Chapter 2
CONSECUTIVE LABELING FOR SUBDIVISION OF WHEEL AND WINDMILL GRAPHS

A graph $G$ consists of a vertex set $V$ and an edge set $E$ with cardinalities $p$ and $q$ respectively. Let $r$ be the number of faces of $G$. A labeling of type $(a, b, c)$ assigns labels from the set $\{1, 2, 3, \ldots, ap+bq+cr\}$ to the vertices, edges and faces of $G$ such that each vertex receives $a$ labels, each edge receives $b$ labels and each face receives $c$ labels and each number is used exactly once as a label. Labeling of type $(1,0,0)$ and $(0,1,0)$ are also called vertex and edge labelings respectively.

The notion of consecutive labeling has its origin in classical Chinese Mathematics of 13th century. For plane graphs it was defined by Ko.Wei Lih [25] where consecutive labeling of type $(1,0,0)$ for wheels, friendship graphs and prisms is given. This type of labeling for subdivision of ladders and lotus inside a circle is given in [07].

In this chapter we describe consecutive labeling of type $(1,0,0)$ for subdivision of wheels and windmill graphs.

The weight of a face under a labeling is the sum of the labels of the face itself together with labels of vertices and edges surrounding that face.
The labeling is said to be consecutive if the weights of all faces constitute a set of consecutive integers.

### 2.1 Consecutive labeling for Subdivision of wheels

For \( n \geq 4 \), the wheels \( W_n \) is defined to be \( K_1 + C_{n-1} \) \[40\]. But for convenience, the wheel \( W_n \) in this chapter, refers to \( W_1 + C_n, n \geq 3 \). Let the vertex of \( K_1 \) be denoted by \( c \).

If \( x = uv \) is an edge of \( G \) and \( w \) is not a vertex of \( G \), then \( x \) is subdivided when \( x \) is replaced by the edges \( uw \) and \( vw \). If every edge of \( G \) is subdivided, the resulting graph is the subdivision graph \( S(G) \).

Let us define the vertex set of \( W_n \) by \( V(W_n) = \{ u_i, i = 1, 2, \ldots, n \} \cup \{ c \} \)
and the edge set by \( E(W_n) = \{ cu_i, i = 1, 2, \ldots, n \} \cup \{ u_i u_{i+1}, i = 1, 2, \ldots, n \} \)
where \( u_{n+1} = u_1 \).

Insert vertices \( v_i \) between \( u_i \) and \( u_{i+1} \) and vertices \( w_i \) between \( c \) and \( u_i \), \( i = 1, 2, \ldots, n \). Thus we obtain the plane graph \( S(W_n) \) with \( 3n+1 \) vertices and \( 4n \) edges.

**Theorem 1:** For \( n \geq 3 \) and \( n \) is odd, the graph \( S(W_n) \) has consecutive labeling of type \((1,0,0)\).

**Proof:** Let us denote the label given to any vertex \( u \) by \( f(u) \) and the weight of the face \( f_i \) by \( \text{wt}(f_i) \). The weight of the \( i^{th} \) face is
\[ \text{wt}(f_i) = f(c) + f(u_i) + f(u_{i+1}) + f(v_i) + f(w_i) + f(w_{i+1}). \]

Define a mapping \( f: V(S(W_n)) \rightarrow \{1, 2, \ldots, 3n+1\} \) in the following way,

\[
\begin{align*}
    f(c) &= 1 \\
    f(u_i) &= \begin{cases} 
        \frac{i+3}{2}, & \text{i is odd} \\
        \frac{i+3+n}{2}, & \text{i is even}
    \end{cases} \\
    f(v_i) &= \begin{cases} 
        \frac{4n+3-i}{2}, & \text{i is odd} \\
        \frac{3n+3-i}{2}, & \text{i is even}
    \end{cases} \\
    f(w_i) &= \begin{cases} 
        \frac{6n+3-i}{2}, & \text{i is odd} \\
        \frac{5n+3-i}{2}, & \text{i is even}
    \end{cases}
\end{align*}
\]

Clearly,

\[
\text{wt}(f_i) = \begin{cases} 
    \frac{16n+17-i}{2}, & \text{i is odd} \\
    \frac{15n+17-i}{2}, & \text{i is even}
\end{cases}
\]

we can see that the weights of all faces assume the consecutive values 7n+9, 7n+10, \ldots, 8n+8. Thus the subdivided wheel graph \( S(W_n) \), \( n \geq 3 \) odd, has consecutive labeling.
Fig. 2.1: Consecutive labeling of $S(W_n)$

Theorem 2: If $n \geq 2$ and $n$ is even, the graph $S(W_n)$ has consecutive labeling of type $(1,0,0)$.

Case 1: $\frac{n}{2}$ is even, $n \geq 4$. Define the mapping

$$f: V(S(W_n)) \rightarrow \{1,2,...,3n+1\}$$

in the following way.

$$f(c) = 1$$

$$f(u_i) =
\begin{cases}
  i + 1, & \text{i is odd} \\
  i + 1 + \frac{n}{2}, & i = 2,4,...,\frac{n}{2} \\
  i + 1 - \frac{n}{2}, & i = \frac{n}{2} + 2,\frac{n}{2} + 4,...,n
\end{cases}$$

$$f(v_i) =
\begin{cases}
  2n + 2 - i, & \text{i is odd} \\
  \frac{3n + 4 - 2i}{2}, & i = 2,4,...,\frac{n}{2} \\
  \frac{5n + 4 - 2i}{2}, & i = \frac{n}{2} + 2,\frac{n}{2} + 4,...,n
\end{cases}$$
3n + 2 - i,  \text{ i is odd} \\
\frac{5n + 4 - 2i}{2},  \quad i = 2, 4, \ldots, \frac{n}{2} \\
\frac{7n + 4 - 2i}{2},  \quad i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n.

Clearly, \\
8n + 9 - i,  \text{ i is odd} \\
\frac{15n + 18 - 2i}{2},  \quad i = 2, 4, \ldots, \frac{n}{2} \\
\frac{17n + 18 - 2i}{2},  \quad i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n.

We can see that the weights of all faces are the consecutive integers of the set \{7n+9, 7n+10, \ldots, 8n\}.

Fig. 2.2: Consecutive labeling of S (W12)

Case 2: \( n \) is odd, \( n \geq 2 \). Choose an even integer \( k \) which is either \( \frac{n-2}{4} \) or \( \frac{n+2}{4} \).
Then define

\[ k_1 = \begin{cases} \frac{n + 2}{2} & \text{if } k = \frac{n - 2}{4} \\ \frac{n - 2}{2} & \text{if } k = \frac{n + 2}{4} \end{cases} \]

Now, the mapping \( f : V(S(W_n)) \rightarrow \{1, 2, ..., 3n+1\} \) is defined as follows:

\[ f(c) = 1 \]

\[ f(u_i) = \begin{cases} 2i, 
& i = 1, 3, ..., \frac{n}{2} \\
& 2i-n, 
& i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n-1. \end{cases} \]

\[ f(v_i) = \begin{cases} \frac{4i + n}{2}, 
& i = 2, 4, ..., k \\
& \frac{4i - n}{2}, 
& i = k + 2, k + 4, ..., k + k_1 \\
& \frac{4i - 3n}{2}, 
& i = k + k_1 + 2, k + k_1 + 4, ..., n. \end{cases} \]

\[ f(v_i) = \begin{cases} \frac{3n + 6 - 4i}{2}, 
& i = 2, 4, ..., k \\
& \frac{5n + 6 - 4i}{2}, 
& i = k + 2, k + 4, ..., k + k_1 \\
& \frac{7n + 6 - 4i}{2}, 
& i = k + k_1 + 2, k + k_1 + 4, ..., n. \end{cases} \]
\[ f(w_i) = \begin{cases} 
3n+3-2i, & i=1,3,\ldots, \frac{n}{2} \\
4n+3-2i, & i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n-1. 
\end{cases} \]
\[ w_i = \begin{cases} 
5n + 6 - 4i & , i = 2, 4, \ldots, k \\
7n + 6 - 4i & , i = k + 2, k + 4, \ldots, k + k_i \\
9n + 6 - 4i & , i = k + k_i + 2, k + k_i + 4, \ldots, n 
\end{cases} \]

It is easy to verify that,
\[ \begin{cases} 
8n+10-2i, & i=1,3,\ldots, \frac{n}{2} \\
9n+10-2i, & i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n-1. 
\end{cases} \]
\[ w_i = \begin{cases} 
15n + 20 - 4i & , i = 2, 4, \ldots, k \\
17n + 20 - 4i & , i = k + 2, k + 4, \ldots, k + k_i \\
19n + 20 - 4i & , i = k + k_i + 2, n + k_i + 4, \ldots, n 
\end{cases} \]

It can be seen, in this case also, that the weights of all faces vary from \(7n+9\) to \(8n+8\).
Thus in both the cases the graph $S(W_n)$ possesses consecutive labeling.

2.2 Consecutive labeling for Subdivision of windmill graph

If in a finite graph $G$, every pair of distinct vertices has a unique neighbor, then $G$ consists of a finite number of triangles with exactly one vertex common.

Such a graph $G$ called a windmill graph [5] is shown in fig. 2.4.
A path of length $n$ is called an $n$-path and is denoted by $P_{n+1}$. Thus the windmill graph $T_n$ can be viewed as $nP_2 + K_1$. This graph $T_n$ is having $2n+1$ vertices and $3n$ edges. The subdivision graph $S(T_n)$ is obtained, as usual, by subdividing each of the $3n$ edges. Clearly $S(T_n)$ has $5n+1$ vertices and $6n$ edges.

**Theorem 3:** For all positive integers $n \geq 2$, the graph $S(T_n)$ has consecutive labeling.

**Proof:** The graph $S(T_n)$ is having $n$ copies of $C_6$ with a common vertex $c$. The mapping $f : V(S(T_n)) \rightarrow \{1, 2, \ldots, 5n+1\}$ is defined as follows: The five vertices of the $i^{th}$ $C_6$ is labeled with $2i-1$, $2i$, $4n+2-2i$, $4n+3-2i$, $4n+1+i$, $i = 1, 2, \ldots, n$, in any order, and $f(c) = 2n+1$. The weight of the $i^{th}$ face, $wt(f_i) = \text{sum of the labels of these five vertices of } i^{th} C_6$ and $f(c) (=2n+1)$.

Clearly $wt(f_i) = 14n+6+i$, $i=1,2,\ldots n$

and $wt(f_{i+1}) - wt(f_i) = 1$. Thus the weights of the $n$ faces of $S(T_n)$ possess the consecutive integers from $14n+7$ to $15n+6$, proving the assertion.
2.3 Consecutive labeling for subdivision of general windmill graph

The generalization of $T_n$, denoted by $T_n^k$, is obtained by having $n$ copies of $P_k$, $k \geq 2$ and joining the two end vertices of each $P_k$ to the vertex $c$ of $K_1$. The vertices of $P_k$ are denoted by $u_1, u_2, \ldots, u_k$. The subdivision graph $S(T_n^k)$ is obtained by inserting $(k-1)$ vertices $v_j$ between $u_ju_{j+1}$ for $j = 1, 2, \ldots, k-1$ and the vertices $x_i$ and $y_i$ between $u_ic$ and $u_{k+1}$ respectively. The subdivision graph $S(T_n^k)$ has $n(2k+1) + 1$ vertices and $2n(k+1)$ edges.

Theorem 4: The subdivision graph $S(T_n^k)$ has consecutive labeling.

Proof: Case 1: $k$ is even

The mapping $f : V(S(T_n^k)) \rightarrow \{1, 2, \ldots, n(2k+1)+1\}$ is defined as follows: $f(c) = nk+1$. For $i = 1, 2, \ldots, n$, the vertices $u_1, u_2, \ldots, u_k$ of the $i$th face are labeled with $i, 2n+i, 4n+i, \ldots, (k-2)n + i, (k+1)n+i, (k+2)n+1+i, \ldots, (2k-2)n+1+i$, the $k-1$
vertices $v_1, v_2, \ldots, v_{k-1}$ are labeled with $2n+1-i, 4n+1-i, \ldots, kn+1-i, (k+2)n+2-i, \ldots, (2k-2)n+2-i; f(x_i) = 2kn+2-i$ and $f(y_i) = 2kn+1+i$.

Thus $wt(f_i) = \text{the sum of the labels of all the vertices of } i^{th} \text{ face.}$

$$= 2(nk+1)(k+1) + nk + i, \quad i = 1, 2, \ldots, n.$$ 

Clearly, $wt(f_{i+1}) - wt(f_i) = 1$ and so the weights of the $n$ faces of $S(T^n_k)$ possess the consecutive integers from $(nk+1)(2k+3)$ to $(2nk + n+2)(k+1)$.

![Diagram](image)

**Fig. 2.7: Consecutive labeling of** $S(T^n_k)$.

**Case 2: $k$ is odd**

The mapping $f : V(S(T^n_k)) \rightarrow \{1, 2, \ldots, n(2k+1)+1\}$ is defined as follows. $f(c) = nk+1$. For all $i = 1, 2, \ldots, n$, the $k$ vertices $u_1, u_2, \ldots, u_k$ of the $i^{th}$ face are labeled with $i, 2n+i, 4n+i, \ldots, (k-1)n+i, (k+1)n+1+i, \ldots, (2k-2)n+1+i; \text{ the } (k-1)$ vertices $v_1, v_2, \ldots, v_{k-1}$ are labeled with $2n+1-i, 4n+1-i, \ldots, (k-1)n+1-i, (k+1)n+2-i, \ldots, (2k-2)n+2-i, f(x_i) = 2kn+2-i, f(y_i) = 2kn+1+i.$
As in Case i, we have \( wt(f_i) = 2(nk+1)(k+1) + nk + i, i=1,2, \ldots, n. \)

Clearly \( wt(f_{i+1}) - wt(f_i) = 1 \) and the weights of \( n \) faces of \( S(T_n^k) \) possess the consecutive integers from \((nk+1)(2k+3)\) to \((2nk+n+2)(k+1)\).

![Fig. 2.8: Consecutive labeling of \( S(T_4^3) \)](image)

In both cases \( S(T_n^k) \) possesses consecutive labeling. Hence the subdivision graph \( S(T_n^k) \) has consecutive labeling for all integers \( n \geq 2, k \geq 2. \)