Chapter 2

An Overview On Latest Optimal And N-Ary Block Designs
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AN OVERVIEW ON LATEST OPTIMAL, A-OPTIMAL AND n-ARY BLOCK DESIGNS

2.1 Introduction

Basing upon the latest three decades’ literature on optimality design, a strong reference is made on all, up-to-date incomplete block designs. We have a note on latest developments in algorithms for construction of i) Continuous and exact designs, ii) designs in both qualitative and quantitative factors, iii) biased-coins designs for sequential clinical trials, iv) response surface designs, v) the designs for off-line quality control associated with Taguchi designs for non-linear models, vi) work on computer-aided designs, vii) Stochastic Processes, viii) Optimal combinatorial design ix) autocorrelated optimal design problems, and x) other related problems, etc.

2.1.1 Algorithms and exact designs

A major use of the General Equivalence Theorem is to provide algorithms for the construction of designs. Computer algorithms for optimal designs take two basic forms. For continuous theory we can improve a measure $\xi$ by augmentation

$$\xi_{n+1} = (1 - \alpha_n) \xi_n + \alpha_n \tilde{x}_n$$  \hspace{1cm} (2.1.1)

where $\tilde{x}_n$ is chosen with regard to the optimality criterion to put mass one at a single point. In the simplest algorithm for the construction of D-Optimum designs, $\alpha_n = (n+1)^{-1}$ and $\xi_n$ puts unit mass at the point where $d(\xi_n)$ is attained. Wu and Wynn (1978) prove convergence of algorithms for regular $\Phi$-optimal criteria. A review and discussions were given by Silvey (1980) and Wynn (1970) worked with the natural case $\alpha_n = \frac{1}{(n+1)}$ which was simultaneous with that of the work of Fedorov (1972). The work took firm hold in the United Kingdom notably with papers Silvey et al. and in the USSR and Eastern Europe. There are several developments of this algorithm. Mitchell (1974) had developed the special DETMAX algorithm involving ‘excursions’ in which new points were added and old points were thrown out. The
intention is to escape form local optimum, which can trap the one-point exchange algorithm. Galil and Kiefer (1980.a) suggest improvements to Mitchell’s algorithm designed to save either computational space or time. A comparison of these algorithms is given by Cook et al. (1980) who include a modified Fedorov algorithm in which for each of the design points, an exchange is made if the design would be improved there by. Johnson et al. (1983) consider the augmentation of n trial designs to yield designs for n + m trials. They find that methods for adding m trials at once, perform no better than exchange algorithms. Their comparison of algorithms leads to a useful modification of the exchange procedure. At each iteration, the ‘s’ points with lowest variance are chosen for deletion. For each candidate for deletion an exchange is made if an improvement is possible. For s = n, this algorithm becomes Cook and Nachtsheim's modification of Fedorov's algorithm.

As even the methods find local optima, it is usual, to repeat the search many times starting the algorithm from a randomly selected starting point. In contrast, Welch (1982) gives a branch-and-bound algorithm which searches systematically over the list of candidate points for the exact optimum design. Because of the structure of the search, the globally optimum design is bounded without the need for an exhaustive search. A subsidiary advantage of this method is that it can provide a list of specified number of the best designs according to the primary criterion, for example D-Optimality. These designs can then be compared using other criteria, for example G-optimality or A-optimality, and a compromise design chosen which does well on all criteria. In this way, over-reliance on a single criterion is avoided.

Because the General Equivalence Theorem does not hold for exact designs, exact G- and D- optimum designs can be appreciably different. Welch (1984) describes the extension of algorithms of the DETMAX type to the calculation of V-and G-optimum designs. These criteria require calculations of the average or the maximum of the variance over a set of r points. If r is large, and it will often correspond to the points of a fine mesh in factor space, a straightforward adaptation of DETMAX is possible, but is computationally inefficient when compared to the D
- Optimum algorithm, which require only one calculation of variance for each candidate point. For G - optimality, Welch (1984) gives bounds on the change in variance, which lead to an algorithm of improved efficiency. However, greater improvement comes from using the criterion of D - optimality for all but the last stage of an excursion. In addition to computational efficiency, this modification avoids the tendency of the G - optimum algorithm to become trapped at a poor local optimum.

The algorithms of this section find optimum designs through search over the grid of candidate points. If the factors are continuous, improved designs will be found by numerical optimization over the design region $\mathcal{S}$. The disadvantage is the appreciable increase in computational effort required, due in part to the presence of local optima. Globally optimum designs can be bound by the use of simulated annealing. This method makes it possible to escape from local optima by accepting steps away from the optimum with a small probability which decreases as a variable analogous to temperature decreases. Haines (1987) finds D- G- and linear-optimum designs for polynomial regression. Bohachevsky, et.al. (1986) demonstrate the advantages of generalized annealing in which the optimum value of the criterion is known, at least approximately. In their application of the method to a non-linear example, the optimization is performed at only one temperature.

Fedorov (1989) deals with the design problem, initiated by Wynn arising in sampling experiments, experiments with spatially distributed observations or containing time type variables. The peculiarity of these experiments is that the number of observations in any given element of the design space cannot exceed a priori prescribed level. In terms of the continuous designs, this means that the density of design measure is restricted. The proposed algorithm are based on simple heuristic idea; to delete, at every step of the iterative procedure, 'bad' (less informative) sets of the supporting points and to include 'good' (most informative) ones in the design. The convergence of the algorithm and its various modifications has been discussed by Fedorov.
Algorithm for the construction of a wide class of block designs including Balanced Incomplete Block (BIB) design is described by Jergaw (1989). The algorithm which allows the experimenter to give weights for a set of treatment contrasts, uses an initial starting design to generate an optimal block design sequentially with examples. A new exchange algorithm for the construction of M.S.-Optimal incomplete block designs (IBD) was developed by Nguyen (1990). This exchange algorithm was used to construct 973 M.S-optimal of IBD (v, k, b) for v=4,5, . . . 12 v (treatments) with arbitrary v(treatments), k (block size) and b (number of blocks). The efficiencies of the "best" M.S.-Optimal IBD constructed by this algorithm were compared with the efficiencies of the corresponding nearly balanced incomplete block (NBIBDs) designs of Cheng, et.al. (1981) and of Mitchell et.al. (1976).

Pukelsheim et.al. (1991) have derived an explicit formula for computing the A-Optimal design weights on linearly independent regression vectors, for the mean parameters in a linear model with homoscedastic variances. The formula emerges as a special case of a general result which holds for a wide class of optimality criteria. There are close links to iterative algorithms for computing optimal weights. Several algorithms have been proposed, but no explicit expressions are available. ABT (1992) paper investigates linear covariance function in one dimension, and shows how exact optimal designs can be formed for several design criteria. Linear in this context means that the obtained predictive function interpolates the observations linearly. Even-though the results may not be of great practical importance, they provide guidance for further work.

Jansen, et.al.,(1992) is concerned with a computer algorithm for searching optimal block designs. The algorithm uses a technique called simulated annealing. Exchange and interchange steps are defined in a way similar to Jones et.al. (1980). They also show the performance of algorithm with two examples.

Recently, there has been increasing use of computers for the construction of experimental designs which are 'good' in some well defined sense. Nguyen et.al (1992) review and compare some well-known exchange algorithms for the construction
of discrete D-Optimal designs. An improved implementation of Fedorov's exchange algorithms is suggested.

Bhaumik' (1993) paper shows that in the presence of a linear trend, an A-optimal BIBD is Cheng's Type 2 $\Psi_f$-Optimal. An algorithm for the construction of an A-optimal BIBD in the presence of a linear trend is provided.

Nguyen's et.al. (1994) article describes an effective algorithm for constructing optimal or near-optimal incomplete block designs with up to 100 treatments. The algorithm is found to perform well when evaluated against 874 optimal or near-optimal incomplete block designs in the literature.

Myers et.al., (1994) describe some alphabetic optimal designs for the logistic algorithm model. Alphabetic optimality has become an important component of experimental design in the case of the standard linear model. The robustness properties of some of the newly created designs are also investigated. Finally, the development of optimal designs for asymmetric regions of the logistic regression model is also presented.

Eccleston et.al., (1994) describe an algorithm for the construction of optimal or near optimal change-over designs for arbitrary numbers of treatments, periods and units. He also explains the performance of the algorithm with examples.

2.1.2 Qualitative and quantitative factors design

So far only qualitative factors have been discussed. In contrast, for the block designs discussed by Paterson (1988) all factors are qualitative. This section is concerned with designs when both kinds of factors are present, a subject which has been comparatively neglected. Pioneering papers are those of Harville, (1975), and Lopes Troya (1982). Wierich (1986) gives several examples; the qualitative factor could be 3 types of catalyst to be tested at different temperatures represented by the single quantitative factor. The gasoline data of Prater, illustrated by Daniel et.al. (1980) demonstrate the occurrence of data for which models are appropriate. Wierich
shows that 'product designs' are indeed optimum, that is, the D-Optimum design for the qualitative factors is repeated at each level of the qualitative factor.

2.1.3. Biased-Coins Designs

Our discussions deal with fixed sample sizes in previous sections. The physical problem considered in this section is the design of clinical trials in which patients arrive sequentially and are assigned to one of \( t \) treatments. Usually the outcome of the treatment is not known before the next allocation is made. If the outcomes are known, an adaptive rule can be used, in which an attempt is made to balance the need for information against the desire to give the best treatment.

A biased coin design used by Efron (1971) in which the under-represented treatment is favoured in a probabilistic allocation. Efron's original proposal was for the comparison of two treatments. Pocock et.al. (1975) and Efron describe extensions to the comparison of any number of treatments with balance for prognostic factors. Wei (1977) uses urn designs to generate the probabilistic allocations. Begg et.al. (1980) employed optimum design theory to yield a deterministic criterion to which they derive an operationally useful approximation. Smith (1984.b) gives a unified description of several biased-coin allocation rules.

The methods of Wei and Atkinson provide algorithms for the construction of biased coin designs in a variety of settings. Some properties of the design have been investigated. Smith (1984.a) studies the distribution of the number of patients receiving each of several treatments when prognostic factors are present. Smith applies the results to a variety of biases that could arise: selection bias, bias due to outliers, and accidental biases due to smooth trends and to correlated errors. Cox, Smith and Wei, et.al. consider the problem of inference after random allocation. Heckman (1985) gives a further local limit theorem for designs with two treatments.
2.1.4 Response Surface Designs

It is assumed here that the response is a smoothly varying function of continuous factors over a well defined experimental region. Box et.al. supposed that instead of assuming that the exact model is not known, it was rather preferable to assume that the model is

$$E(Y) = X_1\beta_1 + X_2\beta_2$$

Least squares is used to fit the model with terms in $X_1$, which model is to be used for prediction over a specified region. The design should protect against bias due to the omitted term $X_2$. The resulting expression for the mean squared error of predictions breaks into two parts, one for variance, the other for bias. In general, the bias, and hence the design, will depend upon the unknown parameters $\beta_2$. Box and Draper argue that, unless the bias is very small, designs which minimize bias lead to a near minimum mean squared error of prediction. A disadvantage of Box and Draper's approach is that, unlike optimum design theory, it does not provide our algorithm for the construction of designs. The symmetrical designs which are specified by the theory are typically shrunken away from the edges of the experimental region.

Many of the recent alternatives due to Box and Draper's work differ both in the nature of the assumed departure from the fitted model and in the method used to fit the model. Pesotchinsky (1982) assumes that the first-order model is distributed by an additive function, a bound on the magnitude which is specified. Sacks et.al. (1984) generalise the problem to the consideration of linear estimates but are concerned only with contrast in the values of the estimated response function. In some cases least-squares estimation with a good design behaves well.

Welch (1983) replaces the parameterized departure by a general departure, subject to an upper bound $Z_{\text{max}}$ on the size of the departure at each point. Let there be $N$ candidate design points for the $n$-trial design. Exact designs are found by a variant of the exchange algorithm. Welch gives an example, the 9 trial design for a two-factor, first-order model over the points of a $3^2$ factorial. When $Z_{\text{max}} = 0$,
the all-variance design is obtained which is, as near as possible, the D-optimum $2^2$
factorial. As $Z_{\text{max}}$ increases, the design changes, by stages, to the uniform all bias
design with one trial at each of the 9 experimental conditions. The design changes
not continuously, but at a few specified points of $Z_{\text{max}}$. From these designs, the one
which is selected now behaves most reasonably for all important values of $Z_{\text{max}}$.

2.1.5. Off-line quality control

The important economic development has been the relative decline in the
importance of manufacturing in the USA and Western Europe, compared with the
Pacific area, especially Japan. Part of the reason for this change is the superior
quality of Japanese goods achieved by the use of statistical techniques. The
calculation of Deming Day' underlines the strength of the belief that statistical
methods have made an important contribution to Japanese economic achievement.

Suppose the required minimum value of the index can be obtained over a
region of factor space, rather than at a single point. The experimental design yielding
this information can be crossed with factors representing further stages in the
manufacturing process or conditions of use. If some replication is present, the
variance at each set of conditions can be considered as a second response. Conditions
can then be found for which $E(Y)$ lies within specification and has acceptably small
variance. Alternatively, the factors representing conditions of use can be regarded as
generating the variability in $Y$. What is required is a product for which the expected
response is high, but the variability is small. Box et.al. (1986) discuss the waste
inherent in replication and suggest how to estimate dispersion effects is unreplicated
functional factorials.

Yet, designs for the two sets of factors, conditions of manufacture and
conditions of use, are often crossed to yield combined experiments which are unduly
large. Simple assumptions about the absence of high-order interactions lead to
smaller designs. More efficient analysis than those associated with Taguchi are also
possible. Box discusses statistically sound alternative to the use of the signal-to-noise
ratio, which gives analysis of an arbitrary combination of mean and variance.
2.1.6. Non-Linear models

Optimum designs for non-linear models depend on the values of the parameters of the model. The purpose of the experiment is usually to estimate the parameters. One solution is to experiment sequentially, starting with some preliminary estimate of the parameter \( \theta_0 \), which is updated as the experimental results become available. Research on designs for non-linear models has been concerned mostly either with the normal theory of non-linear model or with models for binary data.

Ford et.al., (1992) shows that, for a certain class of generalized linear models, the problem can be reduced to a canonical form. This simplifies the underlying problem and designs are constructed for several contents with a single variable using Geometric and other arguments. Huang et.al., (1993) consider the problem of constructing linear-optimal designs for regression models, when some of the factors are not under control of the experiments. Such designs are referred to as marginally restricted (MR for brevity) linear optimal designs.

2.1.7 Computers and optimum designs

Computers are increasingly used not only in the analysis of experiments, but also in the selection of a design. One use of the machines is to store catalogues of designs. Another is for the generation of designs. Paterson and Patterson et.al. (1983) describe an algorithm for the construction of block designs and repeated mention has been made in their paper of the algorithms of optimum design theory.

A review is given by Nachtsheim of packages providing response surface (00) Optimum designs. One advantage of these computer-based methods is that designs can readily be generated for non-standard problems. Snee (1985) gives examples in which seemingly standard problems are complicated by the non-availability of certain treatments combinations. One such example arises where a combination of fertilizers and hormone treatments, each individually beneficial, produce an environment in which a plant cannot grow. Similarly too vigorous
conditions in a chemical experiment may produce a sticky black tar instead of the desired product. Finally, relatively recent developments and more sophisticated programs are discussed by Nachtsheim.

2.1.8 Bayesian design

Most of the work on the Bayesian design of experiments has led to analogues of standard optimality criteria, particularly D-Optimality. If the prior dispersion matrix of the parameters is $H$ then $\det(\chi^T \chi + H)$ is to be maximised, with the appropriate generalization if the observations are not independent and identically distributed. In a sequential design $H$ will incorporate information from earlier trials.

For non-linear regression models, Box et.al. show that non-informative priors lead to D-optimality for the linearized model and so to the procedure of Box et.al. Designs for informative priors are obtained by Smith et.al. with $H$ modelled by hierarchical linear model of Lindley et.al. A full Bayesian approach to the design of experiments with a general decision function can lead to appreciable mathematical difficulties. A review is given by Bandemer (1980). Herrendorfer et.al. (1980) develop a general theory of cost optimal designs which Rasch et.al. (1980) apply to the construction of exact designs. After a series of approximations similar in spirit to those of Box et.al. Brook obtains D-optimality as the appropriate criterion for control. For prediction A-Optimality is appropriate. E-optimality criteria is much useful in Bayesian design.

2.1.9 Stochastic processes

A related development has taken place in the literature of control theory where there has been appreciable interest in design for system identification and parameter estimation. Many of these results are collected by Goodwin et.al. (1977). An introduction to control theory for statisticians and a survey of the design literature in the field is given by Titterington. Dodge et.al. (1988) had developed in great detail the stochastic processes model. A stochastic optimal way is presented
by Singh (1989) where probability of obtaining at least specified yields on components and monetary returns, have been compared for the shared system of sole crops and intercrops, using two practical examples.

2.1.10 Factorial designs

Patterson describes ‘design key’, a method which has been in use for several years at Rothamsted and Edinburgh. This is incorporated in a computer program which generates the design. An introduction, intended for practising statisticians, is given by Patterson and Bailey (1978). This method requires the specification not only of the factorial structure of the treatments but also of the plots factors.

Given the design key, it is relatively easy to identify confounded effects. A more difficult problem is to find the design key from the list of effects to be confounded. This problem is solved by Bailey (1977) using the framework and notation of Bailey et al. An alternative approach to the generation of factorial designs when blocking is required, based on generalized cyclic designs, is summarized by John (1980). The method was developed for factorial structure in a block design and has to be modified for row and column designs or split plot designs. However, the specification of the generalized cyclic design usually appears much simpler than that of the design key for the same design. Cotter (1975) describes a method of obtaining partial confounding with these designs.

The block designs can also be used to provide blocking patterns for factorial experiments. Cotter describes the construction and analysis of designs with both complete and partial confounding. Lewis (1979) studies the construction of resolution III factorial designs from generalized cyclic designs. A very different approach is that of Mitchell et al. who use DETMAX to find optimum fractions of $3^k$-factorials.

The full $2^p$-factorials and their fractions are known to be A-, D- and E-optimum. Their properties are closely related to weighing designs, from which they are formed by squaring the first column of the design matrix. These designs require that $n=2^p$. Designs with similar desirable properties can be obtained if $n$ is a
multiple of 4 by the use of Hadamard matrices. Cheng (1980) reviews this theory and proves the general optimality theorems for designs based on balanced arrays. Galil et.al. (1980) give a thorough discussion of weighing designs for k-objects and hence of two level designs for the p=k-1 factor first order model. A review of the necessary theory of Hadamard matrices is given by Hedayat et.al. (1978) including examples of the matrices generously printed in full. Inspection of these materials shows that cyclic methods of construction can be employed leading to convenient algorithms for computer use.

Kuwada (1988) presents A-optimal partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution V, with $4 \leq m_1 + m_2 \leq 6$. In addition, A-optimal designs with $m_1=m_2=3$ are presented for $42 \leq N \leq 64$, where N is the number of observations of resolution VII. He also described that A-optimal balanced fractional $2^{m}$ factorial designs of resolution V with two blocks, $4 \leq m \leq 6$. Pesotan et. al., (1988), presents an invariance and randomization in factorial designs with applications to D-optimal main effect designs of the symmetrical factorial. Kolyva-Machera (1989) shows that D-optimal functions of three-level factorial designs for k-factors are constructed for factorial effects models. In particular, the information matrix of the main effect model is studied in a result characterizing optimum designs, when N=1, mod (9).

2.2. Miscellaneous Topics

a) Robust designs

Over the past decade there has been enormous interest in robust inferences which do not depend crucially on departures from the assumed model. Box et.al. list 4 properties of a good response surface design. The one they take as a definition of robustness is that the design be insensitive to wild observations. Hedayat et.al. discuss the model robust optimal designs for comparing test treatments with a control. They are simultaneously A-and M.V.-optimal for either one-way or two-way elimination of heterogeneity when the model of response is homoscedastic and linear additive. Bhaumik et.al. (1991) investigates the optimality and robustness to the unavailability of blocks in block designs.
b) Auto correlated models

When the process \( Y(\cdot) \) is autocorrelated, the optimal design problems become more complex and research is proceeding in several directions. Estimation of the parameters \( \theta \) is more usual in the analysis of field experiments in which "uniformity" gives rise to spatial autocorrelation. It has generated a mathematically exciting area of "neighbour designs", that takes into account where treatments appear next to which and how often. For example in block design we may count neighbours inside a block if there is autocorrelation in a block.

Ylvisaker (1987) presents a review of the field of spatial sampling with references to earlier joint work with Sacks and connections with Bayesian experimental design Chaloner, (1984). Applications to computational experiments are described in Sacks et.al. (1987) The strong connection with Kriging in earth sciences is also to be noted.

c) Low dose Carcinogenicity

One problem in low-dose carcinogenicity is that of extrapolation from experimental doses to the levels experienced in daily life. A review of problems and progress is given by Crump (1979). Such extrapolations are, unfortunately, highly dependent model. Hoel et.al. (1979) apply aspects of optimum design theory to extrapolation for a non-linear model. The resulting designs are shown to be over three times as efficient as the equal weighting designs reported in the literature.

d) Designs for discrimination between models

Designs for discrimination between models are concerned with the decision as to which, of two or more, models correctly describes the data. The literature has recently been described by Hill (1978). Even if the models are linear the T-optimum designs of Atkinson et.al. (1975) depend upon the parameters of the unknown true model. Jones et.al. (1978) using this nesting approach, develop nonsequential alternatives to T-optimum designs. The designs which depend upon the functions of
the noncentrality parameter over contours of constant distance between the two models. The resulting criteria come within the framework of optimum design theory.

e) Systematic and spatial designs.

Much of the impetus for the development of new designs continues to come from agricultural field trials. One theme which has recently re-emerged is that of systematic designs in which allowance is made for the structure of the error. In one form of analysis, due to Papadakis, the yield of each plot is adjusted by the average residual of the adjacent plots. Atkinson (1969) investigates the properties of the method for the one-dimensional designs of Williams (1952). Bartlett considers the method for the more challenging two-dimensional case of field trials. In the discussion of Bartlett's paper, several authors present systematic designs for the use with the method in two dimensions.

If two or more species are grown together, by intention or by accident, the effect of spacing may depend also on competition from weeds, from plants of the same genotype or from a crop of a different species. A review of competition experiments is given by Mead (1979).

2.3. Some important concepts and definitions on optimal designs.

Now we present the essential results known so far regarding optimality of certain class of block designs with respect to (particularly to) E-optimality criteria. The class of designs denoted by \( \mathcal{D}(b,v,k) \) will comprise of all connected designs in which \( v \) treatments are compared in \( b \) blocks each of size \( k \). With blocks of unequal sizes, the assumptions of homogeneity of error variances is itself questionable and further, the E-optimality results are seen to depend on too many design parameters.

2.3. Universal optimality of the (Balanced Block Designs) (BBDs)

2.3.1 Definition. Kiefer (1958) defined Balanced Block Designs (BBD) as below:
For a given b, v, and k, a design with incidence matrix $N=(n_{ij})$ is said to be a Balanced Block Design (BBD) if the elements of incidence matrix $N$ satisfy the following conditions:

\[
\begin{align*}
\text{i)} & \quad n_{ij} = \lfloor k/v \rfloor \text{ or } \lfloor k/v \rfloor + 1 \quad (2.3.1) \\
\text{where } \lfloor x \rfloor = \text{largest integer not exceeding } x; \quad x = k/v \\
\text{ii)} & \quad r_i = \sum_{j=1}^{b} n_{ij} \text{ is the same for all } 1 \leq i \leq v \quad (2.3.2) \\
\text{iii)} & \quad \lambda_{ii} = \sum_{j=1}^{b} n_{ij} n_{ij} \text{ is the same for all } 1 \leq i \neq i' \leq v \quad (2.3.3) \\
\text{(i) & (ii) gives} & \quad \lambda_{ii} = \sum_{j=1}^{b} n_{ij}^2 \text{ is the same for all } 1 \leq i \leq v \quad (2.3.4)
\end{align*}
\]

A BBD reduces to a Balanced Incomplete Block Design (BIBD) when $k<v$. Also it reduces to a Randomized Block Design (RBD) when $k=v$. We have,

\[
N_{(v \times b)} = (k/v) J_{(v \times b)} + N^*_{(v \times b)} \text{ where } J_{(p \times q)} = 11' \text{ is a matrix of order } (p \times q) \text{ with all elements unity, and } N^*_{(v \times b)} \text{ is the incidence matrix of a BIBD. Further, } r=b.k/v \text{ is an integer. Extending the above definition and the following definition given by Soundarapandian (1980a), we now define Balanced n-ary Block design as below:}
\]

2.3.2 Definition For a given B, V, and K, a n-ary design with incidence matrix $N=((n_{ij}))$ is said to be a Balanced n-ary Block Design (BNBD), if the elements of $N$ satisfy the following:

\[
\begin{align*}
\text{i)} & \quad n_{ij} = (K/V) \text{ or } (K/V) + 1 \quad (2.3.6) \quad \text{where } (x) = \text{largest integer not exceeding } x; \\
\text{ii)} & \quad R_i = \sum_{j=1}^{B} n_{ij} \text{ is the same for all } 1 \leq i \leq V \quad (2.3.7) \\
\text{iii)} & \quad \Lambda_{ii} = \sum_{j=1}^{B} n_{ij} n_{ij} \text{ is the same for all } 1 \leq i \neq i' \leq V \text{ ; and} \quad (2.3.8)
\end{align*}
\]
iv) \( \Lambda_{ii} = \sum_{j=1}^{p} n_{ij}^2 \) is the same for all \( 1 \leq i \leq V \) \hspace{1cm} (2.3.9)

2.3.3 General optimal design Let us denote by \( \phi \), the class of optimality functionals \( \phi \) defined on the members of \( \xi \). Let \( g \in S_v \), the symmetric group of permutations of order \( v \), and let \( G_g \) denote the corresponding permutation matrix, (i.e.) the matrix obtained by applying \( g \) to the columns of the identity matrix. Let \( C_g = G_g' CG_g \). It is easy to see that \( C_g \) corresponds to a variant of the design based on the permutation \( g \) applied to the set of treatments.

The minimum requirements to be satisfied by the optimality functionals \( \phi \) are:

i) \( \phi(C) = \phi(C_g) \) for every member \( g \) of the symmetric group of permutations. \hspace{1cm} (2.3.10)

ii) \( C_1 \geq C_2 \Rightarrow \phi(C_1) \leq \phi(C_2) \) for two information matrices \( C_1 \) and \( C_2 \). \hspace{1cm} (2.3.11)

iii) \( C_2 \) is \( \phi \) - better than \( C_1 \) iff so are \( t \) copies of the former design compared to \( t \) copies of the latter.

Mathematically,

\[ \phi(C_1) \geq \phi(C_2) \iff \phi(t \ C_1) \geq \phi(t \ C_2) \] for all positive integers \( t \geq 1 \) \hspace{1cm} (2.3.12)

The combination of various forms of \( d \) would do rather better than an exclusive use of \( d \) itself. That is,

iv) \( \phi \left( \sum_{t \ g} G_g' CG_g \right) \leq \phi \left( \sum_{t \ g} C \right) \) for all non-negative integers \( t_g \)'s at least one \( t_g \) being positive, and \hspace{1cm} (2.3.13)

iv') \( \phi \left( \sum_{g \ g} G_g' CG_g \right) \leq \phi \left( v(C) \right) \) \hspace{1cm} (2.3.14)

A design is optimal in a very general sense if the underlying \( C \)-matrix minimizes each optimality functional \( \phi \) (in the class \( \phi \)) satisfying the requirements (i)-(iv).
From (i), (ii) and (iii), the following are seen to be progressively weaker:

1. $\phi$ is convex in the usual sense.
2. $\phi$ is monotone increasing function of a convex function.
3. $\phi$ is weakly convex in the sense (iv)
4. $\phi$ satisfies the symmetry requirement (iv)'

Then we can state

$$\phi_c \subseteq \phi_{ic} \subseteq \phi_{wc} \subseteq \phi_s \quad (2.3.15)$$

where

$$\phi_c = \text{class of convex functionals } = \{\phi : \phi \text{ satisfies (i), (ii), (iii), (iv)'}\}$$
$$\phi_{ic} = \text{class of increasing functions of convex functionals.}$$
$$= \{ (\phi : \phi (C) = g (f(C)) ; \ g \uparrow \text{ and } f \text{ satisfies (i), (ii), (iii), (iv)' } \};$$
$$\phi_{wc} = \text{class of weakly convex functionals}$$
$$= \{ \phi : \phi \text{ satisfies (i), (ii), (iii), (iv)' } \};$$
$$\phi_s = \text{class of symmetric functionals}$$
$$= \{ \phi : \phi \text{ satisfies (i), (ii), (iii), (iv)' } \};$$

2.3.4 Universal Optimality (Kiefer-1975)

Consider $\phi$ defined on the set of all C-matrices which satisfy (i), (ii), (iii), and (iv)' $\phi(tC)$ is non-increasing in $t$, $t \geq 0$ and the condition of convexity in the usual sense i.e.,

(iv)'' $\phi [\alpha C_1 + (1-\alpha) C_2] \leq \alpha \phi (C_1) + (1-\alpha) \phi (C_2) \quad (2.3.16)$

for $0 < \alpha < 1$ and for any pair of C-matrices $C_1$ and $C_2$.

If a design is optimal with respect to all such optimality functionals $\phi$, it is said to be universally optimal.
2.3.5 Extended Universal Optimality

An extended universal optimality design is one among the class of designs, if the underlying C-matrix minimizes every optimality functional \( \phi \) satisfying (i), (ii), (iii) and (iv)' among all C-matrices for designs in that class.

Kiefer's (1975) Proposition I in a modified form may be restated below without proof:

2.3.6 Restated Proposition 1

If there is a completely symmetric feasible C- matrix, then the underlying design is universally optimal in the extended sense of minimizing \( \phi \) simultaneously for all functionals \( \phi \) satisfying (i), (ii), (iii) and (iv)'.

Yeh (1986) suggests another set of sufficient conditions for universal optimality in Kiefer's sense. Historically, Smith (1918) appears to be the first to formally introduce a specific optimality criterion in comparing designs in a given experimental set-up. Smith dealt with regression designs and suggested a response function criterion which we are not interested in.

2.3.7. Extended Universal Optimality of BBDs and BNBDs.

From Kiefer's (1975) Resolution stated above, it is enough to verify that the C-matrix of a BBD or BNBD with parameters \((b,v,k)\) or \((B,V,K)\) respectively is completely symmetric (c.s.) and it maximizes \( \text{tr}(C) \) among all designs in \( \mathcal{D}(b,v,k) \) or \( \mathcal{D}(B,V,K) \) respectively.

For BBD, \( C = r^8 - k^{-1}NN^1 \), where \( r^8 = \text{diag.} \ (r_1, r_2, \ldots, r_v) \) and

\[ l = (1,1, \ldots, 1)^1, \] it follows that for any design in \( \mathcal{D}(b,v,k) \)

\[ C_{ii} = \sum_j n_{ij} = \sum_j r_{ij}/k = r_i - s_i/k \quad \text{and} \]

\[ C_{ir} = \sum_j n_{ir}/k = s_r/k \quad (2.3.17) \]
For BNBD, \( C = R^5 - K^{-1}N' \), where \( R^5 = \text{diag.} (R_1, R_2, \ldots, R_v) \) and \( 1 = (1, 1, \ldots, 1)' \), it follows that for any design in \( \mathcal{D}(B, V, K) \)

\[
C_{ii} = \sum n_{ij} - \frac{\sum n_{ij}^2}{K} = R_i \cdot \frac{\Lambda_{ii}}{K}
\]

and

\[
C_{ij'} = \sum n_{ij} n_{ij'}/K = -\frac{\Lambda_{ij'}}{K}
\]

Thus we have in BBD and BNBD that \( C^* \) has diagonal elements all equal and also off-diagonal elements all equal. Hence it is completely symmetric (c.s.).

### 2.3.8 A Lemma and a Theorem for Generalized optimality

Conniffe et.al. (1974) minimize \( \Sigma f(x_i) \) from a class \( \mathcal{D}(b, v, k) \) by two stages which are presented below in a Lemma and a Theorem without giving the proofs.

#### 2.3.9 Lemma

For a fixed \( A = \sum_{i=1}^{v-1} f(x_i) \) and \( B = \sum_{i=1}^{v-1} f(x_i^2) \) subject to \( A > B > A^2/(v-1) \), for any function \( f \) satisfying

\[
\psi_f = \sum_{i=1}^{v-1} f(x_i) \]

attains its minimum when exactly one of the \( x_i \)'s assumes the value \( \{[A + \delta (v-2)P] / (v-1)\} \) and the rest are each equal to \( \{A - \delta P\} / (v-1) \) where

\[
P^2 = B - A^2/(v-1) \quad \text{and} \quad \delta = \sqrt{(v-1)/(v-2)}.
\]

#### 2.3.10 Theorem

Suppose there exists a design \( d^* \) in \( \mathcal{D}(b, v, k) \) such that its C-matrix has roots

\[
\mu^* = \{A^* + \delta (v-2) P^*\} / (v-1) \quad \text{with multiplicity 1.}
\]

\[
\mu^{**} = \{A^* + \delta P^*\} / (v-1) \quad \text{with multiplicity (v-2)}
\]

where \( A^* \) and \( P^* \) are the corresponding values of \( A \) and \( P \) for the design \( d^* \) and \( \delta = \sqrt{(v-1)/(v-2)} \)

Suppose further that

\[
(i) \quad A^* \geq A
\]
(ii) \( A^* - \delta P^* \geq A - \delta P^* \)

for all pairs \((A, P)\) based on competing designs in \( \mathcal{D} (b, v, k) \)

Then \( d^* \) is optimal with respect to every generalized optimality criterion.

The above lemma and theorem can be extended to BNBD’s also without any difficulty. The statements for BNBD’s are not presented because which are obvious extension of binary designs to n-ary designs.

2.3.11. Optimality of Dual Binary and n-Ary Designs

The designs \( d \) and \( \bar{d} \) with incidence matrices \( N \) and \( \bar{N} \) respectively are said to be duals of each other, when \( N \) and \( \bar{N} \) are transposes of each other.

Among the class \( \mathcal{D} (b, v, k) \) of connected designs, the dual design \( \bar{d} \) and \( d \) belongs to the class \( \mathcal{D} (b=v, v=b, k=r, r=k) \) of connected equireplicate designs. Here \( N=\bar{N} \), then we obtain

\[
C = r I_v - k^{-1} N N'^T
\]

Let the design values of \( C \) be \( x_1, x_2, \ldots, x_{v-1} \) and those of \( \bar{C} \) be \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{b-1} \) for binary designs. Then

\[
\text{for } v < b, \quad \bar{x}_i = \frac{r}{k} x_i \quad 1 \leq i \leq v - 1
\]

\[
= k \quad v \leq i \leq b - 1
\]  

\[
\text{for } v > b, \quad x_i = \frac{v}{k} \bar{x}_i \quad 1 \leq i \leq b - 1
\]  

\[
(2.3.21)
\]  

For BNB design in the class \( \mathcal{D} (B, V, K) \) and \( \mathcal{D} (B=V, V=B, K=R, R=K) \) of connected equireplicated n-ary designs.

Since \( N=\bar{N} \), then we have

\[
C = R I_v - K^{-1} N N'^T
\]

\[
\bar{C} = \bar{R} I_v - \bar{K}^{-1} \bar{N} \bar{N}' = K I_B - R^{-1} N N
\]  

\[
(2.3.22)
\]  

Let the design values of \( C \) be \( x_1, x_2, \ldots, x_{v-1} \) and those of \( \bar{C} \) be \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{b-1} \) for binary designs. Then

\[
\text{for } v < b, \quad \bar{x}_i = \frac{r}{k} x_i \quad 1 \leq i \leq v - 1
\]

\[
= k \quad v \leq i \leq b - 1
\]  

\[
\text{for } v > b, \quad x_i = \frac{v}{k} \bar{x}_i \quad 1 \leq i \leq b - 1
\]  

\[
(2.3.23)
\]
The results for BNB design and its C-matrix can easily be written taking into consideration the eigen values of C and \( \overline{C} \) of n-ary designs.

Cheng (1981) points out an interesting relation between the C-matrix of a design with blocks of size \( k \) and that of a design in which every block of the original design is replaced by \( k.C_2 \) blocks consisting of the pairs formed out of the treatments in that block where \( C_2 \) is the information matrix.

2.3.12 Definition of RGD

For given \( b,v \) and \( k \) the Regular Graph Design is a block design for which

(i) \( n_{ij} = \lfloor k/v \rfloor \) or \( \lfloor k/v \rfloor + 1 \) \( 1 \leq i \leq v, \ 1 \leq j \leq b \);

(ii) \( r_i \)'s are all equal, where \( r_i = \sum_{j=1}^{b} n_{ij} \)

(iii) \( |\lambda_{ii'} - \lambda_{uu'}| \leq 1 \) for all \( i \neq i', u \neq u' \)

Here (ii) requires \( bk/v \) is an integer. When \( bk/v \) is not an integer, we replace (ii) and get (ii)' as:

(ii)' \( |r_i - r_{i'}| \leq 1 \) for all \( 1 < i \neq i' \leq v \)

Jacroux names Semi Regular Graph Designs (SRGDs) to those designs satisfying (i), (ii) and (iii). When \( bk/v \) is an integer, SRGD reduces to an RGD. Cheng et.al. (1981) have named these designs as nearly balanced incomplete block designs.

John et.al. (1977) observe that RGDs are M.S.-optimal, that is

\[
\text{tr.} \ (k.C) = \sum_{i} (kr_i - \lambda_{ii}) \text{ and tr} \ (k^2C^2) = \sum_{i} (kr_i - \lambda_{ii})^2 + \sum_{i \neq i'} \lambda_{ii'}^2
\]

so that an RGD maximizes \( \text{tr}(kC) \) and subject to this, minimizes each of \( \sum (kr_i - \lambda_{ii})^2 \) and \( \sum \lambda_{ii'}^2 \).

In some cases a design other than RGD has the same value for the smallest eigenvalue of the C-matrix as the best RGD and hence an RGD is not uniquely
E-optimal. We also note in that paper that the dual design of an RGD is not an
RGD$^*$ and RGD does not exist. At this context, we can say that further conjectures
have been developed by John et.al. Patterson (1983) and Wild (1987).

Pal et.al (1988), and Lee et.al. (1987a, b, c) have developed some results
on Universal and specific optimality of block designs with unequal block sizes and
in some cases for unequal replicates also. We further discuss these designs with
$(k_1,k_2,...,k_s; b_1,b_2,...,b_s)$ of unequal block sizes and unequal replicates with
E-optimality.

We review the current literature available on BIB designs, BTB designs,
PBIB designs and PBTB designs during 1999. Patwardhan et.al. (1994) have extended
the Nigam's technique of $(p+s-1)$-ary designs from the incidence matrices of balanced
$p$-ary and $s$-ary designs. Here they constructed partially balanced $n$-ary and
generalized partially balanced incomplete block designs. Ghosh et.al.(1994) have
developed various patterned methods for construction of Variance Balanced Ternary
(VBT) designs and Efficiency Balanced Ternary (EBT) designs.

2.4 Brief review of n-Ary Block Designs

Though Tocher (1952) introduced the concept of n-ary designs, no attempt
was made to develop systematic method of construction until 1967. For the first
time, Murthy et.al. (1967) introduced the systematic method of construction of n-ary
design by using a set of MOLS. They have also generalized the concept of BNB
designs. Further Das et.al. (1968) have developed an alternative method of
construction of BNB designs based on incidence matrix of BIB design.

Dey (1970) has presented a method of construction of BIB designs using
affine $\alpha$- resolvable balanced incomplete Block designs. He has constructed the BIB
designs by collapsing the blocks of an affine $\alpha$- resolvable BIB design two by two,
taking one from one $\alpha$- replicate and the second block from another $\alpha$- replicate. By
collapsing the blocks of BIB design evidently means the replacement of varieties of
the GD-PBIB design by the block contents of the BIB design: the rule being that the
i-th treatment of the GD design is replaced by the block contents of the i-th block of the BIB design. As such when the blocks of the BIB design are collapsed in a given number, the treatments of the GD design are in fact replaced by the block contents of the BIB design.

Saha et al. (1973) have made an attempt to construct BIB design through the method of generalized different set. BIB designs may be obtained by developing a single initial block and as such are symmetrical. A general class of BNB designs has also been obtained by developing an initial block. They also have proved that a generalized complement of a BNB design is also a balanced one.

Nigam (1974) has evolved some methods for constructing BNB designs by using the incidence matrix of a BIB. He has shown that BNB design can be obtained through any two p-ary and (n-p+1)-ary balanced block designs. The ternary designs of Dey (1970) turn out to be the particular case of Nigam's ternary designs. Two methods of construction of BIB design (with k<v) have been described by Saha (1975). The first method of construction of BIB design is simple by considering a BIBD with even number of varieties relabelling every pair of subsequent varieties as a single new variety, to get the BIB design for v/2 treatments, while the second method uses, "initial block". The second method is based on a result of Saha et al. (1973) on construction of BTB designs using differences. Some BIBDs in useful range are also reported.

Sharma and Agarwal (1976) have obtained a series of BNB designs by collapsing certain (n-1) tuples of blocks of a BIB design. From the general approach, Dey's results can be obtained. Economy in number of blocks is arrived in Dey's complementary n-ary designs. Morgan (1977) has constructed some families of n-ary designs from (i) t-designs with t≥3; (ii) finite projective planes and (iii) symmetric BIBD's. He has shown that the dual of a SBTB design is a SBTB design with the same parameters. In his second paper, Morgan (1978) has given two methods of construction of B(N+1)B designs from a set of m BIBDs having same set of varieties.
Shafiq et.al. (1979) have extended the concept of N-ary balanced block designs (here N and n has no difference representing the same ary number) where the incidence matrix \( n \) contains the N values, \( 0,1,2,\ldots,N-1 \), to generalized N-ary balanced block designs, where the incidence matrix \( n^* \) contains the N values \( m_a \) for \( a=0, 1, 2,\ldots, N-1 \), for \( m_a = m_1 - (a-1) m_0 \) and for \( m_0 \) and \( m_1 \) satisfying \( 0 < m_0 \leq m_1 \). For ternary designs \( m_2 = 2 m_1 m_0 \) and \( 0 \leq m_0 < m_1 < m_2 \). Given a fixed number \( v \) of treatments and a fixed total number \( N^* \) of experimental units, a class of N-ary balanced block designs with a different set of \( m_a \) is possible. They have also developed criteria to select the designs in the class with the smallest variance of a contrast.

Tyagi et.al. (1979), in their note, have suggested some modifications to Nigam's (1974) method to effect reduction in the number of blocks of the ternary (n-ary) designs, where as Surendran et.al. (1979) have suggested a new method of construction of proper BNB designs from associate designs. Recently in 1980, Kageyama had completely characterized a BNB design with RK=AV of Tocher (1952). It is noted that \( B \geq V \) holds for a non-trivial BNB design, where \( B \) is the number of blocks and \( V \) and number of treatments.

Rao (1955) has shown that the necessary and sufficient condition for any design to be a balanced, is that, the C-matrix of the adjusted intra-block normal equations, shall have equal diagonal elements and all the off-diagonal elements are also equal. John (1964) has given simple proof with examples. Pearce (1964), Calinski (1971) and Calinski et.al. (1974) have associated the use, of the designs in varying replications and unequal block sizes. Kuishreshtha et al., (1972) have presented method of construction of balanced binary and ternary block designs with two block sizes and varying replications. Considering the nearly BIB designs of Nigam, et.al. (1977) have generalized the work of Kuishreshtha et al., (1972) to construct BNB designs with more than two block sizes and unequal replications by using (n-1) BIBD's with some number of treatments. These n-ary designs, however involve too many replications. To overcome these limitations, "nearly balanced" designs have
been suggested by the above authors, which while requiring much less number of replications, are seen to be almost as efficient as the totally balanced designs.

As a second and important review work we consider partially balanced n-ary block design (PBNB). Recollecting basic concepts on PBIB and BNB designs, Paik et.al. (1973) and later Mehta, et.al. (1975) have defined partially balanced n-ary block designs as a generalization of PBIB (binary) designs. The PBIB constructed by the later authors from PBIB designs do not change the type and order of the association scheme of the original PBIB designs. The note of Agarwal (1977) on the construction of PBTB designs has utilized the difference sets concept of Bose et.al (1939).

Soundarapandian (1980 a) studies the methods of construction of partially balanced n-ary block designs using difference sets. Some more new series of PBNB designs were developed from initial blocks. By that method a general class of partially balanced n-ary block designs has also been obtained. In his paper, Soundarapandian (1979a) has given two new methods of construction of 2 associates PBIB and PBNB designs, one on Triangular association scheme and another on Latin square type with the constraints type of family association scheme, whereas Soundarapandian et.al. (1979), in their second paper have given a simple proof of the inequality $b > v$, by generalizing the proof given by Bose (1949) for BIBD.

"On a property of balanced designs", Soundarapandian (1980b) proved the necessary and sufficient condition for a design to be balanced, that is, the adjusted intra-block normal equations and its C-matrix should have all its off-diagonal elements equal and this in turn gives that all the diagonal elements to be equal. Soundarapandian (1980c) developed the concept of N-ary partially balanced incomplete block designs from any binary partially balanced scheme, when the incidence matrix $n$ contains the N values 0,1,2,...,N-1 is extended to generalized N-ary partially balanced block designs where the new incidence matrix $n^*$ contains the N values $m_a$, for $a = 0,1,2,...,N-1$ for $m_a = a - (a-1)$, $m_{b'}$, and for any $m_0$ and $m_1$ satisfying $0 \leq m_0 < m_1$. Criteria are developed to select the two associate scheme
partially balanced n-ary block design(s) in the class with smallest variance of a contrast.

Soundarapandian's (1980d) concept of symmetrical n-ary unequal block (SNUB) arrangements provides extension to binary symmetrical unequal block (SUB) arrangements consisting of the incidence matrix $N$ with $n$ elements $0,1,2,\ldots, (n-1)$ and following the two block association class property, is a completely balanced arrangement with constant $\Lambda$ and $R$, but with different block sizes. Various methods of construction of SNUB arrangements with two unequal block sizes have been studied in detail in this paper. Many new series of SNUB arrangements are obtained first by using two associate class PBIB designs like (i) Group divisible, (ii) Triangular and (iii) Latin square type designs; secondly from (iv) Affine-resolvable BIB designs and (v) GD-PBIB designs with BIB designs and finally from the method of finite differences with two type of initial blocks. Ternary and n-ary block designs have also been used extensively in the construction of SNUB arrangements.

Utilizing the definition and using the association scheme given by Adhikary, (1965), Soundarapandian (1981a) gives a new method of construction of higher associate cyclical partially balanced ternary designs by allowing the treatments to occur more than at least once in any of the initial block. In his paper "Some properties of balanced n-ary designs", Soundarapandian (1981b) states the necessary and sufficient condition for a block design by using pseudo inverses. After giving some of the properties of symmetrical balanced n-ary block designs, this paper shows that an equi-replicated n-ary balanced block design with $B=V$ is symmetrical balanced n-ary block design. A theorem from n-ary to binary designs is stated and proved.

Soundarapandian, et.al. (1981 c) present an extension of Q-method of analysis for binary design given by Rao (1956) to n-ary balanced and partially balanced n-ary stock designs. Here a linked n-ary block (LNB) design is defined as the dual of balanced n-ary block (BNB) design. Having a note on Yate's (1936, 1940) method of P-analysis, Soundarapandian (1981c) further extended the expressions for binary linked block (LB) designs given by Rao (1956) to linked n-ary block (LNB)
designs which admit easy estimation of parameters for these type of all n-ary block designs.

The paper like (i) "New construction of n-ary design", (ii) On linked n-ary block designs", (iii) "On partially linked n-ary block designs", (iv) "Extended generalized balanced designs", (v) "Extended generalized partially balanced designs", (vi) "On doubly balanced n-ary designs", (vii) Analysis of doubly balance n-ary designs", (vii) Analysis of doubly balanced n-ary designs, (viii) On Kronecker product rectangular partially balanced n-ary block designs and rectangular PBIB designs through them, (ix) Construction of rectangular PBTB designs from families of difference sets, were presented by Soundara Pandian (1981.b) in his thesis on "Some contributions to the theory of construction and analysis of n-ary designs" were reviewed by the author and utilized for our present work. The above papers act as the guiding spirit to proceed to the latest developments on A-optimal n-ary block designs, which are currently appreciated by the teachers on optimal designs. The papers presented on various seminars have also been reviewed for our present work.

Some of the papers of Soundarapandian et.al., (1996 a,b,) which were presented in the Indian Science Congress, Statistics session were also reviewed. Shah et.al. (1989) have described a method of construction of balanced n-ary block (BNB) designs, using Mutual Orthogonal Latin Squares (MOLS). To fill the gap between theory and practice of modern statistical designs. Agarwal et.al. (1990) have presented the use of n-ary block designs in the evaluation of balanced incomplete block (BIB) designs for all the four Griffing's complete diallel crosses (GDC) systems.

Pattwardhan et.al. (1994), in their paper "Some construction of BIBDs, BTDs and generalized 4-PBIBDs have extended the Nigam's technique of construction of (p+s-1) - ary designs through incidence matrices, to construct partially balanced n-ary designs and generalized partially balanced incomplete block designs.

Further research works on optimality criteria for block designs have been developed by the author. In these works, A-optimality criteria of blocks designs are given importance. The research work moves a long way from proper designs to
unequal replicate as well as unequal block sized designs. The research work further extended to partially balanced n-ary block (PBNB) designs of Soundarapandian (1980a) as well as to A-optimal n-ary block designs under heteroscedastic and homoscedastic model settings. The various bounds for the smallest positive eigenvalue of C-matrices of our n-ary block designs given in different theorem forms and the various contributions, constitute the important theme of our thesis which is developed step by step in different chapters.