CONSECUTIVE AND FACE MAGIC LABELING OF CARONA

Consider a cycle $C_n$ on $n$ vertices; call it the prime cycle and attach $n$ cycles, each of length $m$, called the auxiliary cycles, at each vertex of the prime cycle. This new graph is called Carona on cycle, denoted by $C_m(C_n)$ (read as $C_m$ on $C_n$). It contains $nm$ vertices, $n(m+1)$ edges and $m+1$ cycles. Similarly the carona on path, denoted by $C_m(P_n)$, is obtained by attaching cycles $C_m$ at each vertex of the path $P_n$. This contains $nm$ vertices, $nm+n-1$ edges and $m$ cycles. The weight of a face under a labeling is taken as the sum of the labels of all the vertices surrounding that face.

A labeling is said to be consecutive if the weights of all the faces constitute a set of consecutive integers. On the other hand, if the weights of all the faces are the same, the labeling is said to be a face magic labeling.

In this chapter, we shall give the consecutive and face magic labeling for both $C_m(C_n)$ and $C_m(P_n)$ for $n \geq 3$. Also we prove that consecutive labeling for $C_m(C_n)$ exists only if $3 \leq m \leq 2n-1$. However, $C_m(P_n)$ has this labeling for all $m \geq 3$ and $n \geq 3$.

3.1 Consecutive labeling of $C_m(C_n)$ and $C_m(P_n)$

Result 1: The graph $C_n(C_n)$, $n=4k -1$ for any $k \geq 2$ has a consecutive labeling.

Proof: Case 1: $n = 8k -1$, $k \geq 1$. The vertices $v_i$, $i=1,2,\ldots,n$ of the prime cycle are labeled by $\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2} -1,\ldots,\frac{n(n+1)}{2} -\frac{(n-3)}{2}$, $\frac{n(n-1)}{2}$, $\frac{n(n-1)}{2} +1,\ldots,\frac{n^2-1}{2}$.
Case 2: $n = 8k + 3$, $k \geq 1$. The vertices $v_i, i = 1, 2, ..., n$ of the prime cycle are labeled by

$$\frac{n^2 + 1}{2}, \frac{n^2 - 1}{2}, \frac{n^2 - 3}{2}, \frac{n(n-1) + 4}{2}, \frac{(n-1)^2}{2}, \frac{n^2 - 1}{2} + 2, \frac{n^2 - 1}{2} + 3, ..., \frac{n(n+1)}{2}. $$

For both cases above, the vertices of the auxiliary cycle $C_i, i = 1, 2, ..., n$ are labeled as follows.

$$4(t-1)n+i, \frac{(8t-5)n+3}{2} - i, \frac{(8t-3)n-1}{2} + i, 4tn+1-i, i = 1, 2, ..., \frac{n-3}{4} $$

$$4tn+i, \frac{(8t+3)n+3}{2} - i, \frac{(8t+5)n-1}{2} + i, i = 1, 2, ..., \frac{n-3}{4} $$

$$\frac{(8t-5)n+3}{2} - i, (4t-3)n + i, (4t-1)n+1-i, \frac{(8t-3)n-1}{2} + i, t = 1, 2, ..., \frac{n-3}{4} $$

$$\frac{(8t+3)n+3}{2} - i, (4t+1)n + i, (4t + 3)n + 1 - i, t = \frac{n-3}{4}$$

It is interesting to note that while labeling the prime cycles as well as the auxiliary cycle, the common vertex gets the same label. The weight of the prime cycle is $\frac{n^3 - 1}{2}$. The weights of auxiliary cycles are $\frac{n^3 - 1}{2} + 1, \frac{n^3 - 1}{2} + 2, ..., \frac{n^3 - 1}{2} + n$. Thus the weights of the faces of all the $(n+1)$ cycles are consecutive integers and so $C_n(C_n)$ for $n=4k-1$, $k \geq 2$ has a consecutive labeling.

32
Remark 1: The graph $C_3(C_3)$ also has a consecutive labeling as given below.

Result 2: The graph $C_n(C_n)$, $n=4k+1$, $k \geq 1$ has a consecutive labeling.

Proof: The vertices of the prime cycle are labeled as follows:
If \( n \) is of the form \( 8k - 3, \ k \geq 1 \), the labels are \( \left( \frac{n-1}{2} \right)n+1, \left( \frac{n-1}{2} \right)n+2, \ldots \),
\[
\left( \frac{n-1}{2} \right)(n+2), \frac{n(n+2)+1}{2}.
\]

If \( n \) is of the form \( 8k+1, \ k \geq 1 \), the labels are \( \left( \frac{n-1}{2} \right)n+1, \left( \frac{n-1}{2} \right)n+2, \ldots \),
\[
\left( \frac{n-1}{2} \right)(n+1), \frac{n^2 + 3}{2}, \frac{n^2 + 3}{2} + 1, \ldots, \frac{n(n+1)+2}{2}.
\]

Now, for both cases above, the vertices of the auxiliary cycles get the following labels:
\[
\begin{align*}
\frac{(8t-5)n+3}{2} - i, (4t-3)n+i, (4t-1)n+1-i, \frac{(8t-3)n-1}{2} + i, t = 1, 2, \ldots, \frac{n-1}{4} \quad &\text{if } i = n, n-1, \ldots, \frac{n+3}{2} \\
\frac{(8t+3)n+3}{2} - i, t = \frac{n-1}{4} \quad &\text{if } i = n.
\end{align*}
\]
\[
4(t-1)n+i, \frac{(8t-5)n+3}{2} - i, \frac{(8t-3)n-1}{2} + i, 4tn+1-i, t = 1, 2, \ldots, \frac{n-1}{4} \quad &\text{if } i = 1, 2, \ldots, \frac{n+1}{2} \\
4tn + i, t = \frac{n-1}{4} \quad &\text{if } i = n.
\]

The weights of the auxiliary cycles are \( \frac{n^3 + 1}{2}, \frac{n^3 + 3}{2}, \ldots, \frac{n^3 + 2n-1}{2} \)
and for the prime cycle the weight is \( \frac{n^3 + 2n+1}{2} \).
Result 3: The graph $C_n(C_n)$, $\frac{n}{2}$ odd, has a consecutive labeling.

Proof: The vertices $v_i$, $i=1,2,\ldots,n$ of the prime cycle are given the labels from $\frac{n(n-1)+2}{2}$ to $\frac{(n-1)(n+2)+2}{2}$. The vertices of the auxiliary cycle $C_i$, $i=1,2,\ldots,n$ are labeled as follows:

\[
\begin{align*}
4(t-1)n+i, \frac{(8t-5)n}{2} + i, t=1,2,\ldots,\frac{n-2}{4} \\
4tn+i, \frac{(8t+3)n}{2} - 1 + i, t = \frac{n-2}{4} \\
4(t-1)n+i, \frac{(8t-5)n}{2} + 1 - i, t=1,2,\ldots,\frac{n-2}{4} \\
4tn+i, \frac{(8t+3)n}{2} + i, t = \frac{n-2}{4}
\end{align*}
\]
The weight of the prime cycle is \( \frac{n(n^2 + 1)}{2} \). The weights of the auxiliary cycles vary from \( \frac{n^3}{2} \) to \( \frac{n(n^2 + 2)}{2} \) excluding \( \frac{n(n^2 + 1)}{2} \). Clearly these weights form a set of consecutive integers. Thus the graph \( C_n \), \( n \) being odd, has a consecutive labeling.
Result 4: The graph $C_n(C_n), \frac{n}{4}$ odd, $n > 4$, has a consecutive labeling.

Proof: The vertices of the prime cycle are given the labels from $\frac{n(n-1)+2}{2}$ to $(n-1)(n+2)+2$. The vertices of the auxiliary cycle $C_i, i=1, 2, \ldots, \frac{n}{2}$ are labeled as indicated below:

\[
\begin{align*}
4(t-1)n + i, & \quad \frac{(8t-5)n}{2} + 1 - i, \quad \frac{(8t-3)n}{2} + i, \quad 4tn + 1 - i, \quad t = 1, 2, \ldots, \frac{n-4}{8} \\
4m + i, & \quad \frac{(8t+3)n}{2} + i, \quad (4t+3)n + 1 - i, \quad t = \frac{n-4}{8} \\
(4t-5)n + i, & \quad \frac{(8t-7)n}{2} + 1 - i, \quad \frac{(8t-5)n}{2} + i, \quad (4t-1)n + 1 - i, \quad t = \frac{n+12}{8} \text{ to } \frac{n}{4} \\
n^2 + 2 - 2i, & \quad t = \frac{n}{4}
\end{align*}
\]
For the cycles $C_i, i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n$, the vertices are labeled by

$$\frac{(8t-5)n}{2} + 1 - i, (4t-3)n + 1 - i, \frac{(8t-3)n}{2} + i, t = 1, 2, \ldots, \frac{n-4}{8}$$

$$\frac{(8t+3)n}{2} + 1 - i, (4t+2)n + 1 - i, \frac{(8t+3)n}{2} + i, t = \frac{n}{2} - 1, \frac{n}{2} + 2, \ldots, n$$

$$\frac{(8t-7)n}{2} + 1 - i, (4t-4)n + 1 - i, (4t-2)n + 1 - i, \frac{(8t-5)n}{2} + i, t = \frac{n+12}{8} \text{ to } \frac{n}{4}$$

$$n^2 - 2n - 1 + 2i, t = \frac{n}{4}$$

The auxiliary cycles get as their weights the consecutive integers from $\frac{n^3}{2}$ to $\frac{n}{2}(n^2 + 2)$ excluding $\frac{n(n^2 + 1)}{2}$ which is the weight of the prime cycle.

Fig. 3.5: Consecutive labeling of $C_{12}$ ($C_{12}$).
Remark 2: The graph $C_4(C_4)$ also has a consecutive labeling as given below:

![Consecutive labeling of $C_4(C_4)$](image)

Fig. 3.6: Consecutive labeling of $C_4(C_4)$.

Result 5: The graph $C_n(C_n)$ for $\frac{n}{4}$ even has a consecutive labeling

Proof: The vertices of the prime cycle are labeled from $\frac{n(n-1)+2}{2}$ to $\frac{(n-1)(n+2)+2}{2}$. The vertices of the auxiliary cycles are labeled in the following manner:

$$
\begin{align*}
4(t-1)n+i, \quad & \frac{(8t-5)n}{2} + 1 - i, \quad \frac{(8t-3)n}{2} + i, \quad 4tn + 1 - i, \quad t = 1, 2, \ldots, \frac{n}{8} \\
(4t+1)n+1-i, \quad & t = \frac{n}{8} \\
(4t-3)n+i, \quad & \frac{(8t-3)n}{2} + 1 - i, \quad \frac{(8t-1)n}{2} + i, \quad (4t+1)n+1-i, \quad t = \frac{n}{8} + 1, \ldots, \frac{n-4}{4} \\
(4t+1)n+i, \quad & \frac{(8t+5)n}{2} + 1 - i, \quad (4t+3)n+1+2i, \quad t = \frac{n-4}{4}
\end{align*}
$$

39
\[
\begin{align*}
\frac{(8t - 5)n}{2} + 1 - i, \frac{(4t - 3)n + 1 - i}{2} + i, t = 1, 2, \ldots, \frac{n}{8} \\
\frac{(8t - 1)n}{2} + i, \ t = \frac{n}{8} \\
\frac{(8t - 3)n}{2} + 1 - i, \frac{(4t - 2)n + 1 - i}{2} + i, t = \frac{n}{8} + 1, \ldots, \frac{n - 4}{4} \\
\frac{(8t + 5)n}{2} + 1 - i, \frac{(4t + 2)n + i}{2} + n^2 + n + 2 - 2i, t = \frac{n - 4}{4}
\end{align*}
\]

Fig. 3.7: Consecutive labeling of \( C_8 \) (C₈)
The weights of these cycles vary from \(\frac{n^3}{2}\) to \(\frac{n(n^2 + 2)}{2}\), excluding \(\frac{n(n^2 + 1)}{2}\), which is the weight of the prime cycle. Hence the graph \(C_n(C_n)\) for \(\frac{n}{4}\) even has a consecutive labeling.

Combining all these five results we get the following theorem.

**Theorem 1:** The graph \(C_n(C_n)\) for all \(n \geq 3\) has a consecutive labeling.

**Theorem 2:** The graph \(C_n(P_n)\) for any \(n \geq 3\) has a consecutive labeling.

**Proof:** The removal of any edge from the prime cycle of \(C_n(C_n)\) results in \(C_n(P_n)\) without affecting the vertices and hence vertex labeling for \(C_n(P_n)\) is the same as the labeling for \(C_n(C_n)\).

### 3.2 Face magic labeling of Carona \(C_n(C_n)\) and \(C_n(P_n)\)

**Result 6:** The graph \(C_n(C_n)\) for \(n = 4k-1, k \geq 2\) has the face magic labeling.

**Proof:** The vertices of the prime cycle get the labels

\[
\frac{1}{2} \left\{ n(n+1) - 4(i-1) \right\} \text{ for } i = 1, 2, ..., \frac{n+1}{2} \quad \text{and} \quad \frac{1}{2} \left\{ n(n-3) + 2(2i-1) \right\} \text{ for } i = \frac{n+3}{2}, \frac{n+5}{2}, ..., n.
\]

The vertices of the auxiliary cycles have the following labeling:

\[
i, 2(2t-1) n+2-2i, 2(2t-1)n-1+2i, 4tn+2-2i, t=1.
\]

\[
4(t-1)n-1+2i, 2(2t-1)n+2-2i, 2(2t-1)n-1+2i, 4tn+2-2i, t=2,3, ..., \frac{n-3}{4}.
\]

\[
4tn-1+2i, 2(2t+1)n+2-2i, \frac{8t+5}{2} n-1 + i, t = \frac{n - 3}{4}
\]
\[
\frac{(8t-5)n+3}{2} - i, (t-1)n+2i, 4tn+2-2i, 2(2t-1)n+2i, ~ t = 1
\]
\[
2(2t-1)n+2i, 4(t-1)n+1+2i, 4tn+2-2i, 2(2t-1)n+2i, ~ t = 2, 3, ..., \frac{n-3}{4}
\]
\[
2(2t+1)n+2i, 4tn+1+2i, ~ n^2+1-i, ~ t = \frac{n-3}{4}
\]

The weight of each face is \(\frac{n(n^2+1)}{2}\). Thus the graph \(C_n\) is face magic for \(n = 4k-1, ~ k \geq 2\).

Fig. 3.8: Face Magic labeling of \(C_7\) (C7)
Result 7: The graph $C_n (C_n)$ for $n=4k+1$, $k \geq 1$, is a face magic graph.

Proof: The vertices of the prime cycle are labeled by $\frac{n(n-1)}{2} - 1 + 2i$ for $i=1,2,...,$

$\frac{n+1}{2}$ and $\frac{n(n+3)}{2} + 2 - 2i$ for $i = \frac{n+3}{2}, \frac{n+5}{2}, ..., n$. The vertices of the auxiliary cycles are labeled as follows:

\[
\begin{align*}
&i, 2(2t-1)n+2-2i, 2(2t-1)n-1+2i, 4tn+2-2i, t=1. \\
&4(t-1)n-1+2i, 2(2t-1)n+2-2i, 2(2t-1)n-1+2i, 4tn+2-2i, t=2,3,..., \frac{n-1}{4} \\
&\left(\frac{8t+1}{2}\right)n-1 + i, t = \frac{n-1}{4} \\
&\left(\frac{8t-5}{2}\right)n+3 - i, (t-1)n-1+2i, 4tn+2-2i, 2(2t-1)n-1+2i, t=1 \\
&2(2t-1)n+2-2i, 4(t-1)n-1+2i, 4tn+2-2i, 2(2t-1)n-1+2i, t=2,3,..., \frac{n-1}{4} \\
&n^2+1-i, t = \frac{n-1}{4} \\
\end{align*}
\]

Clearly all the faces have the same weight equal to $\frac{n(n^2+1)}{2}$. Thus the graph $C_n (C_n)$, for $n = 4k +1$, $k \geq 1$ has a face magic labeling.\[\square\]
Fig. 3.9: Face magic labeling of $C_5$ ($C_5$)

**Result 8:** The graph $C_n (C_n)$ for $\frac{n}{2}$ odd is a face magic graph.

**Proof:** The vertices of the prime cycle are labeled from $\frac{n(n - 1)}{2} + 1$ to $\frac{n(n + 1)}{2}$ and it has the weight $\frac{n(n^2 + 1)}{2}$. The vertices of the auxiliary cycles have the labels as follows:

\[
\begin{align*}
4(t-1)n+i, & \quad \frac{(8t-5)n}{2} + 1 - i, \quad \frac{(8t-3)n}{2} + i, \quad 4tn+1-i, \quad t=1,2,\ldots, \frac{n-2}{4} \quad \text{and} \\
4tn+i, & \quad n^2+1-i, \quad t=\frac{n-2}{4} \\
\frac{(8t-5)n}{2} + 1 - i, (4t-3)n + i, (4t-1)n + 1 - i, & \quad \frac{(8t-3)n}{2} + i, \quad t=1,2,\ldots, \frac{n-2}{4} \quad \text{and} \\
\frac{(8t+3)n}{2} + 1 - i, & \quad \frac{(8t+1)n}{2} + i, \quad t=\frac{n-2}{4}
\end{align*}
\]
Clearly all faces have the same weight \( \frac{n(n^2+1)}{2} \). Thus the graph \( C_n (C_n) \) for \( n \) odd has the face magic labeling.

\[
\text{Fig. 3.10: Face magic labeling of } C_6 (C_6)
\]

**Result 9:** The graph \( C_n (C_n) \) for \( n \) even has a face magic labeling.

**Proof:** The vertices of the prime cycle are labeled from \( \frac{n(n-1)}{2} + 1 \) to \( \frac{n(n+1)}{2} \).

The auxiliary cycles are labeled as given below:

\[
\begin{align*}
4(t-1)n+i, & \quad \frac{(8t-5)n}{2} + 1 - i, \quad \frac{(8t-3)n}{2} + i, \quad 4tn + 1 - i, \quad t = 1, 2, \ldots, \frac{n}{8} \\
\frac{n(n+2)}{2} + 1 - i, & \quad t = \frac{n}{8} \\
(4t-3)n+i, & \quad \frac{(8t-3)n}{2} + 1 - i, \quad \frac{(8t-1)n}{2} + i, \quad (4t+1)n + 1 - i, \quad t = \frac{n}{8} + 1, \quad \frac{n}{8} + 2, \ldots, \frac{n-4}{4} \\
(4t+1)n+i, & \quad \frac{(8t+5)n}{2} + 1 - i, \quad \frac{(8t+7)n}{2} + i, \quad t = \frac{n-4}{4}.
\end{align*}
\]
\[
\frac{(8t-5)n}{2} + 1 - i, (4t - 3)n + i, (4t - 1)n + 1 - i, \frac{(8t - 3)n}{2} + i, \quad t = 1, 2, \ldots, \frac{n}{8}
\]

\[
\frac{n(n-1)}{2} + i, \quad t = \frac{n}{8}
\]

\[
\frac{(8t-3)n}{2} + 1 - i, (4t - 2)n + i, 4tn + 1 - i, \frac{(8t-1)n}{2} + i, \quad t = \frac{n}{8} + 1, \frac{n}{8} + 2, \ldots, \frac{n-4}{4}
\]

\[
\frac{(8t+5)n}{2} + 1 - i, (4t + 2)n + i, 4(t+1)n + 1 - i, \quad t = \frac{n-4}{4}
\]

All the faces including the prime cycle have the same weight \(\frac{n(n^2 + 1)}{2}\).

Thus the graph \(C_n (C_n)\) for \(\frac{n}{4}\) even is face magic.

Fig. 3.11: Face Magic labeling of \(C_8 (C_8)\)
Result 10: The graph $C_n$ ($C_n$), $\frac{n}{4}$ odd, $n > 4$, is a face magic graph.

Proof: The prime cycle has its vertices labeled from $\frac{n(n-1)}{2} + 1$ to $\frac{n(n+1)}{2}$.

The auxiliary cycles have their vertices labeled as given below.

$$4(t-1)n+i, \frac{(8t-5)n}{2} + 1 - i, \frac{(8t-3)n}{2} + i, 4tn + 1 - i, t = 1, 2, ..., \frac{n-4}{8}$$

$$4tn+i, \frac{(8t+3)n}{2} + i, (4t+3)n + 1 - i, t = \frac{n-4}{8}$$

$$\frac{(4t-1)n+i}{2} + 1 - i, \frac{(8t+3)n}{2} + i, (4t+3)n + 1 - i, t = \frac{n+4}{8}, \frac{n+4}{8} + 1, ..., \frac{n-4}{4}$$

$$\frac{(2t-1)n}{2} + 1 - i, t = n$$

$$\frac{(8t-5)n}{2} + 1 - i, (4t-3)n + i, (4t-1)n + 1 - i, \frac{(8t-3)n}{2} + i, t = 1, 2, ..., \frac{n-4}{8}$$

$$\frac{(8t+3)n}{2} + 1 - i, (4t+2)n + 1 - i, \frac{(8t+3)n}{2} + i, t = \frac{n-4}{8}$$

$$\frac{(8t+1)n}{2} + 1 - i, 4tn + i, (4t+2)n + 1 - i, \frac{(8t+3)n}{2} + i, t = \frac{n+4}{8}, \frac{n+4}{8} + 1, ..., \frac{n-4}{4}$$

$$t = \frac{n}{2}, \frac{n}{2} + 1, ..., n$$

The common weight of all the faces is $\frac{n(n^2+1)}{2}$ and so the graph $C_n$ ($C_n$) for $\frac{n}{4}$ odd is a face magic graph.
Remark 3: The graph $C_4 (C_4)$ is face magic as seen below:

![Fig. 3.13: Face Magic labeling of $C_4 (C_4)$](image)

Results 6 to 10 summarize the proof of the next theorem.

**Theorem 3:** The graph $C_n (C_n)$ for all $n \geq 3$ is a face magic graph.
As mentioned in section 3.1, removal of any edge from the prime cycle of $C_n$ does not affect the position of any vertex and this removal results in the graph being reduced to $C_n(P_n)$. Hence the following theorem:

**Theorem 4:** The graph $C_n(P_n)$ for all $n \geq 3$ is a face magic graph.

**Remark 4:** The labelings in figure 3.14 give labelings different from the corresponding labelings given in figures 3.6 and 3.13 respectively. Thus the consecutive labeling and face magic labeling of both the graphs $C_n(C_n)$ and $C_n(P_n)$ need not be unique.

![Fig. 3.14: Consecutive and Face Magic labeling of $C_4(C_4)$](image)

### 3.3 Consecutive labeling of General Corona $C_m(C_n)$ and $C_m(P_n)$

Consider the integers $m = 6$ and $n = 5$ so that $mn = 30$. Arrange the integers 1 to 30 as in matrix $M_1$ and assume the matrix notation $(a_{ij})_{6 \times 5}$.

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**Matrix M_1**

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**Matrix M_3**
If we make the interchanges of the elements (a_{6,1} with a_{6,2}) and (a_{6,3} with a_{6,5}), we get matrix M_2. Matrix M_3 is obtained by making the interchanges (a_{4,1} with a_{5,1}) and (a_{4,3} with a_{5,3}) in matrix M_2. In matrix M_2 and matrix M_3, the column sums and the 4^{th} row sum form a set of consecutive integers from 90 to 95 or 91 to 96 respectively.

Take a cycle C_5 and attach a C_6 at each vertex of C_5 to get the carona C_6(C_5). The vertices of the i^{th} C_6, i = 1 to 5, are given the labels from the i^{th} column elements so that the vertices of C_5 get the labels from the elements of the 4^{th} row. Thus C_6(C_5) has a consecutive labeling.

Thus, by suitably rearranging the elements of (a_{i,j}), the column sums and one row sum can be made to form a set of consecutive integers. That is, suitable rearrangement of the labels of the auxiliary cycles gives a consecutive labeling to the graph C_6(C_5). This gives the theorem:

**Theorem 5:** The graph C_m (C_n) for all m \geq 3 and n \geq 3 has consecutive labeling if 3 \leq m \leq 2n - 1 and has no consecutive labeling if m > 2n - 1.

**Proof:** As the graph C_m (C_n) has mn vertices, its vertices are labeled from 1 to mn. We know that \( \sum_{j=1}^{mn} j = \frac{mn(mn+1)}{2} \) and let \( L = \frac{m(mn+1)}{2} \). Then the weights of the faces corresponding to the auxiliary cycles are given as below:
\[
\begin{align*}
L - n/2 & \text{ to } L + n/2 \quad \text{if } m \text{ and } n \text{ are even} \\
(L - n/2) - \frac{1}{2} & \text{ to } (L + n/2) - \frac{1}{2} \\
\text{Or} & \\
(L - n/2) + \frac{1}{2} & \text{ to } (L + n/2) + \frac{1}{2} \\
\end{align*}
\]

Let \( L' = L - n/2 \) and \( U' = L + n/2 \). The smallest of the weights of the faces are \( L' \) or \( L' - 1/2 \) or \( L' + 1/2 \) and the largest of of the weights are \( U' \) or \( U' - \frac{1}{2} \) or \( U' + \frac{1}{2} \). Assume that the smallest is \( L'' \) and the largest is \( U'' \).

Consider the \( n \) consecutive integers \( m, m-1, m-2, \ldots, m-n+1 \) which are the largest of the \( mn \) integers, may be the labels of the vertices of the prime cycle. Then the weight of the prime cycle is 
\[
\frac{2mn^2 - n^2 + n}{2} = P \quad \text{(say)}.
\]

Let \( f(m, n) = L' - P \). Then,
\[
2f(m, n) = m^2n + m - n - 2mn^2 + n^2 - n.
\]

Now, \( 2f(0, n) = n(n - 2) > 0 \quad \text{(1a)} \)
\[
2f(1, n) = 1 - n^2 - n < 0 \quad \text{(1b)}
\]
\[
2f(2n - 1, n) = -(n - 1)^2 - n < 0 \quad \text{(1c)}
\]
\[
2f(2n, n) = n^2 > 0 \quad \text{(1d)}
\]

From (1b) and (1c), it is clear that the value of \( P \) can be reduced if \( m \) lies between 1 and \( 2n - 1 \). Hence, it is also clear that, by suitably interchanging the labels of the auxiliary cycles, \( P \) can be made to coincide with one of \( L'' \) to \( U'' \), provided the values of \( m \) lie between 1 and \( 2n - 1 \). But, when \( m = 1 \) or 2, the
graph is not simple. Thus, the graph $C_m (C_n)$ has consecutive labeling when $m = 3, 4, \ldots, 2n - 1$.

From (1d), $P < L'$. As $P$ is the maximum possible weight of the prime cycle, it cannot be further increased to coincide with one of $L''$ to $U''$. Thus, a consecutive labeling is not possible when $m = 2n$. The same reason is also valid when $m > 2n$. Thus the graph $C_m (C_n)$ has no consecutive labeling when $m > 2n - 1$.

As the graph $C_m(P_n)$ has no prime cycle, the possibility of consecutive labeling of the auxiliary cycles of $C_m(C_n)$ for all $m$ is applicable to those of $C_m(P_n)$ also. Hence we get the following theorem.

**Theorem 6:** The graph $C_m (P_n)$ has consecutive labeling for all $m, n \geq 3$. 