Chapter 6

Upper and Lower Bound Fuzzy Transportation Problems
In this chapter, a procedure to derive the fuzzy objective value of the fuzzy transportation problem is discussed, in which the cost coefficients of the demand and supply quantities are triangular fuzzy numbers. A pair of mathematical programs is formulated to calculate the upper and lower bounds of the total fuzzy transportation cost at each $\alpha$-level. The membership function of the objective value is also constructed from different values of $\alpha$-level. In conclusion, an application of fuzzy transportation problem is discussed.

6.1. INTRODUCTION

Shiang-Tai Liu and Chiang Kao [42] have already developed a
procedure to derive the fuzzy objective value of the fuzzy transportation problem, in which the cost coefficients and the demand and supply quantities are fuzzy numbers. This concept was based on the extension principle. A pair of mathematical programs were formulated to calculate the upper and lower bounds of the fuzzy total transportation cost at a possibility level, \( a \). From different values of \( a \), the membership function of the objective value was constructed. Two types of fuzzy transportation problems were discussed by Shiang-Tai Liu and Chiang Kao; one with equality constraints and the other with inequality constraints. It was found that the membership function of the objective value of the equality problem is contained in the inequality problem.

In this chapter, an algorithm is developed to express the objective value of a fuzzy transportation problem as a membership function rather than a crisp value, in which the unit shipping cost, demand and supply quantities are triangular fuzzy numbers. A pair of mathematical programs is formulated to calculate the upper and lower bounds of the objective value of
the fuzzy transportation problem with equality constraints at various \( \alpha \)-levels.

The proposed algorithm is found to have the following major advantages:

(i) The membership degree is established directly.
(ii) The upper bound and lower bound of each \( \alpha \)-cut of the fuzzy objective value are calculated.
(iii) The objective value of the problem is expressed by the membership function rather than by a crisp value.
(iv) It provides ample information for making decisions.

6.2. PROBLEM FORMULATION

The considered fuzzy general transportation problem (FGTP) is given below:
Minimize \( z = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{c}_{ij} x_{ij} \),

where \( \bar{s}_i = (s_1, s_2, s_3) \), \( \bar{d}_j = (d_1, d_2, d_3) \), \( \bar{c}_{ij} = (c_1, c_2, c_3) \)

subject to,

\[ \sum_{j=1}^{n} x_{ij} \leq \bar{s}_i \], where \( i = 1, 2, \ldots, m \) and

\[ \sum_{i=1}^{m} x_{ij} \geq \bar{d}_j \], where \( j = 1, 2, \ldots, n \) and

\( x_{ij} \geq 0 \), where \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \)

\( \alpha \)-FGTP:

Minimize \( z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \), where \( c_{ij} \in L_\alpha (\bar{c}_{ij}) \)

subject to,

\[ \sum_{j=1}^{n} x_{ij} \leq s_i \]
\( s_i \in L_\alpha (\bar{s}_i) \)

\[ \sum_{i=1}^{m} x_{ij} \geq d_j \]
\( d_j \in L_\alpha (\bar{d}_j) \)

\( x_{ij} \geq 0 \), for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \)
6.2.1. PROCEDURE FOR COMPUTING LOWER BOUND

\( \alpha \)-FGTP is rewritten in the equivalent form as given below:

\[
\begin{align*}
\text{Minimize } & z_{\alpha}^L = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \text{ where } c_{ij} = c_i + (c_j - c_i)\alpha \\
\text{subject to,} & \\
\sum_{j=1}^{n} x_{ij} & \leq s_i \\
s_i & = s_1 + (s_2 - s_1)\alpha \\
\sum_{i=1}^{m} x_{ij} & \geq b_j \\
d_j & = d_1 + (d_2 - d_1)\alpha \\
x_{ij} & \geq 0, \text{ for all } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n
\end{align*}
\]

6.3. DUAL OF FUZZY TRANSPORTATION MODEL

For a basic feasible solution, if variables \( u_i \) and \( v_j \) are associated with row \( i \), such that \( i = 1,2,\ldots,m \) and column \( j \), such that \( j = 1,2,\ldots,n \) of the
transportation table respectively, then \( u_i \) and \( v_j \) are made to satisfy the equation \( u_i + v_j = c_{ij} \), for each occupied cell \((i,j)\).

These equations yield \( m+n+1 \), equations in \( m+n \) are unknown dual variables. The values of these variables are determined from the above relationship by assigning zero arbitrarily to any one of the variables and obtaining the values of the remaining \( m+n+1 \) variable algebraically. Once the values of \( u_i \) and \( v_j \) are determined, evaluation in terms of opportunity cost of each unoccupied cell is done using the equation:

\[
D_{rs} = c_{rs} - (u_r + v_s); \text{ for each unoccupied cell } (r,s)
\]

In order to prove these two results, the fuzzy general transportation model using sigma notation is considered, as given below:

Minimize \( z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij} \)

subject to,

\[
\sum_{j=1}^{n} x_{ij} = \tilde{z}_i, \text{ where } i = 1, 2, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} = \tilde{d}_j, \text{ where } j = 1, 2, \ldots, n
\]

\( x_{ij} \geq 0 \), for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).
Where, $\tilde{c}_{ij}$, $\tilde{s}_i$, and $\tilde{d}_j$ represent fuzzy cost, fuzzy supply and fuzzy demand, respectively. As all the constraints are considered equalities, each equality constraint is expressed as two equivalent inequalities. The fuzzy general transportation problem is thus:

Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}$

subject to,

$$\sum_{j=1}^{n} x_{ij} \geq \tilde{s}_i, \quad \sum_{j=1}^{n} -(x_{ij}) \leq -\tilde{s}_i,$$

$$\sum_{i=1}^{m} x_{ij} \geq \tilde{d}_j, \quad \sum_{i=1}^{m} -(x_{ij}) \leq -\tilde{d}_j,$$

$$x_{ij} \geq 0, \text{ where } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

If it is considered that $u_i^+$ and $u_i^-$ are the dual variables for each supply constraint $i$, and $v_j^+$ and $v_j^-$ are the dual variables for each demand constraint $j$, then the dual of the fuzzy general transportation model is given by the following equation:
Maximize $z^* = \sum_{i=1}^{m} (u_i^+ - u_i^-) \tilde{s}_i + \sum_{j=1}^{n} (v_j^+ - v_j^-) \tilde{d}_j$

subject to,

$$(u_i^+ - u_i^-) + (v_j^+ - v_j^-) \leq \tilde{c}_{ij} \text{ and}$$

$u_i^+, u_i^-, v_j^+, v_j^- \geq 0$, for all $i$ and $j$.

The values for the variables $u_i^+$ and $u_i^-$ in the objective function, may be positive, negative or zero. Thus, any of these may appear in the optimal basis, as one is the negative of the other. The same argument may be given for $v_j^+$ and $v_j^-$. If the following is considered:

$$u_i = u_i^+ - u_i^- \text{, where } i = 1, 2, \ldots, m$$

$$v_j = v_j^+ - v_j^- \text{, where } j = 1, 2, \ldots, n$$

then $u_i$ and $v_j$ are found to be unrestricted in sign.

Thus, the dual of the fuzzy general transportation problem (FGTP) is given as:

Maximize $z^* = \sum_{i=1}^{m} u_i \tilde{s}_i + \sum_{j=1}^{n} v_j \tilde{d}_j$

subject to,

$u_i + v_j \leq \tilde{c}_{ij}$ for all $(i, j)$ and $u_i^+, v_j^+$ unrestricted in sign for all $i$ and $j$.  

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The solution $x_{ij}$ forms an optimal solution for the given fuzzy general transportation problem, provided:

(i) solution $x_{ij}$ is feasible for all $i$ and $j$ with respect to fuzzy general transportation problem model.

(ii) solution $x_{ij}$ and $u_{i}$, $v_{j}$ is feasible for all $i$ and $j$ with respect to the dual of the original fuzzy general transportation problem.

(iii) $(\tilde{c}_{ij} - u_{i} - v_{j}) x_{ij} = 0$ for all $i$ and $j$

The relationship $(\tilde{c}_{ij} - u_{i} - v_{j}) x_{ij} = 0$ is also known as complementary slackness for a fuzzy general transportation problem. It indicates that,

(iv) if $x_{ij} > 0$ and is feasible, then $\tilde{c}_{ij} - u_{i} - v_{j} = 0$ or $\tilde{c}_{ij} = u_{i} + v_{j}$ for each occupied cell.

(v) if $x_{ij} = 0$ and $\tilde{c}_{ij} > u_{i} + v_{j}$, then it is not desirable to have $x_{ij} > 0$ in the solution mix because it would cost more to transport on a route $(i,j)$.

(vi) if $\tilde{c}_{ij} < u_{i} + v_{j}$ for some $x_{ij} = 0$, then $x_{ij}$ can be brought into the solution mix.
The per unit net contribution to the objective function for a route (i,j) is given by $\tilde{d}_{ij} = \tilde{c}_{ij} - (u_i + v_j)$, for all i and j. It may be noted that $d_{ij} = 0$ for all occupied cells.

6.3.1. PROCEDURE FOR COMPUTING UPPER BOUND

$\alpha$ - FGTP is rewritten in the equivalent form as given below:

$$\begin{align*}
\text{Maximize } z_\alpha^v &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \text{ where } c_{ij} = c_3 - (c_3 - c_2)\alpha \\
\text{subject to, } \\
\sum_{j=1}^{n} x_{ij} &\leq s_i \\
s_i &= s_3 - (s_3 - s_2)\alpha \\
\sum_{i=1}^{n} x_{ij} &\geq d_j \\
d_j &= d_3 - (d_3 - d_2)\alpha \\
\end{align*}$$

where, $x_{ij} \geq 0$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. 

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6.4. NUMERICAL EXAMPLE

For the fuzzy transportation problem given below:

\[
\begin{array}{cccc}
(1,2,3) & (1,3,5) & (9,11,13) & (5,7,9) & (5,6,7) \\
(0,1,2) & (-1,0,1) & (3,6,9) & (0,1,2) & (0,1,2) \\
(3,5,7) & (6,8,10) & (12,15,18) & (7,9,11) & (5,10,15) \\
(5,7,9) & (3,5,7) & (1,3,5) & (1,2,3) & \\
\end{array}
\]

the mathematical model is given below as follows:

Minimize \( \tilde{z} = (1, 2, 3) x_{11} + (1, 3, 5) x_{12} + (9, 11, 13) x_{13} + (5, 7, 9) x_{14} + (0, 1, 2) x_{21} + (-1, 0, 1) x_{22} + (3, 6, 9) x_{23} + (0, 1, 2) x_{24} + (3, 5, 7) x_{31} + (6, 8, 10) x_{32} + (12, 15, 18) x_{33} + (7, 9, 11) x_{34} \)

subject to,

\[
\begin{align*}
\sum_{i=1}^{4} x_{1i} &= (5, 6, 7) \\
\sum_{i=1}^{4} x_{2i} &= (0, 1, 2) \\
\sum_{i=1}^{4} x_{3i} &= (5, 10, 15)
\end{align*}
\]
\[ \sum_{j=1}^{\lambda} x_{ji} = (5, 7, 9) \]
\[ \sum_{j=1}^{\lambda} x_{j2} = (3, 5, 7) \]
\[ \sum_{j=1}^{\lambda} x_{j3} = (1, 3, 5) \]
\[ \sum_{j=1}^{\lambda} x_{j4} = (1, 2, 3) \]

where, \( x_{ij} \geq 0 \).

Using \( \alpha \)-cut, the above problem can be formulated into upper bound problem and lower bound problem.

The lower bound problem is given as follows:

Minimize \[ \hat{z}_a = (1 + \alpha) x_{11} + (1 + 2\alpha) x_{12} + (9 + 2\alpha) x_{13} + (5 + 2\alpha) x_{14} \]
\[ + (0 + \alpha) x_{21} + (-1 + \alpha) x_{22} + (3 + 3\alpha) x_{23} + (0 + \alpha) x_{24} \]
\[ + (3 + 2\alpha) x_{31} + (6 + 2\alpha) x_{32} + (12 + 3\alpha) x_{33} + (7 + 2\alpha) x_{34} \]

subject to,
\[ \sum_{i=1}^{4} x_{ii} = (5 + \alpha) \]
\[ \sum_{i=1}^{4} x_{2i} = (0 + \alpha) \]
\[ \sum_{i=1}^{4} x_{3i} = (5 + 5\alpha) \]
\[ \sum_{j=1}^{1} x_{j1} = (5 + 2\alpha) \]
\[ \sum_{j=1}^{1} x_{j2} = (3 + 2\alpha) \]
\[ \sum_{j=1}^{1} x_{j3} = (1 + 2\alpha) \]
\[ \sum_{j=1}^{1} x_{j4} = (1 + \alpha) \]

where, \( x_{ij} \geq 0 \)

The upper bound problem is given as follows:

\[
\text{Maximize } \bar{z}_{\alpha} = (7 - \alpha) u_1 + (2 - \alpha) u_2 + (15 - 5\alpha) u_3 + (9 - 2\alpha) v_1 + (7 - 2\alpha) v_2 + (5 - 2\alpha) v_3 + (3 - \alpha) v_4
\]

subject to,

\[
\begin{align*}
    u_1 + v_1 & \leq (3 - \alpha) \\
    u_1 + v_2 & \leq (5 - 2\alpha) \\
    u_1 + v_3 & \leq (13 - 2\alpha) \\
    u_1 + v_4 & \leq (9 - 2\alpha) \\
    u_2 + v_1 & \leq (2 - \alpha) \\
    u_2 + v_2 & \leq (1 - \alpha) \\
    u_2 + v_3 & \leq (9 - 3\alpha) \\
    u_2 + v_4 & \leq (2 - \alpha)
\end{align*}
\]
\[ u_3 + v_3 \leq (7 - 2\alpha) \]
\[ u_1 + v_2 \leq (10 - 2\alpha) \]
\[ u_1 + v_3 \leq (18 - 3\alpha) \]
\[ u_2 + v_4 \leq (11 - 2\alpha) \]

where, \( u_1, u_2, u_3, v_1, v_2, v_3, v_4 \) are unrestricted in sign.

Due to the unrestriction in sign for all \( u_i \)'s and \( v_j \)'s, the same are converted into non-negative variables, i.e.,

\[ u_i = u_i^+ - u_i^- \text{, where } i = 1, 2, \ldots, m \text{ and } v_j = v_j^+ - v_j^- \text{, where } j = 1, 2, \ldots, n \]

Maximize

\[ \tilde{z}_\alpha^U = (7 - \alpha) (u_1^+ - u_1^-) + (2 - \alpha) (u_2^+ - u_2^-) + (15 - 5\alpha) (u_3^+ - u_3^-) + (9 - 2\alpha) (v_1^+ - v_1^-) + (7 - 2\alpha) (v_2^+ - v_2^-) + (5 - 2\alpha) (v_3^+ - v_3^-) + (3 - \alpha) (v_4^+ - v_4^-) \]

subject to,

\[ (u_1^+ - u_1^-) + (v_1^+ - v_1^-) \leq (3 - \alpha) \]
\[ (u_1^+ - u_1^-) + (v_2^+ - v_2^-) \leq (5 - 2\alpha) \]
\[ (u_1^+ - u_1^-) + (v_3^+ - v_3^-) \leq (13 - 2\alpha) \]
\[ (u_1^+ - u_1^-) + (v_4^+ - v_4^-) \leq (9 - 2\alpha) \]
\[ (u_2^+ - u_2^-) + (v_1^+ - v_1^-) \leq (2 - \alpha) \]
\[ (u_2^+ - u_2^-) + (v_2^+ - v_2^-) \leq (1 - \alpha) \]
\[ (u_2^+ - u_2^-) + (v_3^+ - v_3^-) \leq (9 - 3\alpha) \]
\[ (u_2^+ - u_2^-) + (v_4^+ - v_4^-) \leq (2 - \alpha) \]
\[
(u_3^+ - u_3^-) + (v_1^+ - v_1^-) \leq (7 - 2\alpha) \\
(u_3^+ - u_3^-) + (v_2^+ - v_2^-) \leq (10 - 2\alpha) \\
(u_3^+ - u_3^-) + (v_3^+ - v_3^-) \leq (18 - 3\alpha) \\
(u_3^+ - u_3^-) + (v_4^+ - v_4^-) \leq (11 - 2\alpha)
\]

where, \( u_i^+, u_i^-, v_j^+, v_j^- \geq 0, \ i = 1,2,3 \) and \( j = 1,2,3,4. \)

The \( \alpha \)-cuts of the total transportation cost at eleven \( \alpha \) values are listed in table 1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^-_\alpha )</td>
<td>32.00</td>
<td>37.17</td>
<td>42.68</td>
<td>48.53</td>
<td>54.72</td>
<td>61.25</td>
</tr>
<tr>
<td>( z^+_\alpha )</td>
<td>203.00</td>
<td>191.17</td>
<td>179.68</td>
<td>168.53</td>
<td>156.08</td>
<td>147.25</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>( z^-_\alpha )</td>
<td>68.32</td>
<td>75.73</td>
<td>83.48</td>
<td>91.57</td>
<td>100.00</td>
<td></td>
</tr>
<tr>
<td>( z^+_\alpha )</td>
<td>137.12</td>
<td>127.33</td>
<td>117.88</td>
<td>108.77</td>
<td>100.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 1