Chapter 1

Introduction

1.1 PREFACE

Ecology is the study of the patterns of abundance and distribution of living organisms in relation to their environment. An ecosystem is the unit of nature in which all living and non-living organisms of a giving region interact with their physical environment and among themselves leading to exchange of energy and materials and bringing about various kinds of diversity. An ecosystem is very complex due to large number of its biotic and abiotic components and the nature of their variety of interactions among the habitat. Interest in ecology has been much concerned with stability of the ecosystems.

In the analysis of ecosystem stability, it is necessary to know the pattern of changes and the driving forces in terms of measurable quantities. When it is possible to translate the complexities of interactions between the living and non-living components of ecosystem in the language of mathematics, then we enter into the system analysis. Therefore, mathematical modelling of an ecological system is a mathematical realization of an underlying ecological system. The research on mathematical ecology exhibited lot of examples on population growth models, particularly the prey-predator ecosystems, where the relationship between the species is identified as either coupled oscillations (for example, the high density of prey leads to rapid increase in predator density, but the over predation cause decrease in prey, then due to the prey shortage, there will be decrease in predator density, which cause less predation and hence increase in prey and so forth.) or independent oscillations. It is now, really an important task for researchers to understand the patterns of species abundance and to describe the differences between the examples. The classical examples isolate the species existence for the convenience of theoretical modeling and analysis, but in reality, we have all multispecies ecosystems subjected to the environmental factors. The major issue is to discover what decides the population growth and to reasonably understand the complexity in real ecosystem by considering the various parameters and its characteristics.

Despite the urgency of the problems in a real ecosystem, the ecology cannot predict the response of the salt marshes to nutrient enrichment and biotic impoverishment. The theory which describes the relative importance of nutrients (bottom-up) or species composition (top-down) on ecosystem function has never been tested in detritus-based aquatic systems earlier. The salt marshes demonstrate detritus-based aquatic ecosystems in which the food-web base is on the consumption of non-living organic matter (detritus). In addition, detritus is important in creating the physical structure of the system by the formation of peat. The combined effects of nutrient enrichment and biotic impoverishment will have its impact on ecosystem function and sustainability, because of nonlinear feedbacks among components of the ecosystem. The term mangrove refers to an ecological system which dominates the world's tropical and subtropical coasts. It is a fact that energy and nutrients are stored in leaves of mangroves. Mangroves are of interest to biologists and scientists, because of their diversity and productivity; and furthermore, they are great source of goodness prove to be beneficial to humans. In addition, they are exploited for timber that is widely used for construction and as firewood. Their use ranges from the casual collection of fallen wood to the large-scale industrial charcoal production. The latter form usually coincides with the use of intensive mangrove plantations, such as the ones in Sundarban, India. The greenery, at times is grazed or harvested for feeding domestic animals.

The perspective of energy flow in ecological systems, comprise of four general types of heterotrophs. Herbivores are called primary consumers because they eat only plants. Carnivores are called secondary consumers because the eat herbivores. There is also a group of tertiary consumers called omnivores because they eat both herbivores and carnivores. The other type of consumer (detritivores) that is critical to energy flow in the ecosystems comprise of organisms that feed on dead plants and animals. Some of the most significant changes which are visible involve in widespread nutrient enrichment, evolution of species, global warming and sea level rise. Anthropogenic nitrogen fixation has recently exceeded the natural biotic fixation and will continue to increase for a predictable future (Vitousek et al, 1997). The dynamics of species through overfishing, extinction and exotic species introduction have large, but partially understood effects on the ecosystems of forests, lakes, streams, and estuaries (Chapin et al, 1998; Enserink, 1999; Williams, 1999; Emeis et

al, 2001; Jackson et al, 2001). Understanding and predicting how multiple stresses affect the sustainability of ecosystems is one of the most crucial challenges in environmental biology and the first step towards management. Mangroves are critical, not only for sustaining biodiversity, but also for their direct and indirect benefits to human activities. Mangroves have long functioned as a storehouse of materials providing food, medicines, shelter and tools. Fish, crabs, shellfish, prawns and edible snakes and worms are found there.

Modelling is an essential tool used throughout to test our predictions and to scale-up the consequences of changes in nutrients and trophic structure to encompass larger spatial and longer temporal-scales. The modeling objectives are indispensable to develop a fine scale model that incorporates dynamic ecological processes for creek watersheds and use this as a basis to predict landscape responses to a varying nutrient and trophic structure regimes. An important challenge is to determine the amount of detail, including the resolution of physical space and biotic interactions, necessary to model landscape responses (Korfmacher, 1998; Deegan et al, 1997, 2001 & 2002; Schneider, 2001). Although it seems intuitive that more a model incorporates, the processes and mechanisms known to occur in the real world situation; the close it should come to predict the ecosystems impact in particular perturbation. Often, more complex, data-intensive models are less stable and more difficult to test (Oreskes et al, 1994). Energy and nutrients are assimilated and stored in the leaves of mangrove trees. Being a detritus-based ecosystem, leaf litter from these trees provides a base for adjacent aquatic and terrestrial food webs. Because most of the energy and nutrients are stored in biomass rather than being free in the water or substrate. The species diversity of these swamps is directly dependent on the primary productivity by mangrove plants.

The productivity of mangrove ecosystems also supports fisheries through the export of carbon. Few fish species are permanent residents in mangroves, but numerous marine species uses the mangroves as nursery grounds. Mangrove swamps also provide feeding grounds for juvenile and sub adult reef fishes. As a result, mangrove-assimilated energy and nutrients are exported to surrounding coral reefs. It is known that an estimated 75 percent of the fish which is caught commercially spend some time in the mangroves or it is dependent on a food chain which can be traced

back to these coastal forests. Fisheries are not the only benefactors of mangrove services for the shrimp trawling industry is also deeply dependent on the nursery function of the mangroves.

The sustainability of coastal ecosystems in the face of widespread environmental change is current issue of pressing concern throughout the world. Coastal ecosystems are a dynamic interface between terrestrial and oceanic systems and happen to be the most productive ecosystems in real world. Coastal systems probably serve more human beings compared to other ecosystem (Emeis et al, 2001). Coastal areas are also the sites of the nation's and the world's most intense commercial activity and population growth worldwide which may approximately amount to 75 percent of human population who live in coastal regions. The coastal environment is continually changing because of the natural variability of environmental drivers because of their sensitivity to small changes in the sea level which is a part of the question in the nonlinear phenomena. The initial research used only linear differential equations. But the reality is that the interactions among the living members of the community and their environment are nonlinear in nature. Due to this complexity, the ecological models must reflect these parameters while modelling to generate the effective patterns of abundance and also to predict the future behaviour of the real ecosystems. Non-linear systems can exhibit models of behaviour that are qualitatively different and much more complicated than the models we know from linear systems theory.

In the recent times, changes are observed in the environment have been accelerated towards nutrient enrichment, species composition and sea level due to man-made alterations at a place that is outside the bounds of natural variability. Nutrient enrichment is a challenging current premier issue for coastal researchers and managers for the last three decades (Cloern, 2001). Although most researchers treat the nutrient enrichment as a stand-alone stress; recent studies suggest that the responses at estuaries to enrichment may depend on species composition (Micheli, 1999; Conley et al, 2000; Cloern, 2001; Sterner and Elser, 2001). At the same time, biotic impoverishment is reducing the abundance of species or the loss of species due to man-made distraction (Ruiz et al, 1997; Cohen and Carlton, 1998). Most of the changes in species composition affect estuaries directly through food web interactions

such as predation or competition or indirectly by altering the rates or pathways of nutrient cycle (Ulanowicz and Tuttle, 1992; Dame and Libes, 1993).

Ecological modelling provides a deep understanding on the patterns and processes underlying the dynamics of population. Mathematical modelling of population dynamics attracted more attention of researchers due to the contribution of Lotka (1925) and Volterra (1931). The Mathematicians and Ecologists recognized the need to work in parallel to gain non-trivial biological insights. But the construction of mathematical models which are biologically eloquent is very essential (Kuang, 2002). The realistic and reasonable mathematical models are structured through the aspects like carrying capacity, intra-specific or inter-specific competitions among species, harvesting of species and functional responses of predators. Ecology also deals with the survival of dissimilar species in the same surroundings and habitation. It is usual that two or more species living in a common residence interact in diverse ways. The interface competition is offensive when species fight for the same limited resource and interference when species exhaust one another's resources by interferences such as insistent displays or fighting. Two models have been developed to describe exploitative and interference competition (MacArthur, 1972; Schoener1976).

A two species commensalism is an ecological relationship between two species where first species known as commensal that receives a gain from the second species referred as host which does not get affected by it. For example Anemonetish, Barnacles and Cattle Egret etc.   The authors (Phanikumar et al, 2011) have been investigated the analytical results on commensalism and the results are justified numerically. Mutualism is a type of interaction between two species where one species provides some benefit to each other (Bronstein, 1994). The benefit exchanged might be good or a service. For example, in insect-mediated pollination, plants exchange carbon energy in the form of nectar or pollen which it has in relative abundance due to its mobility. In this case a resource is traded for a service. This idea of trading goods with differing availability to different species is very similar to the economic concept of comparative advantage (Noe and Hammerstein, 1995; Schwartz and Hoeksema, 1998; Holland, 2002). Neutralism is the most regular type of inter-specific relation. Neither population affects the other. Any interactions that do take place are indirect or incidental. For example, the tarantulas live in a barren region and

the cacti living in a barren region. In ecological sciences, predation explains a biological interaction where a predator feeds on its prey.

The predator functional response is depicted as instantaneous per capita feeding rate of predator and it is categorized (Abrams and Ginzburg, 2000) into three types as: (i) prey density dependent, (ii) ratio dependent and (iii) prey-predator density dependent. In the first type, the consumption rate of predator depends only on prey density (N) and Holling (1959) classified the prey-dependent functional response into three types: (i) The Holling type I functional response is $f(N) = aN$, where, a is the predation rate, (ii) The Holling type II functional response is $f(N) = bN/(1+cN)$, where, b is the capturing rate and c is the handing time on the consumption rate of predator, (iii) The Holling type III functional response is $f(N) = bN^x/(1+cN^x)$, where, $x > 1$ represents the encounter rate between predator and prey before reaching predator's maximum efficiency. In ratio dependence case, the functional response is obtained by putting N/P (prey-predator ratio) for N (prey density) in the Holling type II response (Xiao and Ruan, 2001). That is $f(N,P) = dN/(N+mP)$, where, d is the capturing rate and m is half saturation constant for predator. Due to the inter-specific competition, the time delay deserved by the predator is taken into account in the general from of ratio dependent functional response as $f(N,P) = aN/(1+bN+e(P-1))$, where '$e$' represents the magnitude of the interference among predators (Beddington, 1975; De Angelis et al, 1975). In the prey-predator dependency type, the predation rate depends on the prey and predator densities (Xiao, 2001), i.e. $f(N,P) = aN/(1+bN+e(P-1)+beN(P-1))$. The Holling type-I response is considered in the current study.

The rapid growth of knowledge regarding the mathematical modelling in Biology, Ecology and Epidemics led to new thinking in this field. In all the cases, the major focus is on the positive equilibrium and on its dynamics in the local and global sense. Epidemiology is the study of patterns, causes and effects on health and disease conditions in defined populations. The study aims to identify the risk factors of disease and the main objective is for preventive healthcare. Mathematical epidemiology is the study to understand the contrivance leading to the spread of

disease in populations using mathematical techniques. The basic assumption of mathematical modelling of infectious diseases is to divide the total populations into different classes (like susceptible, infective, recovered etc.) and all transmissions between the classes are measured with suitable parameters. The mathematical disease models are often represented by ordinary, partial, delay and stochastic differential equations. The mathematical study mainly to identify the stable behaviour of equilibrium points (both disease free and endemic). Many authors like Stutzer (1980), Deveney (1989), Inaba (1990), Li and Muldowney (1995), Hilborn (2000), Elaydi (2005), Hilker et al (2006), D'Onofrio et al (2007) and Schaffer and Bronnikova (2007) were studied and analysed the global stability dynamics and chaotic behaviour of various disease models. Dhar and Sharma (2009) investigated the role of incubation class and showed that Hopf-bifurcation occurs with a certain threshold value of disease contact rate. Das and Mukherjee (2004) considered and analysed the S-I Epidemic model with delay in order to investigate the roll of delay on the model studied by Anderson and May (1981). Marion et al (2000) considered the demographic fluctuations arising from the stochastic nature of the epidemic in a reformulation of deterministic model of nematode infection of ruminants proposed by Roberts and Grenfell (1991). In addition, many researchers Pascual et al (2000), Ruan and Wei (2003) Blanchong et al (2006), Webb et al (2006), Codeco et al (2008) and Joh et al (2009) were considered and studied the various delay and stochastic disease models. The current research explores the dynamics of an S-I epidemic model with delay and stochastic effects and also attempted on a discrete disease model which is more effective than continuous models when the population is small. Since the disease data type is discrete in nature, difference equations are very meaningful to represent the disease models.

The objective of the research work is to develop a broad range of illustrative examples of detailed and realistic models. The realistic and reasonable mathematical models are structured through the aspects like carrying capacity, intra-specific or inter-specific competitions among species, harvesting of species and functional responses of predators. The importance is to identify the role of sensitive parameters involving in the system. The special effect of significant attributes delay, noise and harvesting on the stability is mainly focussed in the research. The sophisticated mathematical tools like Routh-Hurvitz criteria, Lyapunov function, Butler's lemma,

Pontryagin's principal and Fourier transforms are used for analysing a nonlinear autonomous models which represent some of the momentous ecological and epidemiological systems. The computer simulation is performed to validate the results. The purpose of the present work will have manifold dimension to look the exchange of scientific ideas between biologist and mathematician in the realistic system. The research also has immense real world practical applications. Our aim is to offer the mathematical analysis of the behaviour of the considered models and so that thee models can be readily utilized by the biologists, mathematicians and also engineers in the realistic scenario.

1.2   BRIEF HISTORY FOCUSING THE SPECIAL ATTRIBUTES

The time delay parameter makes the mathematical models of ecosystems more generous and realistic (Gopalsamy, 1992; Kuang, 1993) because one species cannot contribute to the biomass of other species instantaneously. So it is very meaningful to consider a time lag parameter in the population species growth models. This time lag is demarcated as discrete time delay (Murray, 2002) and is merged in the beneficial term of predator growth equation. The prey-predator model with selective harvesting of mature species by including time delay I the harvesting term has been discussed by (Kar, 2003a). The delay centred models has been deliberated by various researchers like Hastings (1984), Rozen (1987), Martin and Ruan (2001) and Yong-Hong Fan and Wan-Tong Li (2007). The investigation on time delay has shown the maintenance of stability by regulating the influence of time delay and has produced results on the oscillatory behaviour of the model due to increased delay. In accumulation to that the study carried out by Gazi and Bandyopadhyay (2008), Kar and Ashim   Batabyal (2009) and Rui Xu (2011) outspreads the understanding on the role of time in affecting the dynamics of ecosystems. Many other authors (Cushing and Saleem, 1982; Aiello and Freedman, 1990; Beretta and Kuang, 1996; Arino et al, 2001; Ruan 2001) have conversed delay effects on prey-predator models and explored that the delay differential equations approximates better dynamics than ordinary differential equations.

The stochastic analysis gives better insight of complexity in real ecosystem which is always under the random fluctuations of the environment. In a stochastic

model, the parameters of the system fluctuate about their mean values. Therefore the equilibrium point is also oscillating about the mean state. The method of Nisbet and Gurney (1982) to measure the mean-square fluctuations of population helps to describe stability in the mean square sense. Samanta (1994) analysed the stochastic stability using wideband stationary stochastic process. Tapaswi and Mukhopadhyay (1999) considered plankton allelopathy system with fluctuations in population through random white noise perturbations. Malay Bandyopadhyay and Chakrabarti (2003) compared the behaviour of deterministic and stochastic systems. Maiti et al (2007) analysed the deterministic stability and also the stochastic stability using uncorrelated Gaussian white noise. Gazi et al (2010) studied the cholera model by including white noise stochastic perturbations. Nurul Huda Gazi et al (2010a) analyzed the detritus based ecosystem by incorporating stochastic perturbations and shown that the model is stable in mean square sense around coexistence state under certain conditions on system parameters. Nurul Huda Gazi and Kalyan Das (2010b) examined an algal bloom mathematical model with random noise effect and identified the parametric spaces of stability and instability. Mukandavire et al (2011) studied a HIV/AIDS model with delay and stochastic effects. Kalyan Das et al (2012) analysed competitive prey-predator model and explored the chaotic dynamics induced by delay environmental noise. In addition, many authors (Holling, 1973; Carletti, 2006, 2007; Sun et al, 2009, Wang et al, 2011) have studied the ecological dynamics and stability behaviour influenced by noise extensively. Recently, Srinivas et al (2014) have analysed the stochastic dynamics of commensal system and Kalyan Das et al (2014) have studied the chaotic behaviour with noise of a prey predator competitor model. Most of the researchers have used Gaussian white noise perturbations since it provides better approximation of environmental changes. The present work explores the dynamics of some epidemiological and ecological models with the inclusion of Gaussian white noise.

Harvesting of species is a significant area of study in the fishery modelling and it plays a major role on the dynamic evolution of population (Clark, 1985). In the dynamic model of an ecological system, the level of harvesting effort expands or contracts according as the net economic revenue to the owner of the system is positive or negative. Any model which includes this dynamic interaction between the net economic revenue and the harvesting effort is the dynamic reaction model where seasonal harvesting, taxation, lease of property rights etc. are mainly considered as

governing instruments for the regulation of exploitation of biological resources. Out of all such regulating options, taxation plays a major role and it is also to be superior because of its economic flexibility. Many authors Chaudhuri and Johnson (1990), Krishna et al (1998), Dubey et al (2002) and Kar et al (2003b), studied about harvesting problems with taxation as a control instrument.

The harvesting level which optimizes the profit and also allows the species to grow at the maximum growth rate is of major interest. Zhang et al (2007) derived the optimal harvesting policy for the resource dependent fishery model. The optimal harvesting policy, biological and bionomic equilibrium of a multi-species fishery model has been analysed exhaustively by Kar and Chaudhari (2004). The stage structure of species makes the population growth models more meaningful. Kar and Pahari (2005) studied the effect of harvesting in a stage structured prey-predator system. The harvesting subjected to species density and economic constraints are more advantageous and justifiable than the assumption of constant harvesting. Kar and Saroj Kumar Chattopadhyay (2010) emphasised a selective harvesting as a variable.

Generally, random fishing of all fishes is not good for the persistence of the fishery, so the exploitation of mature species is perfect to the economic and biological views of renewable resources management (Matsuda and Nishimori, 2003; Song and Chen, 2001 & 2002). Though harvesting models have been deliberated by many authors (Kar and Chaudhri, 2002; Ragozin and Brown 1985; Mesterton-Gibbons, 1987; Leung, 1995) the stage structure of the species has received very little concentration. Some of the stage-structured models are premeditated by (Arino et al, 2001; Gambell, 1985; Cao et al, 1992; Bosch and Gabriel, 1997) and the orientations therein. The mathematical analysis on harvesting of species have exhibited that harvesting under optimal schemes is a significant control parameter on the population growth. Hence the harvesting attribute attracts more attention in the population dynamics and the current research analyses some of the acceptable ecological models in the presence of harvesting.

1.3   PRELIMINARIES

A Dynamical System: A general n-dim autonomous dynamical system is of the form

$$\dot{x} = f(x), \; x(t_0) = x_0, \; x \in \mathbb{R}^n \tag{1.1}$$

where f has continuous first partial derivatives and f satisfies the standard conditions for existence and uniqueness of solutions.

An Equilibrium Point: A point $x^* \in \mathbb{R}^n$ is an equilibrium point of the dynamical system (1.1) if $f(x^*) = 0$.

Local Stability: An equilibrium point x^* of the dynamical system (1.1) is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left\| x(t) - x^* \right\| < \varepsilon$ whenever $\left\| x(t_0) - x^* \right\| < \delta$ for all $t \geq t_0$.

Local Asymptotic stability: An equilibrium point x^* of the dynamical system (1.1) is said to be locally asymptotically stable if it is locally stable and $\underset{t \to \infty}{Lim} \left\| x(t) - x^* \right\| \to 0$ whenever $\left\| x(t_0) - x^* \right\| < \delta$.

Instability: An equilibrium point x^* of the dynamical system (1.1) is said to be unstable if it is not stable.

Linearization by perturbation technique: The n-dimensional dynamical system (1.1) can explicitly be written as

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots x_n) \\
&\ldots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots x_n)
\end{aligned}
\tag{1.2}
$$

Let $x_i = X_i - x_i^*, i = 1, 2, \ldots n$, where $(x_1^*, x_2^*, \ldots, x_n^*) = x^*$ is the equilibrium point of the system (1.2). Then the linear system of (1.2) is given by $\dot{X} = g(X)$ where g is a linear

function in X .Or equivalently the linear terms of the system (1.2) are obtained from the Jacobian matrix

$$J(x^*) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \cdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{1.3}$$

Routh-Hurwitz criteria: The characteristic equation of order n of (1.3) will be in the form $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n = 0$. The n Hurwitz matrices are given by

$$H_1 = (a_1), H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, ..., H_n = \begin{pmatrix} a_1 & 1 & 0 & ... & 0 \\ a_3 & a_2 & a_1 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & a_n \end{pmatrix}$$

If the determinants of Hurwitz matrices are greater than zero, we get the R-H criteria:

If $n = 2$, then $a_1 > 0, a_2 > 0$. If $n = 3$, then $a_1 > 0, a_3 > 0, a_1 a_2 > a_3$. If $n = 4$, then $a_1 > 0, a_1 a_2 > a_3, a_4 > 0, a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$.

Lyapunov function: Let $x = x^*$ be the equilibrium point of the dynamical system (1.1). Let $D \subset \mathbb{R}^n$ be the region containing x^* and $V : D \to \mathbb{R}^+$ such that $V(x^*) = 0$ and $V(x) > 0$ fro all $x \in D \setminus \{x^*\}$. Then

(i) If $\dot{V}(x) \leq 0$ in D, then x^* is stable.

(ii) If $\dot{V}(x) < 0$ in $D \setminus \{x^*\}$, then x^* is asymptotically stable.

The function V is called as Lyapunov function.

Butler's lemma: Let the characteristic equation of the delayed system $\dot{x} = f\big(x(t), x(t-\tau)\big)$ be $\Delta(\lambda, \tau) = 0$. If $\Delta(\lambda, 0) = 0$ has solutions with negative real parts then the delayed system is locally asymptotically stable for $\tau < \tau_0$ where $\tau_0 > 0$ is the smallest value for which there is a solution to $\Delta(\lambda, \tau) = 0$ with real part zero.

Hopf-Bifurcation: It is a local bifurcation in which an equilibrium point of a system changes its stability (either stable to unstable behaviour or unstable to stable behaviour). In the case of stable to unstable behaviour, the system is expected to have periodic oscillations.

Condition for Hopf-Bifurcation: If $\dfrac{d}{d\tau}\big[\mathrm{Re}(\lambda)\big]\Big|_{(i\omega_0)} \neq 0$, then Hopf-bifurcation occurs at $\tau = \tau_0$. Thus the delayed system is locally asymptotically stable for $\tau < \tau_0$ and at $\tau = \tau_0$ the equilibrium bifurcate into periodic solutions.

Descarte's rule: If the expressions of a single variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is equal to the number of sign differences between consecutive nonzero coefficients.

Optimal harvesting-Pontryagin's principle: To maximize the present value function

$$P = \int_{a}^{b} \phi_1(t, x, h))\,dt \qquad (1.4)$$

where the yield of the resource, where q is the catch ability coefficient, E is the effort used for harvesting, x is the abundance of the resource. The profit function $\phi_1 = pqEx - cE$, where p is the market prize of the resource per unit and the total income is $pqEx$. c is the cost per unit of harvesting effort and the total cost of harvesting is cE.

The equation (1.4) is subjected to the system equations $\dfrac{dx}{dt} = \phi_2(t, x, h)$ \qquad (1.5)

The Hamiltonian function is given by

$$H(t, x, \psi, h) = \phi_1(t, x, h(t)) + \psi(t)\phi_2(t, x, h(t)) \qquad (1.6)$$

The auxiliary equations are given by $\dfrac{\partial H}{\partial x} = -\dfrac{d\psi}{dt}$ \qquad (1.7)

$$H(t, x, \psi, \hat{h}) = \max_{0 \le h \le M} H(t, x, \psi, h)$$ (1.8)

From (1.5), (1.7) and (1.8), the optimal levels $x(t)$, $h(t)$ and $\psi(t)$ can be obtained.

Gaussian white noise: It is basically a random process where all the random variables are following Gaussian distribution with zero mean and are uncorrelated. That is, a n-dimensional Gaussian white noise process is denoted by $\xi(t) = [\xi_1(t), \xi_2(t), ..., \xi_n(t)]$, where each $\xi_i(t)$ follows Gaussian distribution with following properties:

$$E[\xi_i(t)] = 0; i = 1, 2, ..., n \qquad \text{and}$$

$$E[\xi_i(t)\xi_j(t')] = \delta_{ij}\delta(t - t'); i, j = 1, 2, ..., n$$

where δ_{ij} is the Kronecker symbol; δ is the δ-Dirac function.

Random function basics:

If the function $Y(t)$ has a zero mean value then the fluctuation intensity (variance) of its components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$, where $S_Y(\omega)$ is spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T} \to \infty} \frac{|\tilde{Y}(\omega)|^2}{\tilde{T}}$$ (1.9)

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{i\omega\tau} d\omega$$ (1.10)

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega$$ (1.11)

and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \tag{1.12}$$

For a Gaussian white noise process, it is

$$S_{\xi_i \xi_j}(\omega) = \lim_{\hat{T} \to +\infty} \frac{E\left[\tilde{\xi}_i(\omega)\tilde{\xi}_j(\omega)\right]}{\hat{T}}$$

$$= \lim_{\hat{T} \to +\infty} \frac{1}{\hat{T}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} E\left[\tilde{\xi}_i(t)\tilde{\xi}_j(t')\right] e^{-i\omega(t-t')} dt\, dt' = \delta_{ij} \tag{1.13}$$

The n-dim dynamical system with random perturbations will be in the form as follows:

$$\dot{x} = f(x,t) + \alpha\xi(t)$$

Using Perturbation technique, we identify the linear system as $\dot{u} = g(u,t) + \alpha\xi(t)$

Applying Fourier Transform, we get the solution components of linear system as

$$\tilde{u}_i(\omega) = \sum_{j=1}^{n} K_{ij}(\omega)\tilde{\xi}_j(\omega), i = 1, 2, ...n$$

The spectral density of u_i is given by $S_{u_i}(\omega) = \sum_{j=1}^{n} \alpha_j \left|K_{ij}(\omega)\right|^2$; $i = 1, 2, ...n$

The intensities of fluctuations in the variable u_i ; $i = 1, 2, ...n$ are given by

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \eta_j \left|K_{ij}(\omega)\right|^2 d\omega; \ i = 1, 2, ...n$$