Chapter 6

Construction of Partially Balanced Treatment
Incomplete Block and Partially Balanced
Treatment $N$-Ary Block Designs
Chapter-6

CONSTRUCTION OF PARTIALLY BALANCED TREATMENT INCOMPLETE BLOCK AND PARTIALLY BALANCED TREATMENT N-ARY BLOCK DESIGNS

6.1 INTRODUCTION. According to Bechhofer and Tamhane (1981) a balanced treatment incomplete block design (BTIB) is defined to be a design which satisfies

$$\lambda_{01} = \lambda_{02} = \ldots = \lambda_{0p} = \lambda_0 \text{ (say)} \; \lambda_{12} = \lambda_{13} = \ldots = \lambda_{(p-1),p} = \lambda_1 \text{ (say)} \quad (6.1.1)$$

and a balanced treatment n-ary block (BTNB) design is defined to be a design which satisfies,

$$\Lambda_{01} = \Lambda_{02} = \ldots = \Lambda_{0p} = \Lambda_0 \text{ (say)} \; \text{and} \; \Lambda_{12} = \Lambda_{13} = \ldots = \Lambda_{(p-1),p} = \Lambda_1 \text{ (say)} \quad (6.1.2)$$

We proceed for the construction and optimality of partially balanced treatment incomplete block (PBTIB) designs. Tocher (1952) defined a balanced n-ary block (BNB) design as an arrangement of V treatments in B blocks each of size K, such that the i-th treatment occurs in the j-th block n_{ij} times, and altogether R-times, where n_{ij} can take values 0,1,2, \ldots , (n-1). We say, the design is variance balanced if the inner product of any two row vectors of the incidence matrix N_{VxB}

of the n-ary design, \( \sum_{j=1}^{B} n_{ij} n_{kj} \) is a constant and equal to \( \Lambda \) (say), for all \( i \neq k = 1, 2, \ldots V \). This also implies that \( \sum_{j=1}^{B} n_{ij}^2 = \Delta \) (say another constant) for \( i = 1, 2, \ldots V \).

Utilizing the above definition, we have already defined BTNB designs in definition 5..1.1 which satisfies the above conditions (6.1.2) as

$$\Lambda_0 = \sum_{j=1}^{B} n_{0j} n_{kj} \text{ is a constant for } 0 \neq k ; k = 1, 2, 3 \ldots p \text{ and}$$

\[
\Lambda_1 = \sum_{j=1}^{B} n_{ij} n_{kj} \text{ as another constant, } i \neq k ; \ k = 1, 2, \ldots p \quad (6.1.3)
\]

Generalizing the definition of ternary block design of Paik and Federer (1973), Mehta et al. (1975) and extending to n-ary block design of Soundarapandian (1980.a), we now define a partially balanced n-ary block (PBNB) design as follows:

**6.1.1. Definition:** A block design with V treatments and B blocks is said to be a PBNB design with m-associate classes if

(i) the incidence matrix \(N_{VxB}\) has an entries 0,1,2, \ldots \(n-1\)

(ii) \(\sum_{i=1}^{V} n_{ij} = K\) for every \(j: 1,2,\ldots , B;\)

(iii) \(\sum_{j=1}^{B} n_{ij} = B\) and \(\sum_{j=1}^{B} n_{ij}^2 = \Delta\) for every \(i=1,2,\ldots V\) \quad (6.1.4)

(iv) There exists a relationship between two treatments defined as:

(a) any two treatments are either 1st, 2nd, \ldots m-th associates, then the relation of associates being symmetrical,

(b) each treatment, \(\alpha\) has \(m_\alpha\), \(\alpha\)-th associates, then the number of treatments that an \(j\)-th associates of \(\alpha\) and \(k\)-th associate of \(d\) is and is independent of the pairs of \(\alpha\)-th associate \(\alpha\) and \(d\).

(v) The inner product of any two rows of \(N\), ie \(\sum_{j=1}^{B} n_{ij} n_{kj} = \Lambda_\alpha\) if \(i\) and \(k\) are mutually \(\alpha\)-th associates, \(\alpha = 1, 2, \ldots m\).

Paik and Federer (1973) and Soundarapandian (1980.a) introduced PBNB design as a natural extension of BNB designs which has intuitively attract combinational properties and whose algebraic properties enabled efficiency factor to be easily calculated. More attention has been paid to PBNBDs with two associate classes, here after called PBNB(2) designs. More interesting results were arrived at...
6.2. Partially Balanced Treatment Incomplete Block Designs (PBTNBDs) and Partially Balanced Treatment n-Ary Block Designs (PBTNBDs)

We have seen that a BTIB designs and its extension to BTNBD design are such that each treatment appears in the same block with the control the same number of times (= \( \Lambda_o \)) over the design, and any pair of test treatment appears together (in the same block) the same total number of times (= \( \Lambda_1 \)) over the design. Here we are generally considering n-ary block design which will satisfy the binary incomplete block designs also. Taking into consideration of the definition of PBNB design with m-associate classes in definition 6.1.1, we now relax the constant condition

\[
\Lambda_{12} = \Lambda_{13} = \ldots = \Lambda_{(p-1),p} = \Lambda_1 \quad \text{and take then as}
\]

\[
\Lambda_{12} \neq \Lambda_{13} \neq \ldots \neq \Lambda_{(p-1),p} \quad \text{(or) they are in number of different \( \Lambda' \)s} \quad (6.2.1)
\]

ie. \( \Lambda_1, \Lambda_2, \ldots, \Lambda_m \) are of m-associate classes. Then we can define a PBTNBD design including binary design of PBTIB designs as given below taking into consideration of definition 6.1.1. As usual, our purpose is to design an efficient experiment to compare the \( p \) new treatments (labeled 1,2, ..... \( p \)) with the old one (here after called the control and labeled as 0) and to compare the new treatments among themselves. But the emphasis will be on the comparison of the treatments with the control, hence higher precision is desired for these estimates.

6.2.1. Definition. A block design with \( (V+1) \) treatments including the control treatment 0(zero) and \( B \) blocks is said to be a PBTNBD design with m-associate classes if

(i) the incidence matrix \( N_{VxB} \) has n entries 0,1,2,..... (n-1)

\[
V
\]

(ii) \( \sum_{i=1}^{n} n_{ij} = K \) for every \( j = 1, 2, \ldots, B \)
(iii) \( \sum_{j=1}^{B} n_{ij} = R \) and \( \sum_{j=1}^{B} n_{ij}^2 = \Delta \) for every \( i=1,2,\ldots,V \), except for the control zero treatment.

(iv) there exists a relationship between the test treatments defined as:

(a) any two treatments are either 1st, 2nd, \ldots or, \( m \)-th associates, then the relation of association being symmetrical.

(b) each treatment, (any first one, among 1,2,\ldots,p) has \( m_{\alpha} \), \( \alpha \)-th associates, then the number of treatments that are \( j \)-th associates of the above first and \( k \)-th associate of other treatment (any second one among 1,2,\ldots,p) is \( \binom{p_{\alpha}}{j} \) and is independent of the pairs of \( \alpha \)-th associates the first and second treatments.

(v) The inner product of any two rows of \( N \) (ie) if \( i \) and \( k \) are mutually \( \alpha \)-th associate, \( \alpha = 1,2,\ldots,m \) except 0.

(vi) The inner product of any two rows of \( N \) (one row is control treatment) row) \( \sum_{j=1}^{B} n_{ij} n_{kj} = \Lambda_{ij} \) a constant for all \( k = 1,2,\ldots,p \).

Let \( C (B,K,p) \) denote the class of all possible \( n \)-ary block designs with \( B \) blocks of size \( K \) each, and \( (p+1) \) treatments indexed 0,1,2,\ldots,p; 0 being the control treatment.

As usual in PBTN design, for a design \( d \in C (B,K,p) \), the information matrix for estimating all \( (\alpha_0 - \alpha_i) \), \( 1 \leq i \leq p \) is the symmetric non-negative definite \( (p \times p) \) matrix \( M(d) \), whose \((i, L)\)-th entry is

\[
M_{iL}(d) = \begin{cases} 
\frac{1}{K} \sum_{j=1}^{B} n_{ij}^2(d) & (i = L) \\
-\frac{1}{K} \Lambda_{iL}(d) & (i \neq L) 
\end{cases}
\] (6.2.3)
where \( \sum_{j=1}^{B} n_{ij}(d) = R_i(d) \) = number of times i-th treatment appears in the entire design.

This matrix has all diagonal elements equal \( \frac{1}{K} \Lambda_0 + \frac{1}{K} \sum_{i=1}^{m} n_i \Lambda_i \) and \( n_i \) off-diagonal elements equal \( \frac{1}{K} \sum_{i=1}^{m} n_i \Lambda_i \).

Since \( M^{-1} \) is the variance-covariance matrix of the vector of estimates \( (\hat{\theta}_0 - \hat{\theta}_1, \hat{\theta}_0 - \hat{\theta}_2, \hat{\theta}_0 - \hat{\theta}_3, \ldots, \hat{\theta}_0 - \hat{\theta}_p) \), PBTNB design provides BLUEs \( \hat{\theta}_0 - \hat{\theta}_i \) (\( 1 \leq i \leq p \)) with the property that

\[
\text{Var}(\hat{\theta}_0 - \hat{\theta}_i) = \tau^2 \cdot \sigma^2, \quad (1 \leq i \leq p), \quad \tau^2 \text{ is a constant and}
\]

\[
\text{Cov}(\hat{\theta}_0 - \hat{\theta}_i, \hat{\theta}_0 - \hat{\theta}_k) = \rho_i, \quad (1 \leq j, k \leq p \text{ and } i = 1, 2, \ldots, m), \quad (6.2.4)
\]

The values of \( \tau^2 \) and \( \rho_i \) (\( i : 1, 2, \ldots, m \)) are given as

\[
\tau^2 = \frac{K (\Lambda_0 + \sum_{i=1}^{m} \Lambda_i)}{\Lambda_0 (\Lambda_0 + \sum_{i=1}^{m} \rho_i \Lambda_i)} \quad \text{and} \quad \rho_i = \frac{\sum_{i=1}^{m} n_i \Lambda_i}{(\Lambda_0 + \sum_{i=1}^{m} n_i \Lambda_i)} \quad (6.2.5)
\]

Now with all other definitions of association scheme of algebra and conditions of binary partially balanced incomplete block design (PBIBD) is now extended to \( n \)-ary PBTNB designs, with the modification of notations as capital letters instead of small letters.

We recall that a D-optimality criteria minimizes the determinant of the variance-covariance matrix \( M^{-1} \), and E-optimal design minimizes the maximum eigenvalue of \( M^{-1} \), and A-optimal design minimizes the trace of \( M^{-1} \), ie. minimizes

\[
\sum_{i=1}^{p} \text{Var}(\hat{\theta}_0 - \hat{\theta}_i), \quad i = 1, 2, \ldots, p.
\]
Since A-optimality has a natural statistical meaning and it picks up really desirable designs for the problem of multiple comparisons with the control, we will restrict our attention to this A-optimality criterion only.

6.3 Construction of PBTIB and PBTNB designs with illustrative examples.

As stated in chapter-4, in order to choose optimal designs for given $(K, p)$ in BTIB design, we need to construct the minimal complete class of generator designs (MCCGD). It is mentioned that they will not make it possible to construct MCCGD for given $(K, p)$. This problem is left unanswered there. Later Notz and Tamhane (1982) used a method to construct MCCGD for $k=5$, $p=3(1)10$. This method involves so extensive trial and error that, as they pointed out, it will not be practical for $k>3$. Also the inequalities they used to prove that the sets they have constructed are indeed MCCGD will not be adequate to obtain similar results for $k>3$. In chapter-4, section 2 onwards some concepts are introduced by which it becomes possible to construct the MCCGD for $k>3$, at least for moderate $k$ (say, $k \leq w$).

Using same methods of chapter-4, and taking into consideration of PBTIB and PBTNB design concepts given in the previous section, we now present a PBTIB design with parameters, i.e. $p=10$, $K=5$, the MCCGD is as given below:

<table>
<thead>
<tr>
<th>Design</th>
<th>Blocks ($B=5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>$D_1$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

Here $B = 5$, $A_0 = 2$, $A_1 = 1$, $A_2 = 0$, $n_1 = 6$, $n_2 = 3$, ary = 2 (i.e., binary)
Table 6.3.2 PBTIB design with parameters \( p=6, K=5 \). The MCCGD is as given below

<table>
<thead>
<tr>
<th>Design</th>
<th>Blocks (B = 4)</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_2 )</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
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<td>0</td>
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<td>4</td>
<td>4</td>
<td>5</td>
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<td></td>
<td></td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Here \( B = 4, A_0 = 2, A_1 = 1, A_2 = 0, n_1 = 4, n_2 = 1, \) ary = 3 (i.e., ternary)

Table 6.3.3. PBTIB design with parameters \( p=6, K=7 \). The MCCGD is as given below

<table>
<thead>
<tr>
<th>Design</th>
<th>Blocks (B = 4)</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_3 )</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>3</td>
<td>5</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Here \( B = 4, A_0 = 8, A_1 = 4, A_2 = 0, n_1 = 4, n_2 = 1, \) ary = 3 (i.e., ternary)

The necessary tools for determining that a certain set of PBTNB designs constitutes the MCCGD are not given because of non-availability of space in this thesis due to the same reason, only three examples were presented above. The minimal complete class can be provided for \( p = 6(1)10, k=5 \) and \( p=10(1)10, k=7 \) also
for other \( p \) and \( k \) values can be tried and constructed. Many more theorems and results can be worked out for treatment comparisons in our present designs, but for space and other restrictions those results will not be included in our present thesis. They will be published separately as research papers in due course.

In our separate publication, restricting \( m=2 \), we deal with PBTNBD(2) designs of the type Group Divisible-PBTNBD(2), Simple-PBTNBD(2), Triangular-PBTNBD(2), Latin square type-PBTNBD(2) and Cyclical-PBTNBD(2) designs. Their corresponding A-optimality criteria to choose \( R_0 \) and how should the allocation of \( R_0 \) controls to B blocks be performed so that our S-optimal designs can be obtained will be discussed in detail with tables in the subsequent publications. The vital role of computers in the tedious calculations can be appreciated in the future research works.