Chapter 3

A Study of Stochastic Effect on an Ecological System with Predation, Commensalism, Mutualism and Neutralism

3.1 MATHEMATICAL FORMULATION

An ecological system consists of the species interactions like commensalism, neutralism, predation, and mutualism incorporated with prey and predator harvesting together with environmental fluctuations. The main thrust of the chapter is to find out the dynamics of the system in presence of randomly fluctuating driving forces on growth of the species by incorporating random noise into the model. In a conventional ecosystem it is assumed that the presence of randomly fluctuating driving forces influence the growth of the species $S_i, i = 1,2,3,4$ at time 't'. The figure (3.1) represents the system where four species are living together with the following assumptions: (i) The system comprises of a prey (S_1), predator (S_2), two hosts S_3 and S_4 (ii) S_1 is prey of S_2 and S_1 is neutral to S_3 (iii) S_2 is predator of S_1 and S_2 is commensal to S_3 (iv) S_3 is host of S_2 and S_3 is mutual to S_4 (v) S_4 is mutual to S_3 and S_4 is neutral to S_2 (vi) harvesting of S_1 and S_2 results in the following stochastic system with 'additive noise'. The local and global stability of the deterministic model together with its stochastic analysis have been carried out around the co-existent state. Bionomic equilibrium of the system is also derived. Further, MATLAB numerical simulations are performed to validate the results.

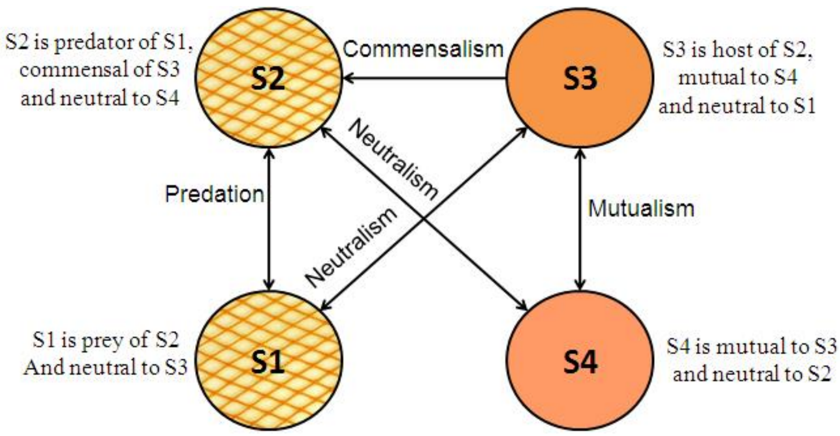

Figure 3.1

Figure (3.1) The representation of four species model

Let $x(t)$, $y(t)$, $z(t)$ and $w(t)$ be the population densities of species S_1, S_2, S_3 and S_4 respectively at time instant 't'. Let a_1, a_2, a_3 and a_4 be the natural growth rates of species S_1, S_2, S_3 and S_4 respectively. Keeping these in view and following (Srilatha 2011), the dynamics of the stochastic system may be governed by the following nonlinear differential equations:

$$\frac{dx}{dt} = a_1 x - a_{11} x^2 - a_{12} xy - q_1 E_1 x + \beta_1 \psi_1(t) \tag{3.1}$$

$$\frac{dy}{dt} = a_2 y - a_{22} y^2 + a_{21} yx + a_{23} yz - q_2 E_2 y + \beta_2 \psi_2(t) \tag{3.2}$$

$$\frac{dz}{dt} = a_3 z - a_{33} z^2 + a_{34} zw + \beta_3 \psi_3(t) \tag{3.3}$$

$$\frac{dw}{dt} = a_4 w - a_{44} w^2 + a_{43} wz + \beta_4 \psi_4(t) \tag{3.4}$$

where a_{11}, a_{22}, a_{33} and a_{44} are self-inhibition coefficients of species S_1, S_2, S_3 and S_4 respectively. a_{12} is the interaction coefficient of S_1 due to S_2, a_{21} is the interaction coefficient of S_2 due to S_1, a_{23} is coefficient of commensal for S_2 due to the host S_3, a_{34} is the rate of increase of S_3 due to the interaction with S_4, a_{43} is the rate of increase of S_4 due to the interaction with S_3, K_1, K_2, K_3 and K_4 are the carrying capacities of species S_1, S_2, S_3 and S_4 respectively, where $K_1 = a_1 / a_{11}$; $K_2 = a_2 / a_{22}$; $K_3 = a_3 / a_{33}$, $K_4 = a_4 / a_{44}$. q_1, q_2 are the catch ability coefficients of species S_1, S_2 respectively. E_1, E_2 are the efforts applied to harvest the species S_1, S_2 respectively. $\beta_i \in R, i = 1,2,3,4$ and $\psi(t) = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)]$ is a four dimensional Gaussian white noise process with $E[\psi_i(t)] = 0$, $i = 1,2,3,4$ and $E[\psi_i(t)\psi_j(t')] = \delta_{ij}\delta(t-t')$, $i, j = 1,2,3,4$, where δ_{ij} and δ are Kronecker and Dirac delta functions respectively. In addition to the variables x, y, z, w the model parameters a_1, a_2, a_3, a_4, a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{21}, a_{23}, a_{34}, a_{34} are assumed to be non-negative constants.

3.2 STABILITY ANALYSIS OF THE DETERMINISTIC SYSTEM

In the absence of randomly fluctuating driving forces on the growth of the species, the model system (3.1) - (3.4) reduces to

$$\frac{dx}{dt} = x\left[\left(a_1 - q_1E_1\right) - \left(a_{11}x + a_{12}y\right)\right] \tag{3.5}$$

$$\frac{dy}{dt} = y\left[\left(a_2 - q_2E_2\right) - \left(a_{22}y - a_{21}x - a_{23}z\right)\right] \tag{3.6}$$

$$\frac{dz}{dt} = z\left(a_3 - a_{33}z + a_{34}w\right) \tag{3.7}$$

$$\frac{dw}{dt} = w\left(a_4 - a_{44}w + a_{43}z\right) \tag{3.8}$$

For the analysis, we assume $a_1 - q_1E_1 > 0$ and $a_2 - q_2E_2 > 0$ (3.9)

3.3 INTERIOR STEADY STATE

The interior equilibrium point $G\left(x^*, y^*, z^*, w^*\right)$ is the solution of $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = \frac{dw}{dt} = 0$.

$$x^* = \frac{\left[a_{22}\left(a_1 - q_1E_1\right) - a_{12}\left(a_2 - q_2E_2\right)\right]\gamma_2 - a_{12}a_{23}\gamma_3}{\gamma_2\gamma_4};$$

$$y^* = \frac{\left[a_{11}\left(a_2 - q_2E_2\right) + a_{21}\left(a_1 - q_1E_1\right)\right]\gamma_2 + a_{11}a_{23}\gamma_3}{\gamma_2\gamma_4};$$

$$z^* = \frac{\gamma_3}{\gamma_2};$$

$$w^* = \frac{\gamma_1}{\gamma_2};$$

where $\gamma_1 = a_4a_{33} + a_3a_{43}$;

$\gamma_2 = a_{33}a_{44} - a_{34}a_{43}$;

$\gamma_3 = a_3a_{44} + a_4a_{34}$;

$\gamma_4 = a_{11}a_{22} + a_{21}a_{12}$;

Further interior equilibrium point $G\left(x^*, y^*, z^*, w^*\right)$ exists if the following inequalities hold: $a_{33}a_{44} > a_{34}a_{43}$ and $\frac{a_{22}\left(a_1 - q_1E_1\right)}{a_{12}} > \frac{a_{23}\gamma_3}{\gamma_2} + \left(a_2 - q_2E_2\right)$.

3.4 LOCAL STABILITY

The community matrix of the system is

$$J = \begin{bmatrix} a_1 - 2a_{11}x - a_{12}y - q_1E_1 & -a_{12}x & 0 & 0 \\ a_{21}y & a_2 - 2a_{22}y + a_{21}x + a_{23}z - q_2E_2 & a_{23}y & 0 \\ 0 & 0 & a_3 - 2a_{33}z + a_{34}y & a_{34}z \\ 0 & 0 & a_{43}w & a_4 - 2a_{44}w + a_{43}z \end{bmatrix}$$

At the interior equilibrium point $G\left(x^*, y^*, z^*, w^*\right)$,

$$a_1 - q_1E_1 = a_{11}x + a_{12}y \, ;$$

$$a_2 - q_2E_2 = a_{22}y - a_{21}x - a_{23}z \, ;$$

$$a_3 = a_{33}z - a_{34}y \, ;$$

$$a_4 = a_{44}w - a_{43}z$$

The community matrix evaluated at $G\left(x^*, y^*, z^*, w^*\right)$ is

$$J = \begin{bmatrix} -a_{11}x & -a_{12}x & 0 & 0 \\ a_{21}y & -a_{22}y & a_{23}y & 0 \\ 0 & 0 & -a_{33}z & a_{34}z \\ 0 & 0 & a_{43}w & -a_{44}w \end{bmatrix} \tag{3.10}$$

The characteristic equation of (3.10) is

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0 \, ,$$

where, $A = a_{11}x + a_{22}y + a_{33}z + a_{44}w > 0$;

$$B = \gamma_4 xy + \gamma_2 zw + \left(a_{11}x + a_{22}y\right)\left(a_{33}z + a_{44}w\right);$$

$$C = \left(a_{11}x + a_{22}y\right) zw\gamma_2 + \left(a_{33}z + a_{44}w\right) xy\gamma_4 > 0;$$

$$D = xyzw\gamma_2\gamma_4 > 0;$$

$$AB - C = m\gamma_4 xy + mn(m+n) + n\gamma_2 zw > 0;$$

$$ABC - C^2 - A^2D = m^3n\gamma_2 zw + m^2n^2(\gamma_4 xy + \gamma_2 zw) + mn^3\gamma_4 xy > 0 \, ;$$

where $m = a_{11}x + a_{22}y; n = a_{33}z + a_{44}w$.

Now using the Routh- Hurwitz criteria the following theorem is stated:

Theorem 3.4.1: The interior equilibrium point $G\left(x^*, y^*, z^*, w^*\right)$ is locally asymptotically stable when $A > 0$; $C > 0$; $D > 0$; $AB - C > 0$; $C\left(AB - C\right) - A^2D > 0$; $\mathrm{D}\left(ABC - C^2 - A^2D\right) > 0$.

3.5 GLOBAL STABILITY ANALYSIS

Theorem 3.5.1: The interior equilibrium point $G(x^*, y^*, z^*, w^*)$ is globally asymptotically stable if $4a_{21}a_{22}a_{33}a_{44} > a_{12}a_{44}a_{23}^2 + a_{21}a_{22}\left(a_{34} + a_{43}\right)^2$.

Proof: To find the conditions for global stability at $G(x^*, y^*, z^*, w^*)$, the Lyapunov function is constructed:

$$V(x,y,z,w) = \left[(x-x^*) - x^*\ln(x/x^*)\right] + l_1\left[(y-y^*) - y^*\ln(y/y^*)\right] + l_2\left[(z-z^*) - z^*\ln(z/z^*)\right] + l_3\left[(w-w^*) - w^*\ln(w/w^*)\right]$$

where l_1, l_2 and l_3 are positive constants.

$$\left(dV/dt\right) = \left[\left(x-x^*\right)/x\right]\left(dx/dt\right) + l_1\left[\left(y-y^*\right)/y\right]\left(dy/dt\right) + l_2\left[\left(z-z^*\right)/z\right]\left(dz/dt\right) + l_3\left[\left(w-w^*\right)/w\right]\left(dw/dt\right)$$

$$\begin{aligned}(dV/dt) = &\left(x-x^*\right)\left[-a_{11}\left(x-x^*\right) - a_{12}\left(y-y^*\right)\right] \\ &+ l_1\left(y-y^*\right)\left[-a_{22}\left(y-y^*\right) + a_{21}\left(x-x^*\right) + a_{23}\left(z-z^*\right)\right] \\ &+ l_2\left(z-z^*\right)\left[-a_{33}\left(z-z^*\right) + a_{34}\left(w-w^*\right)\right] \\ &+ l_3\left(w-w^*\right)\left[-a_{44}\left(w-w^*\right) + a_{43}\left(z-z^*\right)\right]\end{aligned}$$

By choosing $l_1 = \dfrac{a_{12}}{a_{21}}; l_2 = 1; l_3 = 1$

$$(dV/dt) = -\left[\begin{aligned}&a_{11}\left(x-x^*\right)^2 + \left(a_{12}a_{22}/a_{21}\right)\left(y-y^*\right)^2 + a_{33}\left(z-z^*\right)^2 + a_{44}\left(w-w^*\right)^2 \\ &-\left(a_{12}a_{23}/a_{21}\right)\left(y-y^*\right)\left(z-z^*\right) - \left(a_{34}+a_{43}\right)\left(y-y^*\right)\left(w-w^*\right)\end{aligned}\right]$$

$$= -X^T A X$$

where $X = \begin{bmatrix} x-x^* \\ y-y^* \\ z-z^* \\ w-w^* \end{bmatrix}; A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & \dfrac{a_{12}a_{22}}{a_{21}} & -\dfrac{a_{12}a_{23}}{2a_{21}} & 0 \\ 0 & -\dfrac{a_{12}a_{23}}{2a_{21}} & a_{33} & -\dfrac{\left(a_{34}+a_{43}\right)}{2} \\ 0 & 0 & -\dfrac{\left(a_{34}+a_{43}\right)}{2} & a_{44} \end{bmatrix}$

The system is globally stable if the derivative of Lyapunav's function V is negative definite, that is if the matrix A is positive definite, that is if the principal minors of A (say) $M_i, i = 1,2,3,4$ are positive. The principle minors are positive if $4a_{21}a_{22}a_{33}a_{44} > a_{12}a_{44}a_{23}^2 + a_{21}a_{22}\left(a_{34} + a_{43}\right)^2$. Hence, the system is globally stable in the above parametric space.

3.6 BIONOMIC EQUILIBRIUM

It is the combination of biological balance and economic balance. In section (3.3.1), the biological balance is given by $\dot{x} = \dot{y} = \dot{z} = \dot{w} = 0$. When the total profit obtained by selling the yielded biomass equals the total cost utilized in yielding it, then the bionomic balance is achieved. Let c_1 be the constant harvesting cost of species S_1 per unit effort and c_2 be the constant harvesting cost of species S_2 per unit effort. Let p_1 be the constant price of species S_1 per unit biomass and p_2 be the constant price of species S_2 per unit biomass. The revenue at any time is given by $A\left(x, y, z, w, E_1, E_2\right) = \left(p_1q_1x - c_1\right)E_1 + \left(p_2q_2z - c_2\right)E_2$.

Now if $c_1 > p_1q_1x$ and $c_2 > p_2q_2y$, then the economic rent obtained from the system becomes negative and the system will be closed. Hence for the existence of bionomic equilibrium, it is assumed that $c_1 < p_1q_1x$ and $c_2 < p_2q_2y$.

The bionomic equilibrium $\left((x)_\infty, (y)_\infty, (z)_\infty, (w)_\infty, (E_1)_\infty, (E_2)_\infty\right)$ is the positive solution of $\dot{x} = \dot{y} = \dot{z} = \dot{w} = A = 0.$

$$(x)_\infty = c_1/(p_1q_1);$$

$$(y)_\infty = c_2/(p_2q_2);$$

$$(z)_\infty = \gamma_3/\gamma_2;$$

$$(w)_\infty = \gamma_1/\gamma_2;$$

$$(E_1)_\infty = (1/q_1)\left[a_1 - (a_{11}c_1/p_1q_1) - (a_{12}c_2/p_2q_2\right]$$

$$\left(E_2\right)_\infty = (1/q_2)\left[a_2 - (a_{22}c_2/p_2q_2) + (a_{21}c_1/p_1q_1) + (a_{23}\gamma_3/\gamma_2)\right]$$

$(E_1)_\infty > 0$ when $a_1 > a_{11}(x)_\infty + a_{12}(y)_\infty$.

$(E_2)_\infty > 0$ when $a_2 + a_{21}(x)_\infty + a_{23}(z)_\infty > a_{22}(y)_\infty$.

If $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$, then the total cost utilized in harvesting the species population would exceed the total revenues obtained from the ecological system. Hence $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$ cannot be maintained indefinitely. If $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$, then the ecological system is more profitable, and hence in an open access system. This will have an increasing effect on the yielding effort. Hence $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$ cannot be continued indefinitely.

3.7 STOCHASTIC ANALYSIS

It is difficult to describe any natural phenomenon as a deterministic model and particularly the aquatic ecosystem which always has the random fluctuations of the environment. The stochastic analysis helps one to get more insight about the dynamics of any ecosystem. The deterministic model (3.5)-(3.8) with the effect of random noise of the environment results in a stochastic model (3.1)-(3.4) where the parameters of the system oscillate about their mean values. Therefore the equilibrium point which is assumed as fixed will now oscillate about the mean state. The random noise incorporated in the form of additive Gaussian white noise to the model and then any parameter $'p'$ of the system reduces to $'p + \beta\psi(t)'$, where $\beta \in R$ is the amplitude of the noise and $\psi(t)$ is the Gaussian white noise process. This analysis is mainly to depict the dynamics of the system around the equilibrium point; therefore we linearize the model using the following perturbation method:

Let $x(t) = u_1(t) + S^*;\ y(t) = u_2(t) + P^*;\ z(t) = u_3(t) + T^*;\ w(t) = u_4(t) + U^*;$ (3.11)

$$\frac{dx}{dt} = \frac{du_1(t)}{dt};\ \frac{dy}{dt} = \frac{du_2(t)}{dt};\ \frac{dz}{dt} = \frac{du_3(t)}{dt};\ \frac{dw}{dt} = \frac{du_4(t)}{dt}; \tag{3.12}$$

Using (3.11) and (3.12) in (3.1)-(3.4), the respective linear system is identified as

$$\frac{du_1(t)}{dt} = -a_{11}u_1(t)S^* - a_{12}u_2(t)S^* + \beta_1\psi_1(t) \tag{3.13}$$

$$\frac{du_2(t)}{dt} = -a_{22}u_2(t)P^* + a_{21}u_1(t)P^* + a_{23}u_3(t)P^* + \beta_2\psi_2(t) \tag{3.14}$$

$$\frac{du_3(t)}{dt} = -a_{33}u_3(t)T^* + a_{34}u_4(t)T^* + \beta_3\psi_3(t) \tag{3.15}$$

$$\frac{du_4(t)}{dt} = -a_{44}u_4(t)U^* + a_{43}u_3(t)U^* + \beta_4\psi_4(t) \tag{3.16}$$

Using Fourier transform methods on the linear system (3.13) - (3.16),

$$A(\omega)\tilde{u}(\omega) = \tilde{\psi}(\omega) \tag{3.17}$$

Where, $A(\omega) = \begin{pmatrix} A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega) & A_{14}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega) & A_{24}(\omega) \\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega) & A_{34}(\omega) \\ A_{41}(\omega) & A_{42}(\omega) & A_{43}(\omega) & A_{44}(\omega) \end{pmatrix}$;

$$\tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \\ \tilde{u}_4(\omega) \end{bmatrix}; \; \tilde{\psi}(\omega) = \begin{bmatrix} \beta_1\tilde{\psi}_1(\omega) \\ \beta_2\tilde{\psi}_2(\omega) \\ \beta_3\tilde{\psi}_3(\omega) \\ \beta_4\tilde{\psi}_4(\omega) \end{bmatrix};$$

$$A_{11}(\omega) = (i\omega + a_{11}S^*);\; A_{12}(\omega) = a_{12}S^*;\; A_{13}(\omega) = 0;\; A_{14}(\omega) = 0;$$
$$A_{21}(\omega) = -a_{21}P^*;\; A_{22}(\omega) = (i\omega + a_{22}P^*);\; A_{23}(\omega) = -a_{23}P^*;\; A_{24}(\omega) = 0;$$
$$A_{31}(\omega) = 0;\; A_{32}(\omega) = 0;\; A_{33}(\omega) = (i\omega + a_{33}T^*);\; A_{34}(\omega) = -a_{34}T^*;$$
$$A_{41}(\omega) = 0; A_{42}(\omega) = 0; A_{43}(\omega) = -a_{43}U^*; A_{44}(\omega) = (i\omega + a_{44}U^*);$$

Equation (3.17) can also be written as $\tilde{u}(\omega) = B(\omega)\tilde{\psi}(\omega)$ (3.18)

Where, $B(\omega) = \left[A(\omega)\right]^{-1} = \dfrac{Adj\, A(\omega)}{\left|A(\omega)\right|}$

$$\left|A(\omega)\right| = R(\omega) + iI(\omega)$$

$$R(\omega) = \omega^4 - \omega^2 a_{33}a_{44}T^*U^* + \omega^2 a_{34}a_{43}T^*U^* - \omega^2 a_{22}a_{44}P^*U^* - \omega^2 a_{22}a_{33}P^*T^*$$
$$-\omega^2 a_{11}a_{44}S^*U^* - \omega^2 a_{11}a_{33}S^*T^* - \omega^2 a_{11}a_{22}S^*P^* + a_{11}a_{22}a_{33}a_{44}S^*P^*T^*U^*$$
$$-a_{11}a_{22}a_{34}a_{43}S^*P^*T^*U^* - \omega^2 a_{12}a_{21}S^*P^* + a_{12}a_{21}a_{33}a_{44}S^*P^*T^*U^* - a_{12}a_{21}a_{34}a_{43}S^*P^*T^*U^*$$

$$I(\omega) = -\omega^3 a_{11}S^* - \omega^3 a_{22}P^* - \omega^3 a_{33}T^* - \omega^3 a_{44}U^* + \omega a_{22}a_{33}a_{44}P^*T^*U^*$$
$$-\omega a_{22}a_{33}a_{44}P^*T^*U^* + \omega a_{11}a_{33}a_{44}S^*T^*U^* - \omega a_{11}a_{34}a_{43}S^*T^*U^* + \omega a_{11}a_{22}a_{44}S^*P^*U^*$$
$$+\omega a_{11}a_{22}a_{33}S^*P^*T^* + \omega a_{12}a_{21}a_{33}S^*P^*T^* + \omega a_{12}a_{21}a_{44}S^*P^*U^*$$

From (3.18), we have

$$\tilde{u}_i(\omega) = \sum_{j=1}^{4} B_{ij}(\omega)\tilde{\psi}_j(\omega), \; i = 1,2,3,4 \tag{3.19}$$

The corresponding spectrum is

$$S_{u_i}(\omega) = \sum_{j=1}^{4} \beta_j \left|B_{ij}(\omega)\right|^2, \; i = 1,2,3,4 \tag{3.20}$$

The intensities of fluctuations in the variables u_i, $i = 1,2,3,4$ are given by

$$\sigma_{u_i}{}^2 = \frac{1}{2\pi}\sum_{j=1}^{4}\int_{-\infty}^{\infty} \beta_j \left|B_{ij}(\omega)\right|^2 d\omega, \; i = 1,2,3,4 \tag{3.21}$$

That is, the variances of u_i, $i = 1,2,3,4$ are obtained as

$$\sigma_{u_1}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \beta_1 |B_{11}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_2 |B_{12}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_3 |B_{13}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_4 |B_{14}(\omega)|^2 d\omega\right\};$$

$$\sigma_{u_2}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \beta_1 |B_{21}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_2 |B_{22}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_3 |B_{23}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_4 |B_{24}(\omega)|^2 d\omega\right\};$$

$$\sigma_{u_3}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \beta_1 |B_{31}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_2 |B_{32}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_3 |B_{33}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_4 |B_{34}(\omega)|^2 d\omega\right\};$$

$$\sigma_{u_4}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \beta_1 |B_{41}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_2 |B_{42}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_3 |B_{43}(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \beta_4 |B_{44}(\omega)|^2 d\omega\right\}$$

where $B_{jk}(\omega) = \dfrac{X_{jk} + iY_{jk}}{R(\omega) + iI(\omega)}$; $j,k = 1,2,3,4$

$$X_{11} = -\omega^2 a_{22}P^* - \omega^2 a_{33}T^* - \omega^2 a_{44}U^* + \omega a_{22}a_{33}a_{44}P^*T^*U^* - \omega a_{22}a_{33}a_{44}P^*T^*U^*;$$

$$Y_{11} = -\omega^3 + \omega a_{22}a_{33}P^*T^* + \omega a_{33}a_{44}T^*U^* + \omega a_{22}a_{44}P^*U^* - \omega a_{34}a_{43}P^*T^*U^*;$$

$$X_{12} = a_{12}S^*\left(\omega^2 - a_{33}a_{44}T^*U^* + a_{34}a_{43}T^*U^*\right); Y_{12} = -\omega a_{12}a_{33}S^*T^* - \omega a_{12}a_{44}S^*U^*;$$

$$X_{13} = -a_{12}a_{23}a_{44}S^*P^*U^*; Y_{13} = -\omega a_{12}a_{23}S^*P^*; X_{14} = -a_{12}a_{23}a_{34}S^*P^*T^*; Y_{14} = 0;$$

$$X_{21} = a_{21}P^*\left(-\omega^2 + a_{33}a_{44}T^*U^* - a_{34}a_{43}T^*U^*\right); Y_{21} = \omega a_{21}a_{33}P^*T^* + \omega a_{21}a_{44}P^*U^*;$$

$$X_{22} = -\omega^2 a_{11}S^* - \omega^2 a_{33}T^* - \omega^2 a_{44}U^* + a_{11}a_{33}a_{44}S^*T^*U^* - a_{11}a_{34}a_{43}S^*T^*U^*;$$

$$Y_{22} = -\omega^3 + \omega a_{33}a_{44}T^*U^* - \omega a_{34}a_{43}T^*U^* + \omega a_{11}a_{44}S^*U^* + \omega a_{11}a_{33}S^*T^*;$$

$$X_{23} = -\omega^2 a_{23} P^* + a_{11} a_{23} a_{44} S^* P^* U^*; Y_{23} = \omega a_{23} a_{44} P^* U^* + \omega a_{11} a_{23} S^* P^*;$$

$$X_{24} = a_{11} a_{23} a_{34} S^* P^* T^*; Y_{24} = \omega a_{23} a_{34} P^* T^*; X_{31} = 0; Y_{31} = 0; X_{32} = 0; Y_{32} = 0;$$

$$X_{33} = -\omega^2 a_{11} S^* - \omega^2 a_{22} P^* - \omega^2 a_{44} U^* + a_{11} a_{22} a_{44} S^* P^* U^* + a_{12} a_{21} a_{44} S^* P^* U^*;$$

$$Y_{33} = -\omega^3 + \omega a_{22} a_{44} P^* U^* + \omega a_{11} a_{44} S^* U^* + \omega a_{11} a_{22} S^* P^* + \omega a_{12} a_{21} S^* P^*;$$

$$X_{34} = a_{34} T^* \left(-\omega^2 + a_{11} a_{22} S^* P^* + a_{12} a_{22} S^* P^*\right); Y_{34} = a_{34} T^* \left(\omega a_{22} P^* + \omega a_{11} S^*\right);$$

$$X_{41} = 0; Y_{41} = 0; X_{42} = 0; Y_{42} = 0; X_{43} = a_{43} U^* \left(-\omega^2 + a_{11} a_{22} S^* P^* + a_{12} a_{21} S^* P^*\right);$$
$$Y_{43} = a_{43} U^* \left(\omega a_{22} P^* + \omega a_{11} S^*\right);$$

$$X_{44} = -\omega^2 a_{11} S^* - \omega^2 a_{22} P^* - \omega^2 a_{33} T^* + a_{11} a_{22} a_{33} S^* P^* T^* + a_{12} a_{21} a_{33} S^* P^* T^*;$$

$$Y_{44} = -\omega^3 + \omega a_{22} a_{33} P^* T^* + \omega a_{11} a_{33} S^* T^* + \omega a_{11} a_{22} S^* P^* + \omega a_{12} a_{21} S^* P^*;$$

The expressions of population variances can explicitly be written as

$$\sigma_{u_1}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\begin{array}{r} \beta_1\left(X_{11}{}^2 + Y_{11}{}^2\right) + \beta_2\left(X_{12}{}^2 + Y_{12}{}^2\right) + \beta_3\left(X_{13}{}^2 + Y_{13}{}^2\right) \\ + \beta_4\left(X_{14}{}^2 + Y_{14}{}^2\right) \end{array}\right] d\omega\right\};$$

$$\sigma_{u_2}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\begin{array}{r} \beta_1\left(X_{21}{}^2 + Y_{21}{}^2\right) + \beta_2\left(X_{22}{}^2 + Y_{22}{}^2\right) + \beta_3\left(X_{23}{}^2 + Y_{23}{}^2\right) \\ + \beta_4\left(X_{24}{}^2 + Y_{24}{}^2\right) \end{array}\right] d\omega\right\};$$

$$\sigma_{u_3}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\begin{array}{r} \beta_1\left(X_{31}{}^2 + Y_{31}{}^2\right) + \beta_2\left(X_{32}{}^2 + Y_{32}{}^2\right) + \beta_3\left(X_{33}{}^2 + Y_{33}{}^2\right) \\ + \beta_4\left(X_{34}{}^2 + Y_{34}{}^2\right) \end{array}\right] d\omega\right\};$$

$$\sigma_{u_4}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\begin{array}{r} \beta_1\left(X_{41}{}^2 + Y_{41}{}^2\right) + \beta_2\left(X_{42}{}^2 + Y_{42}{}^2\right) + \beta_3\left(X_{43}{}^2 + Y_{43}{}^2\right) \\ + \beta_4\left(X_{44}{}^2 + Y_{44}{}^2\right) \end{array}\right] d\omega\right\}$$

The behaviour of the system (3.1)-(3.4) with either $\beta_1 = 0$ or $\beta_2 = 0$ or $\beta_3 = 0$ or $\beta_4 = 0$, can be analysed to measure the effect of noise on selective species.

If $\beta_1 = \beta_2 = \beta_3 = 0$, then

$$\sigma_{u_1}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty} \frac{X_{14}{}^2}{R^2(\omega) + I^2(\omega)} d\omega; \ \sigma_{u_2}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty} \frac{\left(X_{24}{}^2 + Y_{24}{}^2\right)}{R^2(\omega) + I^2(\omega)} d\omega;$$

$$\sigma_{u_3}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{34}{}^2+Y_{34}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_4}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{44}{}^2+Y_{44}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega.$$

If $\beta_1 = \beta_2 = \beta_4 = 0$, then

$$\sigma_{u_1}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{13}{}^2+Y_{13}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_2}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{23}{}^2+Y_{23}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$$

$$\sigma_{u_3}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{33}{}^2+Y_{33}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_4}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{43}{}^2+Y_{43}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega.$$

If $\beta_1 = \beta_3 = \beta_4 = 0$, then

$$\sigma_{u_1}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{12}{}^2+Y_{12}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_2}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{22}{}^2+Y_{22}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$$

$$\sigma_{u_3}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{32}{}^2+Y_{32}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0;\quad \sigma_{u_4}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{42}{}^2+Y_{42}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0.$$

If $\beta_2 = \beta_3 = \beta_4 = 0$, then

$$\sigma_{u_1}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{11}{}^2+Y_{11}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_2}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{21}{}^2+Y_{21}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$$

$$\sigma_{u_3}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{31}{}^2+Y_{31}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0;\quad \sigma_{u_4}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{41}{}^2+Y_{41}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0.$$

Analytical evaluation of the population variance is difficult, but it can be evaluated numerically for different set of values of parameters.

3.8 NUMERICAL SIMULATIONS

Some simulations were performed in order to validate the dynamical behaviour of the deterministic system (figures 3.2-3.4) and stochastic system (figures 3.5-3.7) using MATLAB.

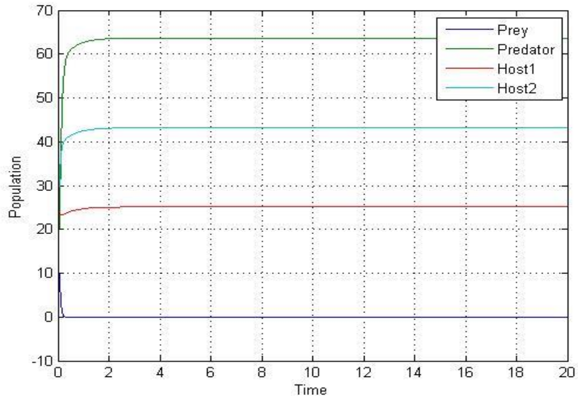

Figure 3.2

Figure (3.2) shows that the variation of populations against time initially with $x=15$; $y=10$; $z=25$; $w=20$ and for the parameters $a_1=5$; $a_{11}=0.21$; $a_{12}=0.5$; $q_1=0.2$; $E_1=20$; $a_2=2$; $a_{22}=0.25$; $a_{21}=0.28$; $a_{23}=0.56$; $q_2=0.01$; $E_2=15$; $a_3=2$; $a_{33}=0.2$; $a_{34}=0.07$; $a_4=1.5$; $a_{44}=0.5$; $a_{43}=0.8$.

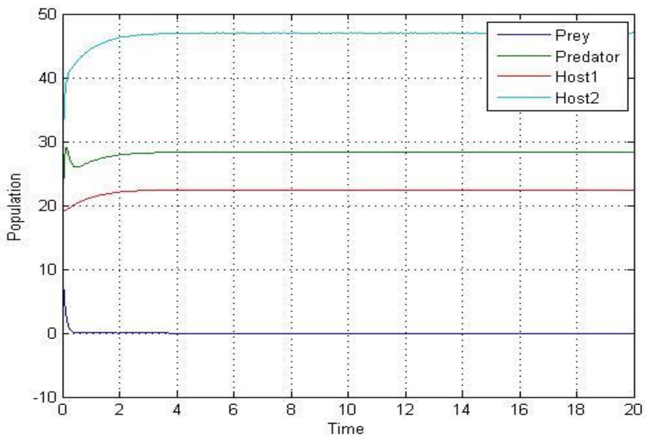

Figure 3.3

Figure (3.3) shows that the variation of populations against time initially with $x=10$; $y=15$; $z=20$; $w=25$ and for the parameters $a_1=4.5$; $a_{11}=0.1$; $a_{12}=0.5$; $q_1=0.12$; $E_1=10$; $a_2=2.6$; $a_{22}=0.5$; $a_{21}=0.8$; $a_{23}=0.56$; $q_2=0.1$; $E_2=10$; $a_3=1.2$; $a_{33}=0.2$; $a_{34}=0.07$; $a_4=1.5$; $a_{44}=0.5$; $a_{43}=0.98$.

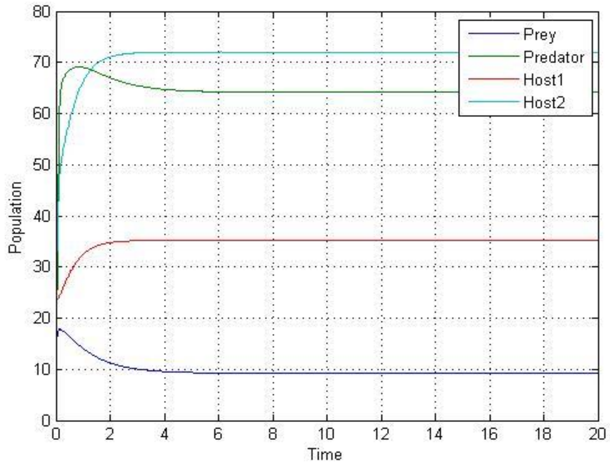

Figure 3.4

Figure (3.4) shows that the variation of populations against time initially with $x = 15$; $y = 10$; $z = 25$; $w = 20$ and for the parameters $a_1 = 4.5$; $a_{11} = 0.01$; $a_{12} = 0.05$; $q_1 = 0.12$; $E_1 = 10$; $a_2 = 6$; $a_{22} = 0.5$; $a_{21} = 0.8$; $a_{23} = 0.56$; $q_2 = 0.1$; $E_2 = 10$; $a_3 = 2$; $a_{33} = 0.2$; $a_{34} = 0.07$; $a_4 = 1.5$; $a_{44} = 0.5$; $a_{43} = 0.98$.

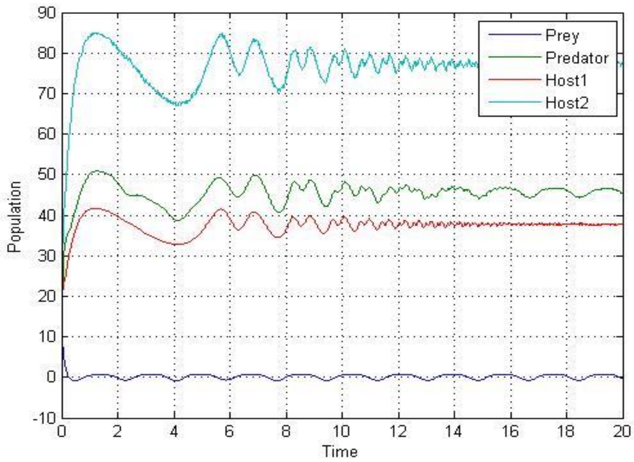

Figure 3.5

Figure (3.5) shows that the variation of populations against time initially with $x = 10$; $y = 15$; $z = 20$; $w = 25$ and for the parameters $a_1 = 4.5$; $a_{11} = 0.1$; $a_{12} = 0.5$; $q_1 = 0.12$; $E_1 = 10$; $a_2 = 2.6$; $a_{22} = 0.5$; $a_{21} = 0.8$; $a_{23} = 0.56$; $q_2 = 0.1$; $E_2 = 10$; $a_3 = 1.2$; $a_{33} = 0.2$; $a_{34} = 0.07$; $a_4 = 1.5$; $a_{44} = 0.5$; $a_{43} = 0.98$.

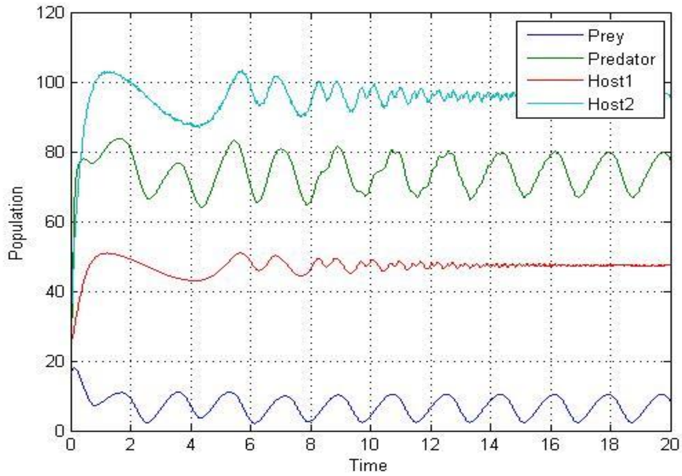

Figure 3.6

Figure (3.6) shows that the variation of populations against time initially with $x=15$; $y=10$; $z=25$; $w=20$ and for the parameters $a_1=4.5$; $a_{11}=0.01$; $a_{12}=0.05$; $q_1=0.12$; $E_1=10$; $a_2=6$; $a_{22}=0.5$; $a_{21}=0.8$; $a_{23}=0.56$; $q_2=0.1$; $E_2=10$; $a_3=2$; $a_{33}=0.2$; $a_{34}=0.07$; $a_4=1.5$; $a_{44}=0.5$; $a_{43}=0.98$.

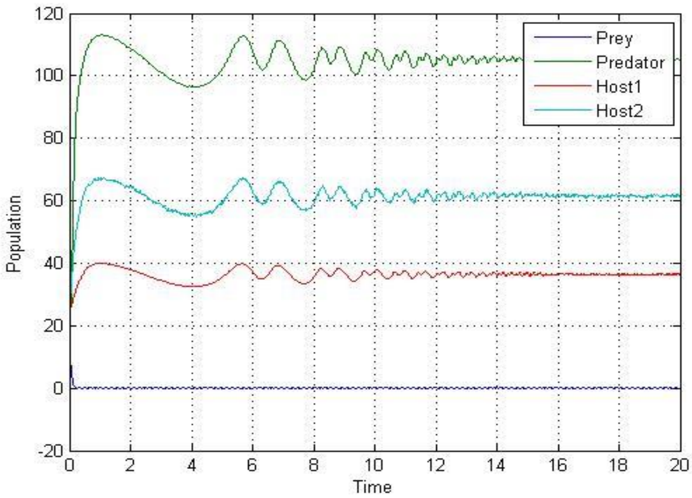

Figure 3.7

Figure (3.7) shows that the variation of populations against time initially with $x=15$; $y=10$; $z=25$; $w=20$ and for the parameters $a_1=2$; $a_{11}=0.21$; $a_{12}=0.5$; $q_1=0.2$; $E_1=20$; $a_2=6$; $a_{22}=0.25$; $a_{21}=0.28$; $a_{23}=0.56$; $q_2=0.01$; $E_2=15$; $a_3=2$; $a_{33}=0.2$; $a_{34}=0.07$; $a_4=1.5$; $a_{44}=0.5$; $a_{43}=0.8$.

3.9 DISCUSSION

The analytical results and numerical simulations of the system suggest that the dynamics of the deterministic model is stable in nature (see figures 3.2 - 3.4). Analysing the stochastic version of the model, it has been identified that the effect of randomness of environment has an important role in the dynamics of the system particularly in an aquatic ecosystem. It is observed that in the parametric space the interior equilibrium point is stable with smaller variances and is unstable with larger variances. Numerical simulations reveal that the trajectories of the system oscillate arbitrarily with remarkable variance of amplitudes with the increasing value of the strength of noises initially but ultimately fluctuating (see figures 3.5 - 3.7). Hence, the researcher concludes that inclusion of stochastic perturbation create a significant change in intensity in the model system due to change of responsive parameters which causes large environmental fluctuations.