CHAPTER - II
CHAPTER 2

BINARY QUADRATIC DIOPHANTINE EQUATIONS

This chapter consists of four sections (A), (B), (C) and (D).

In section (A), the equation to be solved is

\[ ay^2 = bx + c \]

where \( a, b \) and \( c \) are non-zero integers.

Section (B) evaluates the solutions of the equation

\[ x^2 - 5xy + 4y^2 + 8x - 17y + 15 = 0 \]

The following equation is considered in section (C) for its solutions

\[ 3x^2 + xy = 14 \]

Section (D) deals the equation

\[ kxy + 2m(x + y) = x^2 - y^2, \quad k, m \in \mathbb{Z}^+ - \{0\} \]

for its integral solutions.
SECTION - A

The equation under consideration is

$$ay^2 = bx + c$$

(2.1)

where $a, b$ and $c$ are non-zero integers.

Introducing the linear transformations

$$ay = u, ax = v$$

(2.2)

in equation (2.1), it is written as

$$u^2 = bv + ac$$

(2.3)

Assume

$$u = abw + aB$$

(2.4)

where $w$ and $B$ are non-zero parameters.

Substituting (2.4) in (2.3) and using (2.2), we have

$$x = abw^2 + 2aBw + \frac{aB^2 - c}{b}$$

Since our interest centers on finding integral solutions, choose $B = B_0$

such that

$$\frac{aB_0^2 - c}{b} = \alpha \text{ (an integer)}$$

(2.5)

Therefore

$$x = abw^2 + 2aBw_0 + \alpha$$

Now equation (2.4) is expressed as

$$y - bw = B_0$$

The general form of integral solution of the above equation is represented by

$$\begin{align*}
    y &= (b+1)B_0 + bt, \\
    w &= B_0 + t
\end{align*}$$

(2.6)

where $t$ is any non-zero integer.
Using equation (2.6) in (2.5), the value of \( x \) is given by
\[
x = ab(B_0 + t)^2 + 2aB_0(B_0 + t) + \alpha \quad (2.7)
\]
Thus equations (2.6) and (2.7) represent the integral solutions of equation (2.1)

**Example**
Consider the equation  
\[5y^2 = 16x - 3\]
Here \( a = 5, \ b = 16, \ c = -3 \)

The expression \( \frac{aB_0^2 - c}{b} \) is an integer when \( B = B_0 = 2n^2 - 4n + 5 \) and the corresponding integral value is given by
\[
\alpha = \frac{1}{16} [20n^4 - 80n^3 + 180n^2 - 200n + 128]
\]
Hence the values of \( x \) and \( y \) satisfying the equation under consideration are given in Table 2.1 (a)

<table>
<thead>
<tr>
<th>Table 2.1(a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

**Generation of Solutions**
Let \( (x_0, y_0) \) be a solution of equation (2.1) obtained by either following the above analysis or otherwise. The second solution is obtained by setting
\[
\begin{align*}
x_1 &= x_0 + aaby_0h \\
y_1 &= h - y_0
\end{align*}
\] (2.8)
where \( h \) is a non-zero constant given by

\[ h = (2 + \alpha b^2)y_0 \]

The successive solutions are obtained by repeating the above analysis which leads to the general form of the integral solution of equation (2.1) in terms of \((x_0, y_0)\) to be

\[ x_n = x_0 + k y_0 \sum_{s=0}^{n-1} X^s, X = (\alpha b^2 + 1)^2 \]
\[ y_n = (\alpha b^2 + 1)^n y_0, \quad n = 1, 2, 3, \ldots \]

The values of \((x_n, y_n)\) satisfy the following recurrence relations

1) \[ x_{n+1} - x_1 = k y_0^2 X \left( \frac{X^n - 1}{X - 1} \right) \]
2) \[ x_{n+1} - x_1 = X(x_n - x_0) \]
3) \[ x_{n+1} - x_n = k y_n^2 \]
4) \[ y_0 y_{n+k} = y_n y_k \]
SECTIO N - B
To start with, observe that the equation considered
\[ x^2 - 5xy + 4y^2 + 8x - 17y + 15 = 0 \]  \hspace{1cm} (2.9)
represents a hyperbola.

Treating equation (2.9) as quadratic in \( x \), the two values of \( x \) satisfying (2.9) are given by
\[ \begin{aligned}
    x &= 4y - 5 \\
    x &= y - 3
\end{aligned} \] \hspace{1cm} (2.10)

Assuming \( y \) to take arbitrarily the value \( \beta (\neq 0) \), the two sets of non-zero parametric integral solutions of (2.9) are represented by

1) \( x = 4\beta - 5, \ y = \beta \)
2) \( x = \beta - 3, \ y = \beta \)

Some interesting properties of the above solutions are presented below.

**Properties of (1)**

The solutions \( (4\beta - 5, \beta) \) is primitive except for \( \beta \equiv 0(\text{mod} \ 5) \)

The choices of the parameter \( \beta \) and the corresponding m-gonal numbers represented by \( x \) are exhibited in Table 2.2 (a)
Table 2.2(a)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$m$-gonal number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8\alpha^2 - 11\alpha + 5$</td>
<td>$T_n$</td>
</tr>
<tr>
<td>$\frac{1}{8}(3\delta^2 - \delta + 10)$,</td>
<td>$P_n$</td>
</tr>
<tr>
<td>$\delta = \frac{1}{3}(2\alpha^3 - 9\alpha^2 + 34\alpha - 9)$</td>
<td></td>
</tr>
<tr>
<td>$8\alpha^2 - 5\alpha + 2$</td>
<td>$HX_n$</td>
</tr>
<tr>
<td>(i) $40\alpha^2 - 63\alpha + 26$</td>
<td>$HP_n$</td>
</tr>
<tr>
<td>(ii) $40\alpha^2 - 18\alpha + 8$</td>
<td></td>
</tr>
<tr>
<td>$16\alpha^2 - 11\alpha + 3$</td>
<td>$D_n$</td>
</tr>
</tbody>
</table>

The Table 2.2 (b) represents the values of the parameter $\beta$ and the corresponding relation between $x$ values and $m$-gonal numbers.

Table 2.2(b)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Relation between $x$ values and $m$-gonal numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\alpha^2 - \alpha + 2 - N,$</td>
<td>$x + 4N - 3 : O_n$</td>
</tr>
<tr>
<td>$(\alpha, N \in \mathbb{Z}^*)$</td>
<td></td>
</tr>
<tr>
<td>$3\alpha^2 - 4\alpha + 3 - N$</td>
<td>$x + 4N - 2 : O_n$</td>
</tr>
<tr>
<td>$5\alpha^2 - 2\alpha + 2 - N$</td>
<td>$x + 4N - 3 : DD_n$</td>
</tr>
<tr>
<td>$5\alpha^2 - 7\alpha + 4 - N$</td>
<td>$x + 4N - 2 : DD_n$</td>
</tr>
</tbody>
</table>
Properties of (2)

(1) The solution \((\beta - 3, \beta)\) is primitive except for \(\beta = 3t, t > 0\).

The choices of the parameter \(\beta\) and the corresponding \(m\)-gonal numbers represented by \(x\) are exhibited in Table 2.2 (c).

**Table 2.2(c)**

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(m)-gonal numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8\alpha^2 - 5\alpha + 2)</td>
<td>(T_n)</td>
</tr>
<tr>
<td>(\frac{1}{2}(3\delta^2 - \delta + 6))</td>
<td></td>
</tr>
<tr>
<td>(\delta = \frac{1}{3}(2\alpha^3 - 9\alpha^2 + 34\alpha - 9))</td>
<td></td>
</tr>
<tr>
<td>(2\alpha^2 - \alpha + 3)</td>
<td>(HX_n)</td>
</tr>
<tr>
<td>(\frac{1}{2}(5\alpha^2 - 3\alpha + 6))</td>
<td></td>
</tr>
<tr>
<td>(3\alpha^2 - 2\alpha + 3)</td>
<td>(O_n)</td>
</tr>
<tr>
<td>(4\alpha^2 - 3\alpha + 3)</td>
<td>(D_n)</td>
</tr>
<tr>
<td>(5\alpha^2 - 4\alpha + 3)</td>
<td>(DD_n)</td>
</tr>
</tbody>
</table>

**Generation of Solutions**

Let \((x_0, y_0)\) be any given solution of equation (2.9).

Let

\[
\begin{align*}
x_i &= x_0 - 2h, \\
y_i &= y_0 - h
\end{align*}
\]

where \(h\) is a non zero integer.

If (2.11) represents a second solution of (2.9), then the substitution of (2.11) in (2.9) gives
\[ h = \frac{1}{2}(x_0 + 2y_0 + 1) \]  
(2.12)

**Case 1**

The choice \( x_0 = 4\beta - 5, y_0 = \beta \) in (2.12) gives \( h = 3\beta - 2 \) and therefore

\[
\begin{align*}
x_1 &= x_0 - (6\beta - 4), \\
y_1 &= y_0 - (3\beta - 2)
\end{align*}
\]

Repeating the above process, the sequences of values of \( x \) and \( y \) are given by

\[
\begin{align*}
x_n &= 4\beta - 5 - n(6\beta - 4) \\
y_n &= \beta - n(3\beta - 2) \quad n = 0, 1, 2, ...
\end{align*}
\]

(2.13)

Replacing \( \beta \) by \(-n\) in (2.13), we get

\[
\begin{align*}
x_n &= 6n^2 - 5 \\
y_n &= 3n^2 + n
\end{align*}
\]

(2.14)

A few properties observed from (2.14) are presented below.

1) \( x_n(1 - 3n) + 2y_n(3n - 2) = 11n - 5 \)

2) \((x_n, x_{n+1}, x_{n+2})\) forms an A.P. Similarly for \( y \).

3) \( x_n + 5 \) is a Nasty number

4) Relations between the solutions and special polygonal numbers are exhibited below.

(i) \( x_n \equiv 12T_n \pmod{6n + 5} \)

(ii) \( x_n \equiv 3HX_n \pmod{3n - 5} \)

(iii) \( x_n \equiv 4P_n \pmod{2n - 5} \)

(iv) \( x_n \equiv 2O_n \pmod{4n - 5} \)

(v) \( 2x_n \equiv 3D_n \pmod{9n - 10} \)

(vi) \( 5x_n \equiv 6DD_n \pmod{24n - 25} \)

(vii) \( 5x_n \equiv 12HP_n \pmod{18n - 25} \)
(viii) \[ \sum_{i=1}^{n} x_i = (4\beta - 5)n - (6\beta - 4)T_n \]

(ix) \[ \sum_{i=1}^{n} y_i = n\beta - (3\beta - 2)T_n \]

(x) \[ \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} y_i = (2\beta - 5)n \]

(xi) (a) \[ \sum ny_i = \beta T_n - (3\beta - 2)\sum n^2 \]

(b) \[ 6\sum ny_i = 6\beta T_n - (3\beta - 2)(2n + 1)(D_{n+1} - O_{n+1}) \]

(xii) \[ \left( \sum_{i=1}^{n} y_i - n\beta \right)^2 = (3\beta - 2)^2\sum n^3 \]

(xiii) \[ \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = (3\beta - 5)n - (3\beta - 2)T_n \]

(xiv) From the above two equations

\[ \left( \sum_{i=1}^{n} y_i - n\beta \right)^2 = \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i - (3\beta - 5)n \right)^2 \]

(xv) \[ y_n = 6T_n \pmod{2n}. \]

(xvi) \[ y_n = O_n \pmod{3n} \]

(xvii) \[ 4y_n = 3D_n \pmod{13n} \]

(xviii) \[ 5y_n = 3DD_n \pmod{17n} \]

(xix) \[ 5y_n = 6HP_n \pmod{14n} \]

(xx) \[ 2y_n = 3HX_n \pmod{5n} \]

Case 2

The choice \( x_0 = 2\beta - 3, y_0 = 2\beta \) in (2.12) gives \( h = \frac{1}{2}(3\beta - 2) \) which is an integer for even value of \( \beta \).
Replacing $\beta$ by $2\beta$, we have

$$h = 3\beta - 1$$

Thus, from (2.11), we get

$$x_i = x_0 - (6\beta - 2), \quad y_i = y_0 - (3\beta - 1)$$

Repeating the process as in case 1, the general forms of $x$ and $y$ satisfying (2.9) are obtained as

$$\begin{align*}
x_n &= 2\beta(1-3n) + 2n - 3 \\
y_n &= n + \beta(2-3n)
\end{align*}$$

(2.15)

Replacing $\beta$ by $-n$ in (2.15), we get

$$\begin{align*}
x_n &= 6n^2 - 3 \\
y_n &= 3n^2 - n
\end{align*}$$

(2.16)

We present below a few properties satisfied by the above solutions (2.16).

1) $x_n(2-3n) + 2y_n(3n-1) = 11n - 6$

2) $(x_n, x_{n+1}, x_{n+2})$ forms an A.P. Similarly for $y$.

3) Relations between solutions and particular polygonal numbers are projected below

(i) $x_n = 12T_n \pmod{6n+3}$

(ii) $x_n = 3HX_n \pmod{3(n-1)}$

(iii) $x_n = 4P_n \pmod{2n-3}$

(iv) $x_n = 2O_n \pmod{4n-3}$

(v) $2x_n = 3D_n \pmod{9n-6}$

(vi) $x_n = 6DD_n \pmod{24n-15}$

(vii) $5x_n = 12HP_n \pmod{18n-15}$

(viii) $2y_n - x_n - 2\beta = 3$
(ix) \[ \sum_{i=1}^{n} x_i + (6\beta - 2)T_i = (2\beta - 3)n \]

(x) \[ \sum_{i=1}^{n} y_i + (3\beta - 1)T_i = 2\beta n \]

(xi) \[ \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i + (1 - 3\beta)T_i = 3n \]

(xii) \[ 6\sum_{i=1}^{n} n_i = 12\beta T_n + (1 - 3\beta)nHX_{n+1} \]

(xiii) \[ y_n = 6T_n \pmod{4n} \]

(xiv) \[ y_n = O_n \pmod{n} \]

(xv) \[ 4y_n = 3D_n \pmod{5n} \]

(xvi) \[ 5y_n = 3DD_n \pmod{7n} \]

(xvii) \[ 5y_n = 6HP_n \pmod{4n} \]

(xviii) \[ 2y_n = 3HX_n \pmod{n} \]
SECTION - C

The equation to be solved is

\[ 3x^2 + xy = 14 \]  \hspace{1cm} (2.17)

Treating this as a quadratic in \( x \), we obtain

\[ x = \frac{1}{6}[ -y \pm \sqrt{y^2 + 168} ] \]

The square root on the RHS of the above equation is eliminated when

\( y = \pm 19, \pm 41 \)

and the corresponding values of \( x \) are found to be \( x = \mp 7, \mp 14 \)

Also, factoring equation (2.17), it is equivalent to the following systems of equations

1) \( x = 1, 3x + y = 14 \)

2) \( x = 14, 3x + y = 1 \)

3) \( x = 2, 3x + y = 7 \)

4) \( x = 7, 3x + y = 2 \)

It is to be noted that systems (2) and (4) do not provide any new solutions for (2.17).

The solutions obtained from (1) and (3) are \( x = 1, y = 11 \) and \( x = 2, y = 1 \) respectively. As the equation (2.17) is not altered on replacing \( x \) by \( -x \) and \( y \) by \( -y \) simultaneously, the pairs \((-1, -11)\) and \((-2, -1)\) also satisfy (2.17). Thus the equation (2.17) has only the following integral solutions \((\mp 7, \pm 19), (\mp 14, \pm 41), (\pm 2, \pm 1)\) and \((\pm 1, \pm 11)\).

It is worth mentioning here that the equation (2.17) represents a hyperbola which is satisfied by only limited number of integral points whereas the hyperbola \( x^2 + 4xy + y^2 - 2x + 2y - 6 = 0 \) has an infinite number of integral points \([43]\).
SECTION - D

The equation to be solved is

\[ kxy + m(x + y) = x^2 - y^2 \]  \hspace{1cm} (2.18)

where \( k \) and \( m \) are non-zero integers.

Introducing the linear transformations,

\[ x = u + v, \quad y = u - v \]  \hspace{1cm} (2.19)

the equation (2.18) is written as

\[ k(u^2 - v^2) + 4uv = 4uv \]  \hspace{1cm} (2.20)

Multiplying both sides of (2.20) by \( k \), and performing some simple algebra, we have

\[ (k^2 + 4)u^2 + 4mk = (kv + 2u)^2 \]  \hspace{1cm} (2.21)

Again multiplying both sides of (2.21) by \((k^2 + 4)\), it becomes

\[ [(k^2 + 4)u + 2mk]^2 = (k^2 + 4)(kv + 2u)^2 + (2mk)^2 \]  \hspace{1cm} (2.22)

To analyze the nature of solutions, we have to go in for finding solutions of (2.22) when \( k \) and \( m \) take particular values.

For example, we present solutions of (2.22) when \( k = \pm 1 \).

Substituting \( k = 1 \) in (2.22), it is written as

\[ (5u + 2m)^2 = 5(v + 2u)^2 + (2m)^2 \]

The integral values of \( u \) and \( v \) satisfying these equations are found to be

\[ u_s = \frac{m}{5} \left[ (9 + 4\sqrt{5})^s + (9 - 4\sqrt{5})^s - 2 \right] \]

\[ v_s = \frac{m}{5} \left[ (\sqrt{5} - 2)(9 + 4\sqrt{5})^s - (\sqrt{5} + 2)(9 - 4\sqrt{5})^s + 4 \right] \]

where \( s \) takes values 2, 4, 6, ...
In view of (2.19), the sequence of values of $x, y$ satisfying (2.18) are obtained as

$$x_s = \frac{m}{5} \left[ (\sqrt{5} - 1)(9 + 4\sqrt{5})^t - (\sqrt{5} + 1)(9 - 4\sqrt{5})^t + 2 \right]$$

$$y_s = \frac{m}{5} \left[ (3 - \sqrt{5})(9 + 4\sqrt{5})^t - (3 + \sqrt{5})(9 - 4\sqrt{5})^t - 6 \right]$$

where $s = 2, 4, 6, ...$

A few numerical examples are

$x: 80m, 25632m, 8253296m$

$y: 48m, 15840m, 5100816m$

It is observed that the values of $x$ and $y$ satisfy the relation

$$5(x_s + y_s)^2 + 8m(x_s + y_s) = (3x_s + y_s)^2$$

Also,

$$35x_s^2 + 15y_s^2 + 40x_s y_s + 20m(x_s + y_s) + 8m^2$$

is expressed as the sum of two squares and congruent to zero under mod 4.

Further, the values of $x$ and $y$ satisfy the following recurrence relations.

1) $x_{s+4} - 322x_{s+2} + x_s = -128m$

2) $y_{s+4} - 322y_{s+2} + y_s = 384m$

It is worth to mention here that in the above observations $s$ takes only even values.

Now, the substitution $k = -1$ in (2.22) leads to

$$(5u - 2m)^2 = 5(2u - v)^2 + (2m)^2$$
The sequence of integral values of $u$ and $v$ satisfying this equation are given by

$$u_s = \frac{m}{5} \left[(9 + 4\sqrt{5})' + (9 - 4\sqrt{5})' + 2\right]$$

$$v_s = \frac{m}{5} \left[(\sqrt{5} - 2)(9 + 4\sqrt{5})' - (\sqrt{5} + 2)(9 - 4\sqrt{5})' - 4\right]$$

where $s$ takes values 1,3,5,...

Thus, using (2.19), the integral solution of (2.18) are given by

$$x_s = \frac{m}{5} \left[(3 - \sqrt{5})(9 + 4\sqrt{5})' + (3 + \sqrt{5})(9 - 4\sqrt{5})' + 6\right]$$

$$y_s = \frac{m}{5} \left[(\sqrt{5} - 1)(9 + 4\sqrt{5})' - (\sqrt{5} + 1)(9 - 4\sqrt{5})' - 2\right]$$

where $s$ takes values 1,3,5,...

A few numerical examples are

$$x: 4m, 884m, 284260m$$

$$y: 4m, 1428/n, 459940w$$

It is seen that the values of $x$ and $y$ satisfy the relation

$$5(x_s + y_s)^2 = 8m(x_s + y_s) + (3y_s + x_s)^2$$

Also,

$$15x_s^2 + 35y_s^2 + 40x_s y_s - 20m(x_s + y_s) = 0 \text{(mod 4)}$$

and the L.H.S is written as the sum of two squares.

The recurrence relations satisfied by the solutions $x$ and $y$ of (2.18) are found to be

1) $x_{s+4} - 322x_{s+2} + x_s = -384m$

2) $y_{s+4} - 322y_{s+2} + y_s = 128m$

where $s = 1,3,5,...$