Chapter-5

A Prey-Predator Fishery Model with Taxation, Optimal Yield and Environmental Fluctuations

5.1 MATHEMATICAL MODEL

A prey-predator fishery model in a two zones environment is considered here. Zone 1(unreserved) consists of prey and predator. Zone 2 (reserved) consists of only prey. The portrait of the model is given in figure (5.1). In assumption that the following: (i) The unreserved zone is an open access fishery zone where the harvesting of prey is allowed, (ii) The prey in the unreserved zone is also consumed by the predator, (iii) The reserved zone protects the prey and harvesting is prohibited, (iv) The prey (in zone 1 and zone 2) grows logistically in the absence of predator, (v) The predator also follows logistic growth law (vi) The prey can migrate between the two zones. The above postulates are represented by the system of four first order nonlinear differential equations as follows:

$$x'(t) = x\left[\left(r_1 - \sigma_1 - qE\right) - r_1 x K_1^{-1} - mz\right] + \sigma_2 y \tag{5.1}$$

$$y'(t) = y\left[\left(r_2 - \sigma_2\right) - r_2 y K_2^{-1}\right] + \sigma_1 x \tag{5.2}$$

$$z'(t) = z\left[\alpha - \alpha z \gamma^{-1} x^{-1}\right] \tag{5.3}$$

$$E'(t) = E\mu q x\left(p - \sigma\right) - Ec\mu \tag{5.4}$$

where, $x = x(t)$ is the prey population density in the free zone(unreserved zone) at time t, $y = y(t)$ is the prey population density in the reserved zone at time t, $z = z(t)$ is the predator population at time t, $E = E(t)$ is the harvesting effort at time t, σ_1 is the migration rate of prey from free zone to reserved zone, σ_2 is the migration rate of prey from reserved zone to free zone, r_1 and r_2 are the intrinsic growth rates of prey in the free and reserved zone respectively, K_1 and K_2 are the carrying capacities of prey in the free and reserved zone respectively, α is the intrinsic growth rate of predator, γ is the equilibrium ratio of prey to predator biomass, m is the decrease rate of prey in the free zone due to predation, q is the catch ability coefficient, μ is the stiffness parameter used to measure the reaction of harvest effort, p is the fixed price per unit of prey, c is the fixed cost of harvesting per unit effort, σ is the tax per unit of prey. It is assumed that all the parameters of the model are positive and $r_1-\sigma_1-qE>0; r_2-\sigma_2>0.$

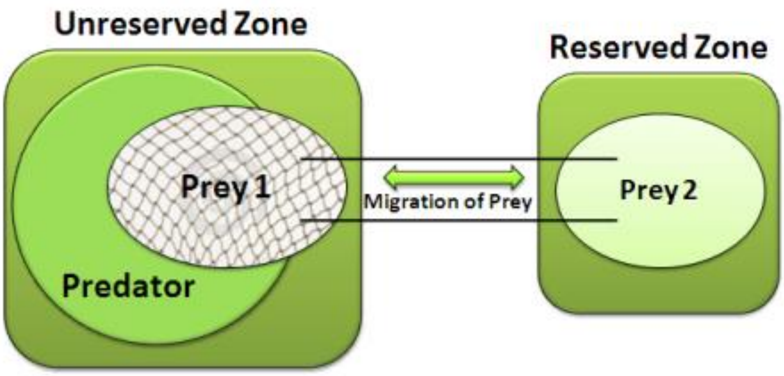

Figure 5.1

Figure (5.1) shows the prey-predator fishery model with prey migration between the two zones (reserved and unreserved)

5.2 STEADY STATES

The model (5.1)-(5.4) has the following four equilibrium points:

Case (i) $S_0(0,0,0,0)$: This trivial equilibrium point is always exists.

Case (ii) $S_1(\bar{x},\bar{y},0,0)$: In the absence of predator and harvesting.

$\bar{x}$ and $\bar{y}$ are solutions of $x'(t)=y'(t)=0$ (5.5)

Eliminating y from (5.5) we get

$$Ax^3+Bx^2+Cx+D=0 \tag{5.6}$$

where, $A=\dfrac{r_2 r_1^2}{K_2\sigma_2^2K_1^2}$; $B=-\dfrac{2r_2r_1(r_1-\sigma_1)}{K_2\sigma_2^2K_1}$;

$$C=\frac{r_2(r_1-\sigma_1)^2}{K_2\sigma_2^2}-\frac{r_1(r_2-\sigma_2)}{K_1\sigma_2};\; D=\frac{(r_2-\sigma_2)(r_1-\sigma_1)}{\sigma_2}-\sigma_1$$

If $r_2K_1(r_1-\sigma_1)^2<r_1K_2\sigma_2(r_2-\sigma_2)$ and $(r_2-\sigma_2)(r_1-\sigma_1)<\sigma_1\sigma_2$ then the equation (5.6) has a unique positive solution $x=\bar{x}$. Substituting $x=\bar{x}$ in $y'(t)=0$,

$$\bar{y}=\frac{\bar{x}}{\sigma_2}\left[\frac{r_1\bar{x}}{K_1}-r_1+\sigma_1\right] \text{ and } \bar{y} \text{ is positive if } \bar{x}>\frac{K_1(r_1-\sigma_1)}{r_1}$$

Case (iii) $S_2\left(x^*, y^*, z^*, 0\right)$: In the absence of harvesting.

x^*, y^* and z^* are solutions of $x'(t) = y'(t) = z'(t) = 0$ (5.7)

Eliminating y and z from (5.7),

$$Px^3 + Qx^2 + Rx + S = 0 \quad (5.8)$$

where, $P = \dfrac{r_2}{\sigma_2^2 K_2}\left(\dfrac{r_1}{K_1} + m\gamma\right)^2$; $Q = \dfrac{-2r_2}{\sigma_2^2 K_2}\left[\left(\dfrac{r_1}{K_1} + m\gamma\right)\left(r_1 - \sigma_1\right)\right]$

$$R = \frac{r_2\left(r_1 - \sigma_1\right)^2}{K_2\sigma_2^2} - \frac{\left(r_2 - \sigma_2\right)}{\sigma_2}\left(\frac{r_1}{K_1} + m\gamma\right);\ S = \frac{\left(r_1 - \sigma_1\right)\left(r_2 - \sigma_2\right)}{\sigma_2} - \sigma_1$$

If $r_2 K_1\left(r_1 - \sigma_1\right)^2 < K_2\sigma_2\left(r_2 - \sigma_2\right)\left(r_1 + m\gamma K_1\right)$ and $\left(r_2 - \sigma_2\right)\left(r_1 - \sigma_1\right) < \sigma_1\sigma_2$ then the equation (5.8) has a unique positive solution $x = x^*$.

Substituting $x = x^*$ in $z'(t) = 0$, $z^* = \gamma x^*$ is obtained.

Substituting $x = x^*$ and $z^* = \gamma x^*$ in $y'(t) = 0$,

$$y^* = \frac{x^*}{\sigma_2}\left(m\gamma x^* + \sigma_1 - r_1 + \frac{r_1 x^*}{K_1}\right) \text{ and}$$

y^* is positive if $x^* > \dfrac{K_1\left(r_1 - \sigma_1\right)}{r_1 + m\gamma K_1}$

Case (iv) $S_3\left(x^\phi, y^\phi, z^\phi, E^\phi\right)$: This is the interior equilibrium point.

x^ϕ, y^ϕ, z^ϕ and E^ϕ are solutions of $x'(t) = y'(t) = z'(t) = E'(t) = 0$

From $E'(t) = 0$, $x^\phi = \dfrac{c}{\left(p - \sigma\right)q}$ is obtained

From $z'(t) = 0$, $z^\phi = \gamma x^\phi$

From $y'(t) = 0$, $\dfrac{r_2}{K_2}y^2 - \left(r_2 - \sigma_2\right)y + -\sigma_1 x = 0$ (5.9)

Equation (5.9) has a unique positive root $y^\phi = \dfrac{\left(r_2 - \sigma_2\right) + \sqrt{\left(r_2 - \sigma_2\right)^2 + 4r_2K_2^{-1}\sigma_1 x^\phi}}{2r_2K_2^{-1}}$

From $x'(t) = 0$, $E^\phi = \dfrac{1}{q}\left[r_1\left(1 - \dfrac{x^\phi}{K_1}\right) - \sigma_1 + \sigma_2\dfrac{y^\phi}{x^\phi} - m\gamma x^\phi\right]$

5.3 LOCAL STABILTY

In this section, the local stability of S_3 is verified, that is, the stability of the model (5.1)-(5.4) in the neighbourhood of S_3. The community matrix of the model (5.1)-(5.4) is

$$C(x,y,z)=\begin{bmatrix} r_1-\frac{2r_1x}{K_1}-\sigma_1-mz-qE & \sigma_2 & -mx & -qx \\ \sigma_1 & r_2-\frac{2r_2y}{K_2}-\sigma_2 & 0 & 0 \\ \frac{\alpha z^2}{\gamma x^2} & 0 & \alpha-\frac{2\alpha z}{\gamma x} & 0 \\ \mu E\left(p-\sigma\right)q & 0 & 0 & \mu\left[\left(p-\sigma\right)qx-c\right] \end{bmatrix} \tag{5.10}$$

At the interior equilibrium point $S_3\left(x^{\phi},y^{\phi},z^{\phi},E^{\phi}\right)$,

$$r_1-\sigma_1-qE=\frac{r_1x}{K_1}-\frac{\sigma_2y}{x}+mz\,;\, r_2-\sigma_2=\frac{r_2y}{K_2}-\frac{\sigma_1x}{y}\,;\ \gamma=\frac{z}{x}\,;\ c=\left(p-\sigma\right)qx$$

Then (5.10) evaluated at S_3 is

$$C(x,y,z)=\begin{bmatrix} -\frac{r_1x}{K_1}-\frac{\sigma_2y}{x} & \sigma_2 & -mx & -qx \\ \sigma_1 & -\frac{r_2y}{K_2}-\frac{\sigma_1x}{y} & 0 & 0 \\ \frac{\alpha z^2}{\gamma x^2} & 0 & -\alpha & 0 \\ \mu E\left(p-\sigma\right)q & 0 & 0 & 0 \end{bmatrix} \tag{5.11}$$

The characteristic equation of $C(x,y,z)$ is given by

$$\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0 \tag{5.12}$$

where, $b_1=\alpha+\frac{r_1x}{K_1}+\frac{\sigma_2y}{x}+\frac{r_2y}{K_2}+\frac{\sigma_1x}{y}>0;$

$$b_2=\mu E\left(p-\sigma\right)q^2x+\frac{\alpha z^2m}{\gamma x}+\frac{\alpha r_1x}{K_1}+\frac{\alpha\sigma_2y}{x}+\frac{\alpha r_2y}{K_2}+\frac{\alpha\sigma_1x}{y}+\frac{r_1r_2xy}{K_1K_2}+\frac{r_1\sigma_1x^2}{K_1y}+\frac{r_2\sigma_2y^2}{K_2x};$$

$$b_3 = \mu E\left(p-\sigma\right)q^2\left(\alpha x + \frac{r_2 xy}{K_2} + \frac{\sigma_1 x^2}{y}\right) + \frac{\alpha z^2 m}{\gamma x}\left(\frac{r_2 y}{K_2} + \frac{\sigma_1 x}{y}\right) + \alpha\left(\frac{r_1 r_2 xy}{K_1 K_2} + \frac{r_1\sigma_1 x^2}{K_1 y} + \frac{r_2\sigma_2 y^2}{K_2 x}\right) > 0;$$

$$b_4 = \mu E\left(p-\sigma\right)q^2\alpha\left(\frac{r_2 xy}{K_2} + \frac{\sigma_1 x^2}{y}\right) > 0;$$

$$\begin{aligned} b_1 b_2 - b_3 &= (g+a)[hqx + \alpha lmx + \alpha(a+b+f+g) + (ab+af+gb)] + f\alpha g \\ &\quad + (\alpha+b+f)(ab+af+gb) + \alpha^2(lmx+a+b+f+g) + \alpha(b+f)^2 > 0; \end{aligned}$$

$$\begin{aligned} b_3\left(b_1 b_2 - b_3\right) - b_1^2 b_4 &= hqx(g+a)[(\alpha+b+f)hqx + \alpha lmx(b+f) + \alpha(ab+af+gb)] \\ &\quad + (a+g+\alpha)\alpha lmx[(\alpha+b+f)hqx + \alpha lmx(b+f) + \alpha(ab+af+gb)] \\ &\quad + \alpha^2 hqx(a+g)^2 + \alpha^2 hqx(a+g)(b+f) + \alpha^2 lmx(b+f)(a+b+f+g)^2 \\ &\quad + \alpha^2(ab+af+gb)(a+b+f+g)^2 + \alpha^3 lmx(b+f)(a+b+f+g) \\ &\quad + (a+b+f+g)(ab+af+gb)[(\alpha+b+f)hqx + \alpha lmx(b+f)] \\ &\quad + \alpha(ab+af+gb)(a+b+f+g)(ab+af+gb+\alpha^2) + \alpha^3 hqx(g+a) > 0 \end{aligned}$$

Where, $a = \frac{r_1 x}{K_1}; b = \frac{r_2 x}{K_2}; f = \frac{\sigma_1 x}{y}; g = \frac{\sigma_2 y}{x}; h = \mu E(p-\sigma)q; l = \frac{z^2}{\gamma x^2}$.

The interior equilibrium point S_3 is locally asymptotically stable if and only if all the eigenvalues of (5.12) have negative real part, that is, iff the Routh-Hurwitz criteria $b_1 > 0,\ b_3 > 0,\ b_4 > 0, b_1 b_2 - b_3 > 0,\ b_3\left(b_1 b_2 - b_3\right) > b_1^2 b_4,\ b_4\left(b_1 b_2 b_3 - b_1^2 b_4 - b_3^2\right) > 0$ hold. Here, the above criteria are verified and hence the interior steady state S_3 of the system (5.1)-(5.4) is locally asymptotically stable.

5.4 GLOBAL STABILITY

To verify the global stability phenomena, the classical Lyapunov's approach is adapted. The following real valued function $V: R^4 \to R$ which is a positive definite in the neighborhood of the interior steady state S_3 but zero at the point S_3.

$$V\left(x,y,z,E\right) = \sum_{n=x,y,z,E} L_n\left[\left(n-n^{\phi}\right) - n^{\phi}\log\left(\frac{n}{n^{\phi}}\right)\right] \tag{5.13}$$

where, $L_n, n = x, y, z, E$ are positive constants.

Differentiating (5.13) with respect to t,

$$V'(t) = \sum_{n=x,y,z,E} L_n \left(\frac{n - n^{\phi}}{n} \right) \frac{dn}{dt} \tag{5.14}$$

It can be easily verified that at the point $S_3\left(x^{\phi}, y^{\phi}, z^{\phi}, E^{\phi}\right)$, (5.14) becomes

$$V'(t) = -L_x\left(x - x^{\phi}\right)^2 \left[\frac{r_1}{K_1} + \frac{\sigma_2 y^{\phi}}{xx^{\phi}}\right] - L_y\left(y - y^{\phi}\right)^2 \left[\frac{r_2}{K_2} + \frac{\sigma_1 x^{\phi}}{yy^{\phi}}\right] - L_z\left(z - z^{\phi}\right)^2 \frac{\alpha}{\gamma x}$$

$$+\left(x - x^{\phi}\right)\left(y - y^{\phi}\right)\left[\frac{\sigma_2}{x} + \frac{L_y \sigma_1}{y}\right] + \left(x - x^{\phi}\right)\left(z - z^{\phi}\right)\left[-m + \frac{L_z \alpha z^{\phi}}{\gamma x x^{\phi}}\right]$$

$$+\left(x - x^{\phi}\right)\left(E - E^{\phi}\right) L_E \mu\left(p - \sigma\right) q \tag{5.15}$$

Taking $L_x = 1, L_y = \dfrac{y^{\phi}}{\sigma_1 x^{\phi}}, L_z = \dfrac{\gamma x^{\phi}}{\alpha z^{\phi}}, L_E = \dfrac{1}{\mu\left(p - \sigma\right) q}$ and using $2ab \le a^2 + b^2$,

(5.3) becomes

$$V'(t) \le -\left(x - x^{\phi}\right)^2 \left[\frac{r_1}{K_1} + \frac{\sigma_2 y^{\phi}}{xx^{\phi}}\right] - \left(y - y^{\phi}\right)^2 \left[\frac{y^{\phi} r_2}{\sigma_1 x^{\phi} K_2} + \frac{1}{y}\right] - \left(z - z^{\phi}\right)^2 \frac{x^{\phi}}{xz^{\phi}}$$

$$+\left(x - x^{\phi}\right)^2 \left[\frac{\sigma_2}{2x} + \frac{y^{\phi}}{2yx^{\phi}}\right] + \left(y - y^{\phi}\right)^2 \left[\frac{\sigma_2}{2x} + \frac{y^{\phi}}{2yx^{\phi}}\right] + \left(x - x^{\phi}\right)^2 \left[\frac{-m}{2} + \frac{1}{2x}\right]$$

$$+\left(z - z^{\phi}\right)^2 \left[\frac{-m}{2} + \frac{1}{2x}\right] + \frac{\left(x - x^{\phi}\right)^2}{2} + \frac{\left(E - E^{\phi}\right)^2}{2} \tag{5.16}$$

From (5.16), it can be easily verified that $V'(t)$ is negative if $M < x < N$, where

$M = \max\left\{n_1, n_2\right\}$, $n_1 = \dfrac{\sigma_1 \sigma_2 K_2 y x^{\phi}}{2yy^{\phi} r_2 + K_2 \sigma_1\left(2x^{\phi} - y^{\phi}\right)}$, $n_2 = \dfrac{z^{\phi} - 2x^{\phi}}{mz^{\phi}}$,

$N = \dfrac{K_1 y\left(2\sigma_2 y^{\phi} - x^{\phi}\left(\sigma_2 + 1\right)\right)}{K_1 y x^{\phi}\left(1 - m\right) + K_1 y^{\phi} - 2yr_1 x^{\phi}}$.

Hence by Lyapunov theorem, one can conclude that the interior steady state S_3 is globally asymptotically stable.

5.5 SCHEME OF OPTIMAL YEILD

To optimize the present value function $\int_0^{\infty} E(t)\left[pqx(t) - c\right] e^{-\varepsilon t} dt$ (5.17)

where ε is the instantaneous annual rate of discount, subject to the equations (5.1)-(5.4).

$\sigma(t)$ is the control variable subjected to $\sigma_l \le \sigma \le \sigma_u$, where $\sigma_l = p - \frac{\gamma c}{qz^{\phi}}$;

$\sigma_m = p - \frac{cK_2\sigma_1}{q\left[r_2 y^{\phi^2} - K_2\left(r_2 - \sigma_2\right)y^{\phi}\right]}$ and $p > \frac{\gamma c}{qz^{\phi}} > \frac{cK_2\sigma_1}{q\left[r_2 y^{\phi^2} - K_2\left(r_2 - \sigma_2\right)y^{\phi}\right]}$.

By Pontryagin's maximum principle, the Hamiltonian is

$$H = E(t)\left[pqx(t) - c\right]e^{-\varepsilon t} + \lambda_1\left[x\left[\left(r_1 - \sigma_1 - qE\right) - r_1 x K_1^{-1} - mz\right] + \sigma_2 y\right]$$
$$+\lambda_2\left[y\left[\left(r_2 - \sigma_2\right) - r_2 y K_2^{-1}\right] + \sigma_1 x\right] + \lambda_3\left[z\left[\alpha - \alpha z\gamma^{-1}x^{-1}\right]\right] + \lambda_4\left[E\mu qx\left(p - \sigma\right) - Ec\mu\right] \quad (5.18)$$

Where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are adjoint variables.

A singular control is optimal when $\frac{\partial H}{\partial \sigma} = 0$, that is when $\lambda_4(t) = 0$. (5.19)

The adjoint equations evaluated at the interior equilibrium point are given below:

$$\frac{d\lambda_1}{dt} = -E^{\phi}pqe^{-\varepsilon t} + \lambda_1\left(\frac{\sigma_2 y^{\phi}}{x^{\phi}} + \frac{r_1 x^{\phi}}{K_1}\right) - \lambda_2\sigma_1 + \frac{\lambda_3\alpha}{x^{\phi}} - \frac{\lambda_4 E^{\phi}\mu c}{x^{\phi}} \quad (5.20)$$

$$\frac{d\lambda_2}{dt} = -\lambda_1\sigma_2 + \lambda_2\left(\frac{\sigma_1 x^{\phi}}{y^{\phi}} + \frac{r_2 y^{\phi}}{K_2}\right) \quad (5.21)$$

$$\frac{d\lambda_3}{dt} = \lambda_1 m x^{\phi} + \lambda_3\alpha \quad (5.22)$$

$$\frac{d\lambda_4}{dt} = -\left(pqx^{\phi} - c\right)e^{-\varepsilon t} + \lambda_1 q x^{\phi} \quad (5.23)$$

From (5.23),

$$\lambda_1 = Q_1 e^{-\varepsilon t}, \text{ where } Q_1 = \left(p - \frac{c}{qx^{\phi}}\right) \quad (5.24)$$

Using (5.24) in (5.21) and by putting the arbitrary constant of integration is zero,

$$\lambda_2 = Q_2 e^{-\varepsilon t}, \text{ where } Q_2 = Q_1\sigma_2\left[\varepsilon + \left(\frac{\sigma_1 x^{\phi}}{y^{\phi}} + \frac{r_2 y^{\phi}}{K_2}\right)\right] \quad (5.25)$$

Using (5.24) and (5.25) in (5.20),

$$\lambda_3 = Q_3 e^{-\varepsilon t}, \text{ where } Q_3 = \frac{x^{\phi}}{\alpha}\left[Q_2\left(\sigma_1 - \frac{1}{\sigma_2}\right) + E^{\phi}pq\right] \quad (5.26)$$

Using (5.24) and (5.26) in (5.22), the equation of singular path is

$$\psi\left(x^{\phi}\right) = \left(p - \frac{c}{qx^{\phi}}\right) - \frac{(\varepsilon + \alpha)}{\alpha m}\left[Q_2\left(\frac{1}{\sigma_2} - \sigma_1\right) - E^{\phi}pq\right] = 0 \quad (5.27)$$

The equation (5.27) has a unique positive root $x^{\phi} = x_{\varepsilon}$ in $0 < x^{\phi} < K_1$ when $\psi(0) < 0$, $\psi(K_1) > 0$ and $\psi'(x^{\phi}) > 0 \; \forall \; x^{\phi} > 0$. For $x^{\phi} = x_{\varepsilon}$,

$$y_{\varepsilon} = \frac{(r_2 - \sigma_2) + \sqrt{(r_2 - \sigma_2)^2 + 4r_2K_2^{-1}\sigma_1 x_{\varepsilon}}}{2r_2K_2^{-1}}; \; z_{\varepsilon} = \gamma x_{\varepsilon};$$

$$E_{\varepsilon} = \frac{1}{q}\left[r_1\left(1 - \frac{x_{\varepsilon}}{K_1}\right) - \sigma_1 + \sigma_2\frac{y_{\varepsilon}}{x_{\varepsilon}} - m\gamma x_{\varepsilon}^2\right]; \; \sigma_{\varepsilon} = p - \frac{c}{qx_{\varepsilon}}.$$

From (5.19), (5.24), (5.25) and (5.26), it is clear that $\lambda_i(t)e^{\varepsilon t}$ is independent of time. Hence, they are bounded at ∞.

From (5.24), it is also clear that the discounted rate of future price is same as the total users cost of harvest per unit effort at the equilibrium point since $\frac{\partial \pi}{\partial E}e^{-\varepsilon t} = (pqx^{\phi} - c)e^{-\varepsilon t} = \lambda_1 qx^{\phi}$, where $\pi(x, y, z, E, \sigma, t) = (pqx - c)E$ is the net economic revenue.

5.6 STOCHASTIC STABILITY

The deterministic model (5.1)-(5.4) is extended, to analyse the role of random environmental fluctuations on stability. The random fluctuations make the parameters of the model to oscillate about their average values. One can consider such randomness to the model (5.1)-(5.4) by incorporating additive white noises. The white noise perturbation included will change any parameter ν of the model as $\nu + \alpha\xi(t)$, where α is the amplitude of the noise and $\xi(t)$ is a Gaussian white noise process at time t, but the deterministic and stochastic models have same equilibria which will also now fluctuate about their mean states. By considering the randomly fluctuating driving forces in the form of additive noise to the model (5.1)-(5.4), one gets the stochastic model as follows:

$$x'(t) = x\left[(r_1 - \sigma_1 - qE) - r_1 x K_1^{-1} - mz\right] + \sigma_2 y + \alpha_1\xi_1(t) \tag{5.28}$$

$$y'(t) = y\left[(r_2 - \sigma_2) - r_2 y K_2^{-1}\right] + \sigma_1 x + \alpha_2\xi_2(t) \tag{5.29}$$

$$z'(t) = z\left[\alpha - \alpha z\gamma^{-1}x^{-1}\right] + \alpha_3\xi_3(t) \tag{5.30}$$

$$E'(t) = E\mu qx(p - \sigma) - Ec\mu + \alpha_4\xi_4(t) \tag{5.31}$$

Where α_i , $i = 1, 2, 3, 4$ are real constants and $\xi(t) = [\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)]$

is a four dimensional Gaussian white noise process agreeable

$E[\xi_i(t)] = 0;\ i = 1, 2, 3, 4$

$E[\xi_i(t)\xi_j(t')] = \delta_{ij}\delta(t - t');\ i, j = 1, 2, 3, 4$

Where δ_{ij} and δ are Kronecker and Dirac delta functions respectively.

This analysis focuses on the dynamics of the model (5.28)-(5.31) at the interior equilibrium point only. So it is necessary to compute the population variances around $S_3(x^\phi, y^\phi, z^\phi, E^\phi)$.

Let $x(t) = u_1(t) + S^*$; $y(t) = u_2(t) + P^*$; $z(t) = u_3(t) + T^*$; $E(t) = u_4(t) + U^*$

The system (5.28)-(5.31) is cantered on S_3; hence, only the effect of linear stochastic perturbations is considered. Hence, the respective linear system is

$$u_1'(t) = -\frac{r_1 S^*}{K_1} u_1(t) - mS^* u_3(t) + \alpha_1 \xi_1(t) \tag{5.32}$$

$$u_2'(t) = -\frac{r_2 P^*}{K_2} u_2(t) + \alpha_2 \xi_2(t) \tag{5.33}$$

$$u_3'(t) = -\frac{\alpha T^*}{\gamma S^{*2}} u_1(t) + \alpha_3 \xi_3(t) \tag{5.34}$$

$$u_4'(t) = \mu q(p - q)U^* u_1(t) + \alpha_4 \xi_4(t) \tag{5.35}$$

Taking the Fourier transform of (5.32)-(5.35), the respective algebraic system is obtained as

$$M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega) \tag{5.36}$$

where

$$M(\omega) = \begin{pmatrix} i\omega + \frac{r_1 S^*}{K_1} & 0 & mS^* & 0 \\ 0 & i\omega + \frac{r_2 P^*}{K_2} & 0 & 0 \\ -\frac{\alpha T^*}{\gamma S^{*2}} & 0 & i\omega & 0 \\ -\mu q(p - \sigma) & 0 & 0 & i\omega \end{pmatrix};$$

$\tilde{u}(\omega) = [\tilde{u}_1(\omega) \quad \tilde{u}_2(\omega) \quad \tilde{u}_3(\omega) \quad \tilde{u}_4(\omega)]^T$;

$\tilde{\xi}(\omega) = \left[\alpha_1\tilde{\xi}_1(\omega) \quad \alpha_2\tilde{\xi}_2(\omega) \quad \alpha_3\tilde{\xi}_3(\omega) \quad \alpha_4\tilde{\xi}_4(\omega)\right]^T$

Hence, the solution of (5.36) is given by

$$\tilde{u}(\omega) = \left[M(\omega)\right]^{-1}\tilde{\xi}(\omega) = K(\omega)\tilde{\xi}(\omega) \tag{5.37}$$

where $K(\omega) = \left[M(\omega)\right]^{-1}$, inverse matrix of $M(\omega)$.

The components of the solution (5.37) are given by

$$\tilde{u}_i(\omega) = \sum_{j=1}^{4} K_{ij}(\omega)\alpha_j\tilde{\xi}_j(\omega); i = 1, 2, 3, 4 \tag{5.38}$$

where $K_{ij}(\omega)$ are the elements of the matrix $K(\omega)$ and $\tilde{u}_i(\omega)$ are the mean values of the populations. The intensities of fluctuations of u_i ; $i = 1, 2, 3, 4$ are given by

$$\sigma_{u_i}{}^2 = \frac{1}{2\pi}\sum_{j=1}^{4}\int_{-\infty}^{\infty}\alpha_j \left|K_{ij}(\omega)\right|^2 d\omega;\ i = 1, 2, 3, 4 \tag{5.39}$$

where $K_{ij}(\omega) = \dfrac{G_{ij}(\omega)}{\det M(\omega)}; i, j = 1, 2, 3, 4$ (5.40)

Using (5.40), the population variances of the model (5.32)-(5.35) as follows:

$$\sigma_{u_i}{}^2 = \frac{1}{2\pi}\left\{\alpha_1\int_{-\infty}^{\infty}\left|\frac{G_{i1}(\omega)}{|M(\omega)|}\right|^2 d\omega + \alpha_2\int_{-\infty}^{\infty}\left|\frac{G_{i2}(\omega)}{|M(\omega)|}\right|^2 d\omega + \alpha_3\int_{-\infty}^{\infty}\left|\frac{G_{i3}(\omega)}{|M(\omega)|}\right|^2 d\omega + \alpha_4\int_{-\infty}^{\infty}\left|\frac{G_{i4}(\omega)}{|M(\omega)|}\right|^2 d\omega\right\};$$

$$= \frac{1}{2\pi}\left\{\alpha_1\int_{-\infty}^{\infty}\frac{|G_{i1}(\omega)|^2}{|\det M(\omega)|^2}d\omega + \alpha_2\int_{-\infty}^{\infty}\frac{|G_{i2}(\omega)|^2}{|\det M(\omega)|^2}d\omega + \alpha_3\int_{-\infty}^{\infty}\frac{|G_{i3}(\omega)|^2}{|\det M(\omega)|^2}d\omega + \alpha_4\int_{-\infty}^{\infty}\frac{|G_{i4}(\omega)|^2}{|\det M(\omega)|^2}d\omega\right\} \tag{5.41}$$

where $|G_{11}(\omega)|^2 = \omega^6 + \left(\dfrac{\omega^2 r_2 P^*}{K_2}\right)^2$; $|G_{12}(\omega)|^2 = 0$;

$|G_{13}(\omega)|^2 = \left(m\omega^2 S^*\right)^2 + \left(\dfrac{m\omega r_2 S^* P^*}{K_2}\right)^2$; $|G_{14}(\omega)|^2 = 0$; $|G_{21}(\omega)|^2 = 0$;

$|G_{22}(\omega)|^2 = \left(\dfrac{\omega^2 r_1 S^*}{K_1}\right)^2 + \left(\dfrac{\omega m\alpha T^*}{\gamma S^*} - \omega^3\right)^2$; $|G_{23}(\omega)|^2 = 0$; $|G_{24}(\omega)|^2 = 0$;

$|G_{31}(\omega)|^2 = \left(\dfrac{\omega^2\alpha T^*}{\gamma S^{*2}}\right)^2 + \left(\dfrac{\omega\alpha r_2 T^* P^*}{\gamma K_2 S^{*2}}\right)^2$; $|G_{32}(\omega)|^2 = 0$;

$$|G_{33}(\omega)|^2 = \left(\frac{\omega^2 r_2 P^*}{K_2} + \frac{\omega^2 r_1 S^*}{K_1}\right)^2 + \left(\frac{\omega r_1 r_2 S^* P^*}{K_1 K_2} - \omega^3\right)^2; \ |G_{34}(\omega)|^2 = 0;$$

$$|G_{41}(\omega)|^2 = \left(\omega^2 \mu q (p-\sigma)\right)^2 + \left(\frac{\omega \mu q (p-\sigma) r_2 P^*}{K_2}\right)^2; |G_{42}(\omega)|^2 = 0;$$

$$|G_{43}(\omega)|^2 = \left(\frac{\mu q m S^* (p-\sigma) r_2 P^*}{K_2}\right);$$

$$|G_{44}(\omega)|^2 = \left(\frac{m\alpha r_2 P^* T^*}{\gamma K_2 S^*} - \frac{\omega^2 r_2 P^*}{K_2} - \frac{\omega^2 r_1 S^*}{K_1}\right)^2 + \left(\frac{\omega r_1 r_2 S^* P^*}{K_1 K_2} + \frac{\omega m \alpha T^*}{\gamma S^*} - \omega^3\right)^2;$$

$$|\det M(\omega)|^2 = M_R^2 + M_I^2$$

$$M_R = \frac{\omega^2 r_1 r_2 S^* P^*}{K_1 K_2} + \frac{\omega^2 m \alpha T^*}{\gamma S^*} - \omega^4 \ ; \ M_I = \frac{\omega^3 r_2 P^*}{K_2} + \frac{\omega^3 r_1 S^*}{K_1} - \frac{\omega m \alpha r_2 P^* T^*}{\gamma K_2 S^*}$$

If only the effect of noise, on the harvesting effort is considered, that is if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then the population variances are given by

$$\sigma_{u_i}{}^2 = \frac{\alpha_4}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{i4}(\omega)|^2}{|\det M(\omega)|^2} d\omega; i = 1,2,3,4$$

If the effect of noise, on the predators is taken into account, that is if $\alpha_1 = \alpha_2 = \alpha_4 = 0$ then the population variances are given by

$$\sigma_{u_i}{}^2 = \frac{\alpha_3}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{i3}(\omega)|^2}{|\det M(\omega)|^2} d\omega; i = 1,2,3,4$$

If the effect of noise, is on the prey in the reserved zone, that is if $\alpha_1 = \alpha_3 = \alpha_4 = 0$ then the population variances are given by

$$\sigma_{u_i}{}^2 = \frac{\alpha_2}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{i2}(\omega)|^2}{|\det M(\omega)|^2} d\omega$$

If the effect of noise, is on the prey in the unreserved zone, that is if $\alpha_2 = \alpha_3 = \alpha_4 = 0$ then the population variances are given by

$$\sigma_{u_i}{}^2 = \frac{\alpha_1}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{i1}(\omega)|^2}{|\det M(\omega)|^2} d\omega$$

The equation (5.18) gives the variances of the two populations. The integrals in (5.18) can be evaluated both analytically and numerically.

5.7 NUMERICAL SIMULATIONS

To validate the analytical results, numerical simulation is a powerful tool. The population dynamics is observed for the chosen set of parameters using MATLAB simulations. The significant changes in the prey population densities, predator population densities and harvesting efforts with different tax levels are portrayed. The variance of fluctuations in the population densities due to the change in amplitude of noise is depicted in following numerical examples. The smaller variance in fluctuations indicates the stable behaviour of the system where as the high intensity fluctuations denote the instability of populations.

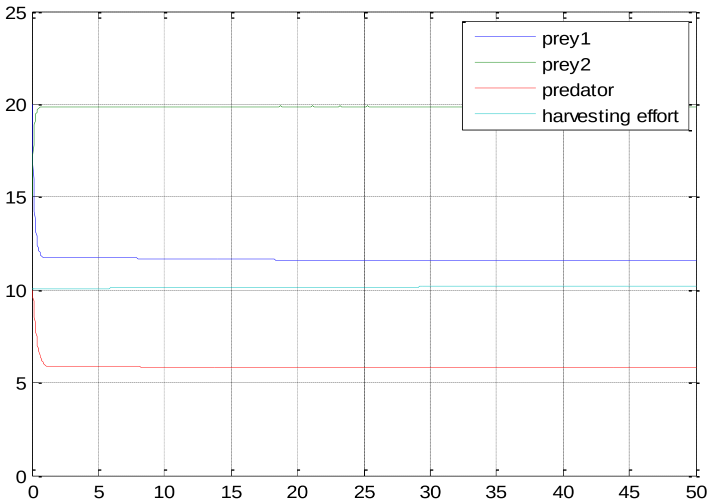

Figure 5.2

Figure (5.2) shows the variation in population densities against time with the initial condition $x = 20; y = 15; z = 10; E = 10$ and for the parameters $r_1 = 8;\ K_1 = 50;$

$\sigma_1 = 0.08;\ q = 0.5;\ r_2 = 8;\ K_2 = 20;\ \sigma_2 = 0.1;\ m = 0.2;\ \alpha = 5;\ \gamma = 0.5;\ \mu = 0.05;\ p = 15;$ $\sigma = 13.7;\ c = 7.5.$

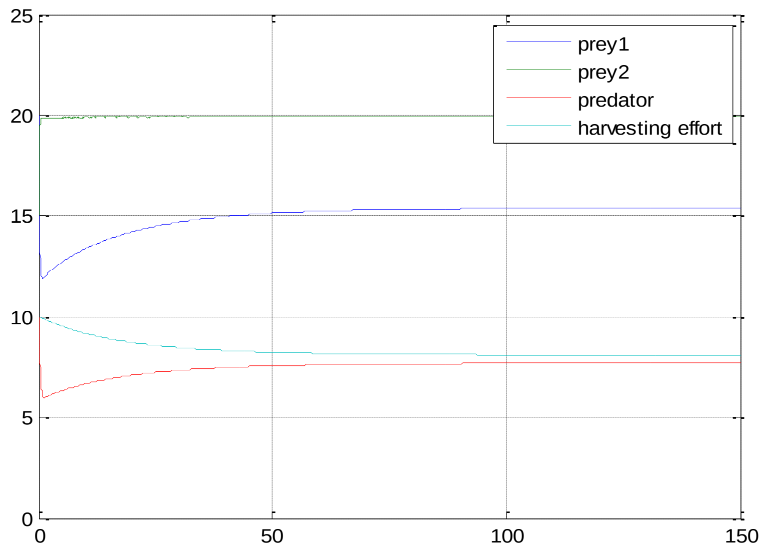

Figure 5.3

Figure (5.3) shows the variation in population densities against time with the initial condition $x = 20; y = 15; z = 10; E = 10$ and for the parameters $r_1 = 8;\ K_1 = 50;$ $\sigma_1 = 0.08;\ q = 0.5;\ r_2 = 8;\ K_2 = 20;\ \sigma_2 = 0.1;\ m = 0.2;\ \alpha = 5;\ \gamma = 0.5;\ \mu = 0.05;\ p = 15;$ $\sigma = 13.7;\ c = 10.$

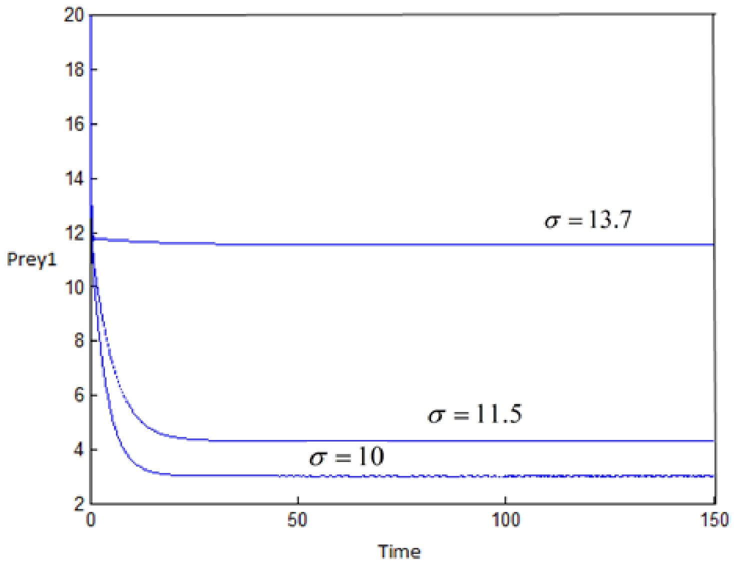

Figure 5.4

Figure (5.4) shows the variation in prey1 densities against time with the initial condition $x = 20; y = 15; z = 10; E = 10$ and for the parameters $r_1 = 8;\ K_1 = 50;$ $\sigma_1 = 0.08;\ q = 0.5;\ r_2 = 8;\ K_2 = 20;\ \sigma_2 = 0.1;\ m = 0.2;\ \alpha = 5;\ \gamma = 0.5;\ \mu = 0.05;\ p = 15;$ $c = 7.5$ with different tax levels.

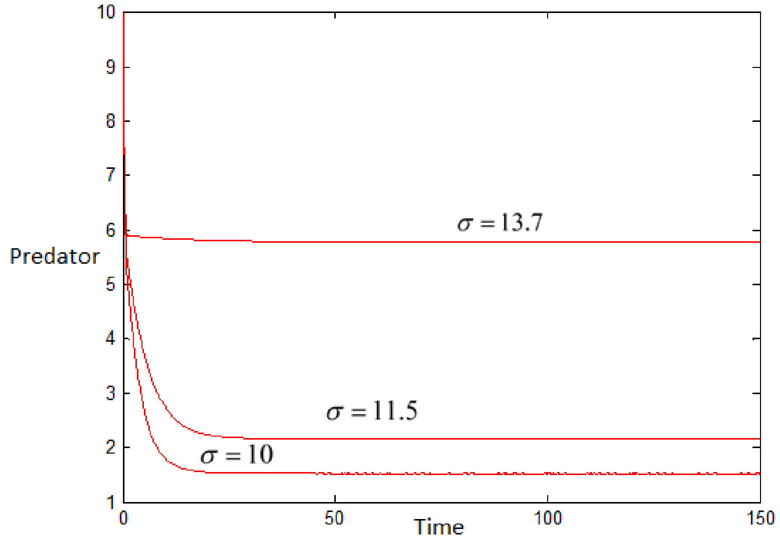

Figure 5.5

Figure (5.5) shows the variation in predator densities against time with the initial condition $x = 20; y = 15; z = 10; E = 10$ and for the parameters $r_1 = 8;\ K_1 = 50;$ $\sigma_1 = 0.08;\ q = 0.5;\ r_2 = 8;\ K_2 = 20;\ \sigma_2 = 0.1;\ m = 0.2;\ \alpha = 5;\ \gamma = 0.5;\ \mu = 0.05;\ p = 15;$ $c = 7.5$ with different tax levels.

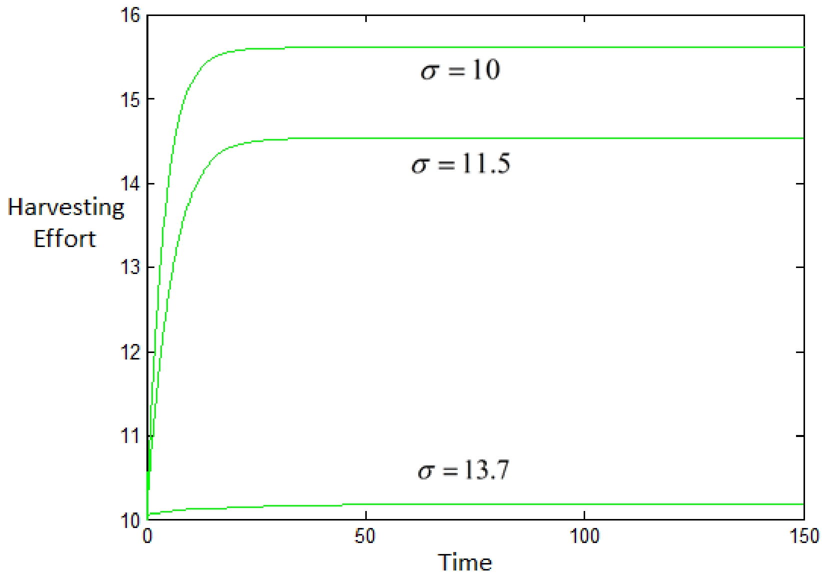

Figure 5.6

Figure (5.6) shows the variation in harvesting efforts against time with the initial condition $x = 20; y = 15; z = 10; E = 10$ and for the parameters $r_1 = 8;\ K_1 = 50;$ $\sigma_1 = 0.08;\ q = 0.5;\ r_2 = 8;\ K_2 = 20;\ \sigma_2 = 0.1;\ m = 0.2;\ \alpha = 5;\ \gamma = 0.5;\ \mu = 0.05;\ p = 15;$ $c = 7.5$ with different tax levels.

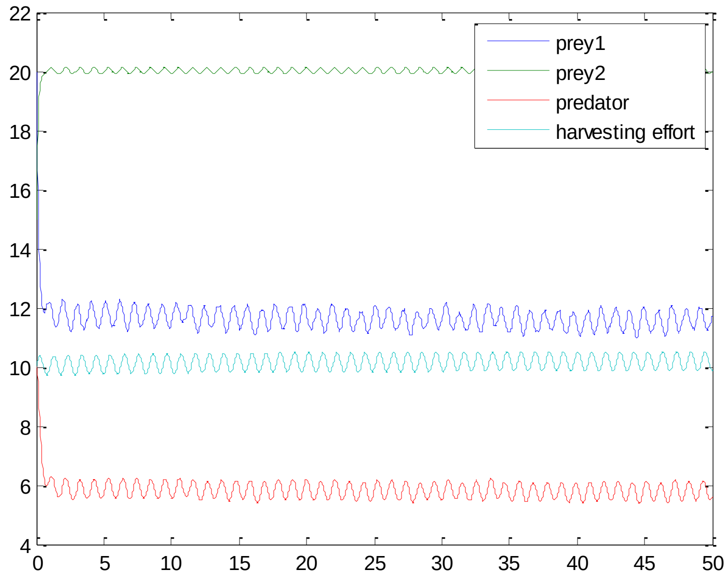

Figure 5.7

Figure (5.7) shows the variation in population densities against time with the same parameters of figure (5.2) and $\alpha_1 = 1;\ \alpha_2 = 1;\ \ \alpha_3 = 2;\ \ \alpha_4 = 2$.

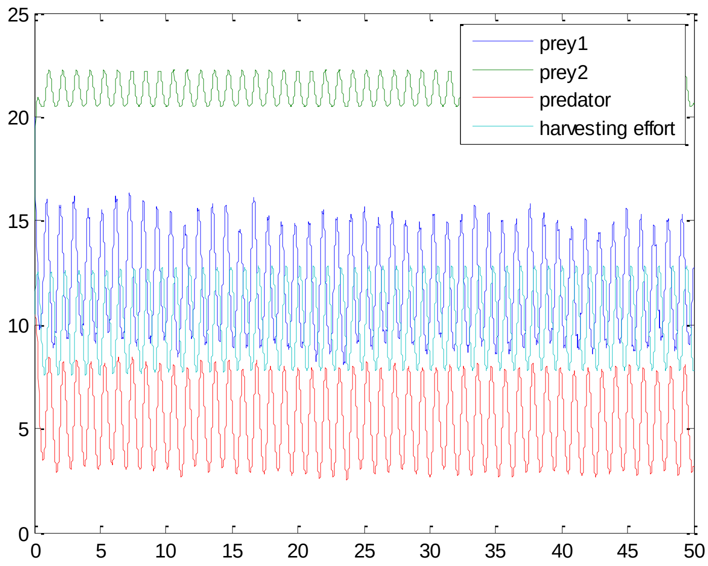

Figure 5.8

Figure (5.8) shows the variation in population densities against time with the same parameters of figure (5.2) and $\alpha_1 = 10;\ \alpha_2 = 10;\ \alpha_3 = 15;\ \alpha_4 = 15$.

5.8 CONCLUSION

The research mainly focused to understand the real complexity in the natural phenomena especially in the presence of harvesting. It has analysed the local and global stability of a bio-economic model. The ecological balance is maintained by controlling the harvesting effort through tax. It is observed that, in the presence of predation and harvesting, populations can be preserved at a stable equilibrium state, only if the number of prey in the unreserved zone lies in the interval (M, N). We also obtained the feasible harvesting condition which is optimal. The mathematical analysis on harvesting of species have exhibited that harvesting under optimal schemes really influences the species exaggeration. The stability behavior of the equilibrium points (11.5385, 19.8662, 5.7692, 10.1843) and (15.3846, 19.9046, 5.6923, 8.0988) are shown in figures (5.2) and (5.3).The dynamics due to various values of tax parameter is exhibited in the figures (5.4), (5.5) and (5.6). The harvesting efforts are decreasing with increase in tax levels. Hence the prey and predator densities are improving with the decrease in harvesting efforts. In the presence of Gaussian white noise, the fluctuation in population densities are exhibited in figures (5.7) and (5.8).