_1, u^{f(n)}_3, \sigma^{(n)}_{33}, \sigma^{(n)}_{31}, \sigma^{(n)}, \phi^{s(n)}, \phi^{f(n)}, D^{s(n)}_3, D^{f(n)}_3 \) in subsequent
expressions are evaluated at $x_3 = h$, we can write

\[
T^s_{41} u^{s(n)}_1 + T^s_{42} u^{s(n)}_3 + T^s_{43} u^{s(n)}_3 + T^s_{44} \sigma^{s(n)}_3 + T^s_{45} \phi^{s(n)} + T^s_{46} D^{s(n)}_3 + T^s_{47} \sigma^{s(n)} + T^s_{48} \sigma^{s(n)}_3 + T^s_{49} \phi^{s(n)} + T^s_{410} D^{s(n)}_3 + T^s_{411} \theta^{s(n)} + T^s_{412} \theta^{f(n)} + T^s_{413} \phi^{s(n)} = (1 - f) x'_1 [U^{us}_1 + U^{us}_2],
\]

\[
T^s_{71} u^{s(n)}_1 + T^s_{72} u^{s(n)}_3 + T^s_{73} u^{s(n)}_3 + T^s_{74} \phi^{s(n)} + T^s_{75} \phi^{s(n)} + T^s_{76} D^{s(n)}_3 + +T^s_{77} \sigma^{s(n)} + T^s_{78} \sigma^{s(n)}_3,\]

\[
T^s_{81} u^{s(n)}_1 + T^s_{82} u^{s(n)}_3 + T^s_{83} u^{s(n)}_3 + T^s_{84} \phi^{s(n)} + T^s_{85} \phi^{s(n)} + T^s_{86} D^{s(n)}_3 + +T^s_{87} \sigma^{s(n)} + T^s_{88} \sigma^{s(n)}_3,\]

\[
T^s_{89} \phi^{s(n)} + T^s_{810} D^{f(n)} + T^s_{811} \theta^{s(n)} + T^s_{812} \theta^{f(n)} + T^s_{813} \theta^{s(n)} + T^s_{814} \theta^{f(n)} = 0,
\]

\[
[(1 - f) T^s_{21} + f T^{s(n)}_{31}] u^{s(n)}_1 + [(1 - f) T^s_{22} + f T^{s(n)}_{32}] u^{s(n)}_3 + [(1 - f) T^s_{23} + f T^{s(n)}_{33}] u^{s(n)}_3 +
\]

\[
[(1 - f) T^s_{24} + f T^{s(n)}_{34}] \sigma^{s(n)}_3 + [(1 - f) T^s_{25} + f T^{s(n)}_{35}] \phi^{s(n)} + [(1 - f) T^s_{26} + f T^{s(n)}_{36}] D^{s(n)}_3 +
\]

\[
[(1 - f) T^s_{27} + f T^{s(n)}_{37}] \sigma^{s(n)} + [(1 - f) T^s_{28} + f T^{s(n)}_{38}] \sigma^{s(n)}_3 + [(1 - f) T^s_{29} + f T^{s(n)}_3 \phi^{s(n)} +
\]

\[
[(1 - f) T^s_{210} + f T^{s(n)}_{310}] D^{f(n)}_3 + [(1 - f) T^s_{211} + f T^{s(n)}_{311}] \theta^{s(n)} + [(1 - f) T^s_{212} + f T^{s(n)}_3 \theta^{f(n)} +
\]

\[
[(1 - f) T^s_{213} + f T^{s(n)}_{313}] \theta^{s(n)}_3 + [(1 - f) T^s_{214} + f T^{s(n)}_{314}] \theta^{f(n)} + y'_1 [U^{us}_1 - U^{us}_2],
\]

\[
T^s_{61} u^{s(n)}_1 + T^s_{62} u^{s(n)}_3 + T^s_{63} u^{s(n)}_3 + T^s_{64} \sigma^{s(n)}_3 + T^s_{65} \phi^{s(n)} + T^s_{66} D^{s(n)}_3 + +T^s_{67} \sigma^{s(n)} + T^s_{68} \sigma^{s(n)}_3
\]

\[
+T^s_{69} \phi^{s(n)} + T^s_{610} D^{f(n)} + T^s_{611} \theta^{s(n)} + T^s_{612} \theta^{f(n)} + T^s_{613} \theta^{s(n)} + T^s_{614} \theta^{f(n)} = 0,
\]

\[
T^s_{10} u^{s(n)}_1 + T^s_{10} u^{s(n)}_3 + T^s_{10} \phi^{s(n)}_3 + T^s_{10} \phi^{s(n)}_3 + T^s_{10} D^{s(n)}_3 + +T^s_{10} \sigma^{s(n)} + T^s_{10} \sigma^{s(n)}_3
\]

\[
+T^s_{10} \phi^{s(n)} + T^s_{10} D^{f(n)} + T^s_{10} \theta^{s(n)} + T^s_{10} \theta^{f(n)} + T^s_{10} \sigma^{s(n)}_3 + T^s_{10} \theta^{f(n)}_3 = 0,
\]

\[
T^s_{11} u^{s(n)}_1 + T^s_{11} u^{s(n)}_3 + T^s_{11} \phi^{s(n)}_3 + T^s_{11} \phi^{s(n)}_3 + T^s_{11} \phi^{s(n)} + T^s_{11} \phi^{s(n)} + T^s_{11} \theta^{s(n)} + T^s_{11} \theta^{f(n)} + T^s_{11} \sigma^{s(n)} + T^s_{11} \sigma^{s(n)}_3
\]

\[
+T^s_{11} \phi^{s(n)} + T^s_{11} D^{f(n)} + T^s_{11} \theta^{s(n)} + T^s_{11} \theta^{f(n)} + T^s_{11} \sigma^{s(n)}_3 + T^s_{11} \theta^{f(n)}_3 = 0,
\]

\[
T^s_{12} u^{s(n)}_1 + T^s_{12} u^{s(n)}_3 + T^s_{12} \phi^{s(n)}_3 + T^s_{12} \phi^{s(n)}_3 + T^s_{12} \phi^{s(n)} + T^s_{12} \phi^{s(n)} + T^s_{12} \theta^{s(n)} + T^s_{12} \theta^{f(n)} + T^s_{12} \sigma^{s(n)} + T^s_{12} \sigma^{s(n)}_3
\]

\[
+T^s_{12} \phi^{s(n)} + T^s_{12} D^{f(n)} + T^s_{12} \theta^{s(n)} + T^s_{12} \theta^{f(n)} + T^s_{12} \sigma^{s(n)}_3 + T^s_{12} \theta^{f(n)}_3 = 0,
\]

(7.30)
7.5 Reflection and transmission coefficients

In order to derive expressions for reflection and transmission coefficients, the following notations are defined:

\[
\begin{align*}
\frac{u_1^{s(n)}}{U_{1us}} &= W_1, \quad \frac{u_3^{s(n)}}{U_{1us}} = W_2, \quad \frac{u_3^{f(n)}}{U_{1us}} = W_3, \quad \frac{\sigma_{33}^{(n)}}{U_{1us}} = W_4, \quad \frac{\phi^{s(n)}}{U_{1us}} = W_5, \\
\frac{D_3^{s(n)}}{U_{1us}} &= W_6, \quad \frac{\sigma^{(n)}}{U_{1us}} = W_7, \quad \frac{\sigma_{13}^{(n)}}{U_{1us}} = W_8, \quad \frac{\phi^{f(n)}}{U_{1us}} = W_9, \quad \frac{D_3^{f(n)}}{U_{1us}} = W_{10}, \\
\frac{\theta^{s(n)}}{U_{1us}} &= W_{11}, \quad \frac{\theta^{f(n)}}{U_{1us}} = W_{12}, \quad \frac{\theta_3^{(n)}}{U_{1us}} = W_{13}, \quad \frac{\theta_3^{f(n)}}{U_{1us}} = W_{14}, \quad \frac{U_{us}}{U_{1us}} = R_C, \quad \frac{U_{ls}}{U_{1us}} = T_C,
\end{align*}
\]

\(T_i^{fs} = (1 - f)T_{2i}^{*} + fT_{3i}^{*}, \quad (i = 1, 2, ..., 14). \tag{7.31}\)

Using the above defined notations, the equations (7.29) and (7.30) become

\[
\begin{align*}
T_{41}^{*}W_1 + T_{42}^{*}W_2 + T_{43}^{*}W_3 + T_{44}^{*}W_4 + T_{45}^{*}W_5 + T_{46}^{*}W_6 + T_{47}^{*}W_7 + T_{48}^{*}W_8 \\
+ T_{49}^{*}W_9 + T_{410}^{*}W_{10} + T_{411}^{*}W_{11} + T_{412}^{*}W_{12} + T_{413}^{*}W_{13} + T_{414}^{*}W_{14} = (1 - f)x_1'[1 + R_C], \\
T_{71}^{*}W_1 + T_{72}^{*}W_2 + T_{73}^{*}W_3 + T_{74}^{*}W_4 + T_{75}^{*}W_5 + T_{76}^{*}W_6 + T_{77}^{*}W_7 + T_{78}^{*}W_8 \\
+ T_{79}^{*}W_9 + T_{710}^{*}W_{10} + T_{711}^{*}W_{11} + T_{712}^{*}W_{12} + T_{713}^{*}W_{13} + T_{714}^{*}W_{14} = f x_1'[1 + R_C], \\
T_{81}^{*}W_1 + T_{82}^{*}W_2 + T_{83}^{*}W_3 + T_{84}^{*}W_4 + T_{85}^{*}W_5 + T_{86}^{*}W_6 + T_{87}^{*}W_7 + T_{88}^{*}W_8 \\
+ T_{89}^{*}W_9 + T_{810}^{*}W_{10} + T_{811}^{*}W_{11} + T_{812}^{*}W_{12} + T_{813}^{*}W_{13} + T_{814}^{*}W_{14} = 0, \\
T_{11}^{*}W_1 + T_{12}^{*}W_2 + T_{13}^{*}W_3 + T_{14}^{*}W_4 + T_{15}^{*}W_5 + T_{16}^{*}W_6 + T_{17}^{*}W_7 + T_{18}^{*}W_8 \\
+ T_{19}^{*}W_9 + T_{110}^{*}W_{10} + T_{111}^{*}W_{11} + T_{112}^{*}W_{12} + T_{113}^{*}W_{13} + T_{114}^{*}W_{14} = y_1'[1 - R_C], \\
T_{61}^{*}W_1 + T_{62}^{*}W_2 + T_{63}^{*}W_3 + T_{64}^{*}W_4 + T_{65}^{*}W_5 + T_{66}^{*}W_6 + T_{67}^{*}W_7 + T_{68}^{*}W_8 \\
+ T_{69}^{*}W_9 + T_{610}^{*}W_{10} + T_{611}^{*}W_{11} + T_{612}^{*}W_{12} + T_{613}^{*}W_{13} + T_{614}^{*}W_{14} = 0,
\end{align*}
\]
\[ T_{10}^* W_1 + T_{10}^* W_2 + T_{10}^* W_3 + T_{10}^* W_4 + T_{10}^* W_5 + T_{10}^* W_6 + T_{10}^* W_7 + T_{10}^* W_8 + T_{10}^* W_9 + T_{10}^* W_{10} + T_{10}^* W_{11} + T_{10}^* W_{12} + T_{10}^* W_{13} + T_{10}^* W_{14} = 0, \]
\[ T_{11}^* W_1 + T_{11}^* W_2 + T_{11}^* W_3 + T_{11}^* W_4 + T_{11}^* W_5 + T_{11}^* W_6 + T_{11}^* W_7 + T_{11}^* W_8 + T_{11}^* W_9 + T_{11}^* W_{10} + T_{11}^* W_{11} + T_{11}^* W_{12} + T_{11}^* W_{13} + T_{11}^* W_{14} = 0, \]
\[ T_{12}^* W_1 + T_{12}^* W_2 + T_{12}^* W_3 + T_{12}^* W_4 + T_{12}^* W_5 + T_{12}^* W_6 + T_{12}^* W_7 + T_{12}^* W_8 + T_{12}^* W_9 + T_{12}^* W_{10} + T_{12}^* W_{11} + T_{12}^* W_{12} + T_{12}^* W_{13} + T_{12}^* W_{14} = 0, \]
\[ W_4 = (1 - f)x_2' T_C, \]
\[ W_7 = f x_2' T_C, \]
\[ W_8 = 0, \]
\[ (1 - f)W_2 + fW_3 = y_2' T_C, \]
\[ W_6 = 0, \]
\[ W_{10} = 0, \]
\[ W_{11} = 0, \]
\[ W_{12} = 0. \] (7.32)

Making use of the equations (7.32), the reflected \((R_C)\) and transmitted \((T_C)\) amplitude ratios are obtained as

\[ R_C = \frac{a_{51}a_{44} - a_{41}a_{54}}{a_{51}a_{42} - a_{41}a_{52}}, \] (7.33)

and

\[ T_C = \frac{(a_{51}a_{44} - a_{41}a_{54})(a_{61}a_{52} - a_{51}a_{62}) - (a_{51}a_{42} - a_{41}a_{52})(a_{61}a_{54} - a_{51}a_{64})}{(a_{51}a_{42} - a_{41}a_{52})a_{51}a_{63}}, \] (7.34)

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where expressions of $a_{ij}$ are written as

$$
\begin{align*}
\theta = & \ a_{11} = T''_{42} + T''_{45} \hat{z}_4 + T''_{49} \hat{z}_2 + T''_{41} \hat{z}_6, \\
\theta = & \ a_{12} = T''_{43} + T''_{45} \hat{z}_3 + T''_{49} \hat{z}_1 + T''_{41} \hat{z}_5, \\
\theta = & \ a_{21} = T''_{72} + T''_{75} \hat{z}_4 + T''_{79} \hat{z}_2 + T''_{71} \hat{z}_6, \\
\theta = & \ a_{22} = T''_{73} + T''_{75} \hat{z}_3 + T''_{79} \hat{z}_1 + T''_{71} \hat{z}_5, \\
\theta = & \ a_{31} = T''_{52} + T''_{55} \hat{z}_4 + T''_{59} \hat{z}_2 + T''_{51} \hat{z}_6, \\
\theta = & \ a_{32} = T''_{53} + T''_{55} \hat{z}_3 + T''_{59} \hat{z}_1 + T''_{51} \hat{z}_5, \\
\theta = & \ a_{41} = a_{12}a_{21} - a_{11}a_{22}, \\
\theta = & \ a_{42} = f a_{11} x'_1 - (1 - f) a_{21} x'_1, \\
\theta = & \ a_{44} = -f a_{11} x'_1 + (1 - f) a_{21} x'_1, \\
\theta = & \ a_{51} = a_{12}a_{31} - a_{11}a_{32}, \\
\theta = & \ a_{52} = -a_{11} y'_1 - (1 - f) a_{31} x'_1, \\
\theta = & \ a_{54} = -a_{11} y'_1 + (1 - f) a_{31} x'_1, \\
\theta = & \ a_{61} = a_{12}(1 - f) - a_{11} f, \\
\theta = & \ a_{62} = -(1 - f)^2 x'_1, \\
\theta = & \ a_{63} = a_{11} y'_2, \\
\theta = & \ a_{64} = (1 - f)^2 x'_1.
\end{align*}
$$
\[
\begin{align*}
\hat{a}_{12} &= \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \quad \hat{a}_{13} = \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \\
\hat{a}_{15} &= \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \quad \hat{a}_{19} = \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \\
\hat{\beta}_{11} &= \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \quad \hat{\beta}_{12} = \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \\
\hat{\beta}_{13} &= \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \quad \hat{\beta}_{15} = \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \\
\hat{\beta}_{19} &= \frac{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}{T_{11}^r T_{12}^{r \prime} - T_{11}^r T_{12}^{r \prime}}, \\
\end{align*}
\]

\[
\begin{align*}
&z_1 = \frac{x_{12} x_{13} - x_{11} x_{23}}{x_{12} x_{13} - x_{11} x_{23}}, \quad z_2 = \frac{x_{21} x_{14} - x_{11} x_{24}}{x_{21} x_{14} - x_{11} x_{24}}, \quad z_3 = \frac{x_{13} - x_{12} z_1}{x_{11}}, \quad z_4 = \frac{x_{14} - x_{12} z_2}{x_{11}}, \\
&z_5 = \frac{T_{63} \hat{T}_{65} - T_{63} \hat{T}_{65}}{T_{61}}, \quad z_6 = \frac{T_{62} + T_{63} \hat{T}_{65} + T_{63} \hat{T}_{65}}{T_{61}}, \\
x_{11} = T_{81} T_{65} - T_{61} T_{85}, x_{12} = T_{81} T_{69} - T_{61} T_{89}, x_{13} = T_{61} T_{83} - T_{63} T_{81}, \\
x_{14} = T_{83} T_{61} - T_{63} T_{81}, x_{21} = T_{10} T_{19} - T_{10} T_{19} T_{10} 5, x_{22} = T_{10} T_{19} - T_{10} T_{19} T_{10} 9, \\
x_{23} = T_{81} T_{10} - T_{81} T_{10} T_{10}, x_{24} = T_{10} T_{10} - T_{62} T_{62}, \\
T_{41} = T_{81} T_{42} + (1 - f) T_{41} T_{42} T_{41} x_{21} / y_{21} + f (1 - f) T_{42} x_{21} / y_{21}, \\
T_{43} = T_{43} + (1 - f) T_{43} x_{21} / y_{21} + f (1 - f) T_{43} x_{21} / y_{21}, T_{45} = T_{45}, T_{49} = T_{49}, \\
T_{71} = T_{71} T_{72} + (1 - f) T_{71} x_{21} / y_{21} + f (1 - f) T_{71} x_{21} / y_{21}, \\
T_{73} = T_{73} T_{74} + (1 - f) T_{73} x_{21} / y_{21} + f (1 - f) T_{73} x_{21} / y_{21}, \quad T_{75} = T_{75}, T_{79} = T_{79}, \\
T_{13} = T_{71} T_{72}, T_{14} = T_{73}, \\
T_{51} = T_{81} T_{82} + (1 - f) T_{81} T_{82} x_{21} / y_{21} + f (1 - f) T_{82} x_{21} / y_{21}, \\
T_{53} = T_{81} T_{82} + (1 - f) T_{81} T_{82} x_{21} / y_{21} + f (1 - f) T_{82} x_{21} / y_{21}, T_{55} = T_{81} T_{82}, T_{59} = T_{81} T_{82} T_{13} = T_{13}, T_{514} = T_{14}, \quad T_{81} = T_{81} T_{82} + (1 - f) T_{81} T_{82} x_{21} / y_{21} + f (1 - f) T_{82} x_{21} / y_{21}, \\
T_{83} = T_{83} + (1 - f) T_{83} x_{21} / y_{21} + f (1 - f) T_{83} x_{21} / y_{21}, T_{85} = T_{85}, T_{89} = T_{89}, T_{813} = T_{813}, T_{814} = T_{814}.
\end{align*}
\]
\[
T'_{10\ 1} = T_{10\ 1}^*; T'_{10\ 2} = T_{10\ 2}^* + (1 - f)^2 T_{10\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{10\ 1}^* \times \frac{x_2'/y_2'}{y_2'}, T_{10\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{10\ 5} = T_{10\ 5}^*, T'_{10\ 3} = T_{10\ 3}^* + (1 - f)f T_{10\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{10\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{10\ 5} = T_{10\ 5}^*, T'_{10\ 9} = T_{10\ 9}^*; T'_{10\ 13} = T_{10\ 13}^*; T'_{10\ 14} = T_{10\ 14}^*, T'_{11\ 1} = T_{11\ 1}^*; T'_{11\ 2} = T_{11\ 2}^* + (1 - f)^2 T_{11\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{11\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{11\ 5} = T_{11\ 5}^*, T'_{11\ 3} = T_{11\ 3}^* + (1 - f)f T_{11\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{11\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{11\ 14} = T_{11\ 14}^*, T'_{11\ 9} = T_{11\ 9}^*; T'_{11\ 13} = T_{11\ 13}^*; T'_{12\ 1} = T_{12\ 1}^*; T'_{12\ 2} = T_{12\ 2}^* + (1 - f)^2 T_{12\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{12\ 2}^* \times \frac{x_2'/y_2'}{y_2'}, T_{12\ 5} = T_{12\ 5}^*, T'_{12\ 3} = T_{12\ 3}^* + (1 - f)f T_{12\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{12\ 3}^* \times \frac{x_2'/y_2'}{y_2'}, T_{12\ 9} = T_{12\ 9}^*, T'_{12\ 13} = T_{12\ 13}^*; T'_{12\ 14} = T_{12\ 14}^*.
\]

The power reflection \((E_R)\), transmission \((E_T)\) and absorption coefficients \((E_{abs})\) are defined as

\[
E_R = |R_C|^2, E_T = |T_C|^2, E_{abs} = 1 - (|R_C|^2 + |T_C|^2).
\]

### 7.6 Discussion of Numerical Results

In order to study the effects of frequency, angle of incidence and number of layers on the reflection, transmission and power absorption coefficients for the considered model, numerical calculations have been done. For the purpose of numerical calculations, the water is considered as the ambient medium on the top and bottom of the laminated plate. The thickness of each layer is assumed to be 0.0015 m. The materials Barium titanate and PZT-6B are coded here as 1 and 3, respectively. Following Auld (1973) and Kar-Gupta and Venkatesh (2006), Zhou et al. (2012) Vashishth and Gupta (2012, 2015) the elastic, piezoelectric, dielectric, thermoelastic and pyroelectric coefficients for these ceramics are listed in the Tables 7.1-7.4. The dynamical coefficients and other parameters, used in the numerical study, are listed in the Table 7.5.
Table 7.1: Elastic constants, piezoelectric constants and dielectric constants of BaTiO₃ material (code 1) (Vashishth and Gupta, 2015)

<table>
<thead>
<tr>
<th>Elastic constants (GPa)</th>
<th>Piezoelectric constants (C/m²)</th>
<th>Dielectric constants (C²N⁻¹m⁻²) × 10⁻⁹</th>
</tr>
</thead>
<tbody>
<tr>
<td>c₁₁ = 150.4</td>
<td>η₃₁ = -4.32</td>
<td>ξ₁₁ = 10.8</td>
</tr>
<tr>
<td>c₁₂ = 65.63</td>
<td>η₃₃ = 17.4</td>
<td>ξ₃₃ = 13.1</td>
</tr>
<tr>
<td>c₁₃ = 65.94</td>
<td>η₁₅ = 11.4</td>
<td>ξ₁₁ = 11.8</td>
</tr>
<tr>
<td>c₃₃ = 145.5</td>
<td>ζ₃₁ = -1.728</td>
<td>ξ₃₃ = 13.9</td>
</tr>
<tr>
<td>c₅₅ = 43.86</td>
<td>ζ₃₃ = 6.96</td>
<td>A₃₃ = 15.1</td>
</tr>
<tr>
<td>m₁₁ = 8.8</td>
<td>ζ₁₅ = 4.56</td>
<td>A₁₁ = 12.8</td>
</tr>
<tr>
<td>m₃₃ = 5.2</td>
<td>ζ₃ = -7.5</td>
<td></td>
</tr>
<tr>
<td>R = 20</td>
<td>e₃ = -3.6</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.2: Expansion coefficients, Thermal conductivity constants and pyroelectric constants and some thermal parameters of BaTiO₃ (Zhou et al. (2012), Vashishth and Gupta (2012,2015))

<table>
<thead>
<tr>
<th>Expansion coefficients (K⁻¹)</th>
<th>Thermal conductivity constant (Wm⁻¹K⁻¹)</th>
<th>Pyroelectric constants (CK⁻¹m⁻²)</th>
<th>other parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>α₁₁ = 8.53 × 10⁻⁶</td>
<td>K¹¹⁺ = 1.1</td>
<td>p₃ = 5.53 × 10⁻⁴</td>
<td>K(Kgm⁻²s⁻¹K⁻¹) = 10</td>
</tr>
<tr>
<td>α₃₃ = 1.99 × 10⁻⁶</td>
<td>K₃₃⁺ = 3.5</td>
<td>p₃ = -2.27 × 10⁻⁴</td>
<td>T₀ = 300K</td>
</tr>
<tr>
<td>α⁽¹⁾₁₁ = -4.27 × 10⁻⁶</td>
<td>K⁽¹⁾₁₁⁺ = 0.582</td>
<td>p₃⁺ = 1 × 10⁻⁴</td>
<td>Cₑ = 5000</td>
</tr>
<tr>
<td>α⁽³⁾₃₃ = -0.99 × 10⁻⁶</td>
<td>K⁽³⁾₃₃⁺ = 0.6</td>
<td>p₃⁺ = -0.5 × 10⁻⁴</td>
<td></td>
</tr>
<tr>
<td>α⁽¹⁾₃₃ = -1.5 × 10⁻⁴</td>
<td></td>
<td></td>
<td>r(Kgm⁻³) = 5700</td>
</tr>
<tr>
<td>α⁽¹⁾ = 3 × 10⁻⁴</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3: Elastic constants, piezoelectric constants and dielectric constants of PZT-6B crystal (code 3) material (Vashishth and Gupta, 2015)

<table>
<thead>
<tr>
<th>Elastic constants (GPa)</th>
<th>Piezoelectric constants (C/m²)</th>
<th>Dielectric constants (C²N⁻¹m⁻²) × 10⁻⁹</th>
</tr>
</thead>
<tbody>
<tr>
<td>c₁₁ = 148</td>
<td>η₃₁ = -2.324</td>
<td>ξ₁₁⁺ = 3.984</td>
</tr>
<tr>
<td>c₁₂ = 76.2</td>
<td>η₃₃ = 10.99</td>
<td>ξ₃₃⁺ = 11.8</td>
</tr>
<tr>
<td>c₁₃ = 74.2</td>
<td>η₁₅ = 9.3</td>
<td>ξ₁₁⁺ = 2.081</td>
</tr>
<tr>
<td>c₃₃ = 131</td>
<td>ζ₃₁ = -0.48</td>
<td>ξ₃₃⁺ = 13.9</td>
</tr>
<tr>
<td>c₅₅ = 43.86</td>
<td>ζ₃₃ = 5.32</td>
<td>A₁₁ = 12.8</td>
</tr>
<tr>
<td>m₁₁ = 8.8</td>
<td>ζ₁₅ = 2.72</td>
<td>A₃₃ = 15.1</td>
</tr>
<tr>
<td>m₃₃ = 5.2</td>
<td>ζ₃ = -7.5</td>
<td></td>
</tr>
<tr>
<td>R = 20</td>
<td>e₃ = -3.6</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.4: Expansion coefficients, Thermal conductivity constants and pyroelectric constants and some thermal parameters of PZT-6B crystal (Zhou et al. (2012), Vashishth and Gupta (2012, 2015));

<table>
<thead>
<tr>
<th>Expansion coefficients $(K^{-1})$</th>
<th>Thermal conductivity constant $(Wm^{-1}K^{-1})$</th>
<th>Pyroelectric constants $(CK^{-1}m^{-2})$</th>
<th>other parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{11} = 7 \times 10^{-6}$</td>
<td>$K_{11}^{s*} = 1.2$</td>
<td>$p_3^s = 3.7 \times 10^{-4}$</td>
<td>$K(Kgm^{-2}s^{-1}K^{-1}) = 10$</td>
</tr>
<tr>
<td>$\alpha_{33} = 7 \times 10^{-6}$</td>
<td>$K_{33}^{s*} = 1.2$</td>
<td>$p_3^{sf} = -1.7 \times 10^{-4}$</td>
<td>$T_o = 300K$</td>
</tr>
<tr>
<td>$\alpha_{11}^{sf} = -3.24 \times 10^{-6}$</td>
<td>$K_{11}^{sf} = 0.582$</td>
<td>$p_3^{sf} = 1 \times 10^{-4}$</td>
<td>$C_o^{s}(JKg^{-1}K^{-1}) = 420$</td>
</tr>
<tr>
<td>$\alpha_{33}^{sf} = -2.8 \times 10^{-6}$</td>
<td>$K_{33}^{sf} = 0.6$</td>
<td>$p_3^{fs} = -0.5 \times 10^{-4}$</td>
<td>$C_o^{f}(JKg^{-1}K^{-1}) = 4186$</td>
</tr>
<tr>
<td>$\alpha^{fs} = -1.5 \times 10^{-4}$</td>
<td></td>
<td></td>
<td>$\rho( Kgm^{-3} ) = 7600$</td>
</tr>
<tr>
<td>$\alpha^f = 3 \times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Table 7.5: Dynamical coefficients, permeability tensor and other parameters (Vashisht and Gupta (2012, 2015))

<table>
<thead>
<tr>
<th>ρ_{us} = 1000 Kg/m³</th>
<th>ρ_{ls} = 1000 Kg/m³</th>
<th>\nu_{us} = 1500 m/s²</th>
<th>\nu_{ls} = 1500 m/s²</th>
</tr>
</thead>
<tbody>
<tr>
<td>\mu = 1 \times 10^{-3} Ns/m²</td>
<td>v = 1 MHz</td>
<td>f = 0.2</td>
<td>s_{33} = 1.0 \times 10^{-10} m²</td>
</tr>
<tr>
<td>ρ_{11}/\rho = \rho_{33}/\rho = 0.66</td>
<td>ρ_{12}/\rho = \rho_{32}/\rho = -0.15</td>
<td>ρ_{22}/\rho = \rho_{33}/\rho = 0.64.</td>
<td></td>
</tr>
</tbody>
</table>

Using the coefficients listed in these tables, the thermoelastic coupling coefficients can be calculated as follows:

\[
\begin{align*}
\beta_{11}^t &= (c_{11} + c_{13})\alpha_{11} + c_{13}\alpha_{33} + \alpha_{sf} m_{11}; \\
\beta_{33}^t &= 2c_{13}\alpha_{11} + c_{33}\alpha_{33} + \alpha_{sf} m_{33}; \\
C_{11}^t &= (c_{11} + c_{13})\alpha_{sf}^{11} + c_{13}\alpha_{33}^{sf} + \alpha_{sf} m_{11}; \\
C_{33}^t &= 2c_{13}\alpha_{11}^{sf} + c_{33}\alpha_{33}^{sf} + \alpha_{sf} m_{33}; \\
M &= 2\alpha_{11} m_{11} + \alpha_{33} m_{33} + \alpha_{sf} R; \\
\tilde{N} &= \alpha_{sf} R + 2\alpha_{11}^{sf} m_{11} + \alpha_{33}^{sf} m_{33}; \\
U &= -\left(\alpha_{11}(c_{11}^t + C_{33}^t) + \alpha_{33}(c_{11}^t + C_{33}^t) + \alpha_{sf} \tilde{N}\right)300; \\
\end{align*}
\]

(7.41)

The variation of power reflection and transmission coefficients with the angle of incidence for a Barium Titanate plate is observed in the Figure 7.2. The effects of thickness of the plate on the variation of these coefficients with angle of incidence are also observed. The frequency is assumed as 1MHz. The power reflection and transmission coefficients show maxima and minima for different angles in the angle range 0° – 38°. The reflection minima are due to constructive interference effects within the layers. The maxima (minima) in transmission (reflection) coefficient spectra correspond to the excitation of surface wave modes. The number of maxima and minima increases with increase in the thickness of the layer. It can therefore, be concluded that, more guided wave modes can be excited as the thickness of plate increase due to increase the number of pass bands. The angle of incidence, after which all incident energy is reflected back, decreases as the thickness of the layer is increased. This technique can
potentially be applied to assist transducer design in underwater NDE systems. The amount of energy, trapped within the layered structure, increases with increase in the thickness of the laminated plate which is in agreement with general physical laws and earlier studies (Sastry and Munjal, 1998).

Figure 7.2: Variation of reflection coefficient \((E_R)\) and transmission coefficient \((E_T)\) with the angle of incidence at frequency = 1 MHz; (i) 1 PPTE layer (ii) 2 PPTE layers (iii) 3 PPTE layers (iv) 4 PPTE layers

Figure 7.3 shows the effects of addition of periodic stack, consisting of layers 1 and 3, on the variation of reflection, transmission and absorption coefficients with the angle of incidence. The number of maxima and minima in the reflection and transmission coefficients also increases with the addition of stack of layers in the angle range \(0^\circ - 40^\circ\). Comparison of the figures 7.2 and 7.3 gives the effects of the angle of incidence and material of the layers on the reflection and transmission coefficients.

The effects of number of layers and frequency on the reflection, transmission co-
Figure 7.3: Variation of reflection, transmission and absorption coefficients with the angle of incidence at frequency = 1MHz; (i) ← |1|3| → (ii) ← |1|3|1|3| → (iii) ← |1|3|1|3|1|3| → (iv) ← |1|3|1|3|1|3|1|3|
Efficients and absorption coefficients have been investigated and are shown in Figure 7.4. The angle of incidence is taken as $\theta = 5^\circ$. The frequency dependence of the power coefficients shows oscillating behaviour which confirms that the reflection and transmission coefficients for layered structures are sensitive to changes in the frequency. The minima in the reflection coefficients correspond to principle Bragg reflection and they are further accompanied by secondary minima. It is observed that the number of maxima (minima) in power transmission (reflection) coefficients increases in number as the layers are added to the array. The frequencies at which reflection is very low are of practical importance as maximum energy get transmitted corresponding to such frequencies.

**Figure 7.4:** Variation of reflection, transmission and absorption coefficients with the frequency at $\theta = 5^\circ$; (i) 1 PPTE layer (ii) 2 PPTE layers (iii) 3 PPTE layers (iv) 4 PPTE layers
The effects of periodic stack consisting of layers 1 and 3 on the variation of reflection and transmission coefficients with frequency are illustrated in Figure 7.5. When more than two media are involved, interference effects produce frequency dependent transmission and reflection coefficients for waves travelling through media of finite thickness. The development of the band structures with the increase of stacks can be seen in the Figure 7.5. Further, it is also observed that the minima in the reflection coefficient get deepen and increase in number with the addition of periodic stacks of layers. The Bragg minima do not have same amplitudes since the length of the structure is finite and therefore is not perfectly periodic. The number of stop bands \((E_R = 1)\) also increases as the stacks of layers are added to the array. The existence of these stop bands are of great importance in design and construction of piezo-composite transducers (Otero et al., 2004).
**Figure 7.5:** Variation of power reflection, transmission and absorption coefficients with the frequency at $\theta = 5^\circ$; (i) ← |1⟩|3⟩ → (ii) ← |1⟩|3⟩|1⟩|3⟩ → (iii) ← |1⟩|3⟩|1⟩|3⟩|1⟩ → (iv) ← |1⟩|3⟩|1⟩|3⟩|1⟩|3⟩ →
In order to study the effects of the angle of incidence on the variation of reflection and transmission coefficients with frequency for a layered structure consisting of two periodic stacks of layers 1 and 3, results are shown in the Figure 7.6. It is observed that the minima in the power reflection coefficients shift towards higher frequency as the angle of incidence increases.

**Figure 7.6:** Variation of power reflection, transmission and absorption coefficients with the frequency; ← |3|1|3| → (i) $\theta = 2^\circ$ (ii) $\theta = 5^\circ$ (iii) $\theta = 10^\circ$ (iv) $\theta = 20^\circ$

The variation of reflection and transmission losses with the angle of incidence and frequency for a laminated porous piezo-thermoelastic structure is shown in the Figures 7.7 and 7.8, respectively. The shallow tightly spaced maxima correspond roughly to the thickness resonance of the entire array and they increases in number as stacks are added to the array. These results have practical importance in the sense to judge the angles and frequencies at which the reflection loss is minimum.
Figure 7.7: Variation of reflection and transmission losses with the angle of incidence at Frequency = 1MHz; (i) ← |1|3| → (ii) ← |1|3|1|3| → (iii) ← |1|3|1|3|1|3| → (iv) ← |1|3|1|3|1|3|1|3| →
Figure 7.8: Variation of reflection and transmission losses with frequency at $\theta = 5^\circ$; (i)$\leftarrow|1|3|\rightarrow$ (ii)$\leftarrow|1|3|1|3| \rightarrow$ (iii)$\leftarrow|1|3|3|1|3| \rightarrow$ (iv)$\leftarrow|1|3|1|3|1|3|1|3| \rightarrow$
The effects of porosity on the variation of power reflection, transmission and absorption coefficient with frequency at $\theta = 5^\circ$ are shown in the Figure 7.9. The plate consisting of 4 alternate layers of materials 1 and 3 is considered for computation and plotting curves. It is observed that at a particular frequency, less amount of energy get reflected back from fluid-solid interface with increase in the porosity of porous piezo-thermoelastic materials. This implies that the acoustic impedance of porous piezo-thermoelastic materials becomes closer to ambient media as compared to dense piezoelectric materials.

![Figure 7.9:](image)

**Figure 7.9:** Variation of reflection, transmission and absorption coefficients with the frequency at $\theta = 5^\circ$; (i) $f = 0.2$ (ii) $f = 0.4$ (iii) $f = 0.6$ (iv) $f = 0.8$. 

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To observe the thermal effects on the reflection-transmission phenomena, the results are further reduced for the porous piezoelectric material (PPM) laminated structures (Figures (7.10)-(7.13)). Comparison of the figures (7.10-7.13) with the corresponding Figures (7.2-7.5) for the porous piezo-thermoelastic case show that the behaviour of variation of power reflection and transmission coefficients with frequency and angle of incidence is not much affected due to addition of thermal effects. In case of porous piezoelectric laminated structure, at a fixed frequency, the power reflection and transmission coefficients show maxima and minima for different angles in the angle range \(0^\circ - 40^\circ\). The critical angles corresponding to excitation of surface modes also changes due to inclusion of thermal effects. The amount of incident energy absorbed in laminated plate is more in case of porous piezo-thermoelastic case in comparison to porous piezoelectric structure. These results, obtained as particular cases of the present study, agree with those of (Vashishth and Gupta, 2012) and thus validate the generality and accuracy of the results.
Figure 7.10: Variation of reflection, transmission and absorption coefficients with the angle of incidence at frequency = 1 MHz; (i) 1 PPM layer (ii) 2 PPM layers (iii) 3 PPM layers (iv) 4 PPM layers
Figure 7.11: Variation of reflection, transmission and absorption coefficients with the angle of incidence at frequency = 1MHz in reduced case of porous piezoelectric periodic layers; (i) $\left|1\right|3\left|1\right|3\left|\right|$ (ii) $\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|\right|$ (iii) $\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|\right|$ (iv) $\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|1\right|3\left|\right|$
Figure 7.12: Variation of reflection, transmission and absorption coefficients with the frequency at $\theta = 5^\circ$; (i) 1 PPM layer (ii) 2 PPM layers (iii) 3 PPM layers (iv) 4 PPM layers
Figure 7.13: Variation of reflection, transmission and absorption coefficients with the frequency at $\theta = 5^\circ$ in reduced case of porous piezoelectric periodic layers; (i) $\leftarrow [1|3]$ $\rightarrow$ (ii) $\leftarrow [1|3|1|3]$ $\rightarrow$ (iii) $\leftarrow [1|3|1|3|1|3]$ $\rightarrow$ (iv) $\leftarrow [1|3|1|3|1|3|1|3]$ $\rightarrow$