Chapter 1
MEAN LABELING OF THE GRAPH < $K_{1,n}$ : $K_{1,m}$ >

Somasundaram and Ponraj [40] have introduced the notion of mean labelings of graphs. Several general graphs such as $P_n$, $C_n$, $K_{2,n}$, $P_m \times P_n$, $P_m \times C_n$, $K_{1,n}$ if and only if $n < 3$, bistars $B_{m,n}$ ($m > n$) if and only if $m < n+2$ and the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$ are proved to be mean graphs. It is also proved that the complete bipartite graphs $K_{1,n}$ and the wheels $W_n$ are not mean graphs if $n > 3$ [40].

In this chapter, we prove that the graph < $K_{1,n}$ : $K_{1,m}$ > for $m = n$ is a mean graph if $m \leq n + 2$ and it is not a mean graph if $m > n + 2$.

**Definition 1:** A graph $G = (V, E)$ with $p$ vertices and $q$ edges is called a mean graph if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $0, 1, \ldots, q$ in such a way that when each edge $e = uv$ is labeled with $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even, and $(f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct. $f$ is called a mean labeling.

**Definition 2:** The graph < $K_{1,n}$ : $K_{1,m}$ > is defined as the graph obtained by joining the center $u$ of the star $K_{1,n}$ and the center $v$ of another star $K_{1,m}$ to a new vertex $w$.

Clearly the number of vertices in this graph is $n + m + 3$ and the number of edges is $n + m + 2$; $u_1$ and $v_i$, $i = 1, 2, \ldots, n$ denote the vertices adjacent to $u$ and $v$ respectively.
Observation 1: If G is a mean graph, then the edges get labels 1, 2, ..., q.

Observation 2: If f(u) is even and x is an even integer such that 0 ≤ x ≤ q, then x and x-1 are not simultaneously labels of vertices adjacent to u.

Observation 3: If f(u) is odd and x is an even integer such that 0 ≤ x ≤ q, then x and x + 1 are not simultaneously labels of vertices adjacent to u.

1.1 Existence of mean labeling

Theorem 1: The graph <K₁,n:K₁,m> for m = n is a mean graph.

Proof: Since m = n, the number of vertices is 2n + 3 and the number of edges is 2n + 2. For the mean labeling of this graph, the vertices are labeled from 0, 1, ..., 2n + 2 as follows:

\[ f(u) = 0, \quad f(u_i) = 2i, \quad f(v_i) = 2i - 1, \quad \text{for } i = 1, 2, \ldots, n, \]

\[ f(w) = 2n + 1 \quad \text{and} \quad f(v) = 2n + 2. \]

Clearly the labeling uses all the numbers from 0 to 2n + 2. By Observation 2 and Observation 3, the edge labels are all distinct. Thus, the graph <K₁,n:K₁,m> for m = n is a mean graph.
Theorem 2: The graph \( K_{1,n} : K_{1,m} \) for \( m = n + 1 \) is a mean graph.

Proof: Since \( m = n + 1 \), the number of vertices is \( 2n + 4 \) and the number of edges is \( 2n + 3 \). For the mean labeling of this graph, the vertices are labeled from 0, 1, \ldots, \( 2n + 3 \) as follows:

\[
\begin{align*}
f(u) &= 0, \\
f(u_i) &= 2i, \\
f(v_i) &= 2i - 1, \\
f(w) &= 2n + 1, \\
f(v) &= 2n + 3, \\
f(v_{n+i}) &= 2n + 2 \quad \text{and} \quad f(v_{n+2}) = 2n + 3.
\end{align*}
\]

Clearly all the numbers from 0 to \( 2n + 3 \) are used in labeling this graph. By observation 2 and observation 3, the edge labels are all distinct. Thus, the graph \( K_{1,n} : K_{1,m} \) for \( m = n + 1 \) is a mean graph.

Theorem 3: The graph \( K_{1,n} : K_{1,m} \) for \( m = n + 2 \) is a mean graph.

Proof: Since \( m = n + 2 \), the number of vertices is \( 2n + 5 \) and the number of edges is \( 2n + 4 \). For the mean labeling of this graph, we use the numbers from 0, 1, \ldots, \( 2n + 4 \) as follows:

\[
\begin{align*}
f(u) &= 0, \\
f(u_i) &= 2i, \\
f(v_i) &= 2i - 1, \\
f(w) &= 2n + 1, \\
f(v) &= 2n + 3, \\
f(v_{n+i}) &= 2n + 2, \\
&\quad \text{and} \quad f(v_{n+2}) = 2n + 4.
\end{align*}
\]
Clearly the labeling uses all the numbers from 0 to \(2n + 4\) and the edge labels are all distinct. Thus, the graph \(<K_{1,n}:K_{1,m}>\) for \(m = n + 2\) is a mean graph.

**Observation 4:** For some vertex \(x\), if \(f(x) > 2\), then any vertex \(y\) adjacent to \(x\) cannot have \(f(y) = 0\). For, if it were so, then no edge will get the label 1.

**Observation 5:** For some vertex \(x\), if \(f(x) = 2n + 5\), then any one of the vertices \(y\) adjacent to \(x\) must have the label \(f(y) = 2n + 4\). Otherwise, no edge will get the label \(2n + 5\).

### 1.2 Non-existence of mean labeling

**Theorem 4:** The graph \(<K_{1,n}:K_{1,m}>\) for \(m > n + 2\) is not a mean graph.

**Proof:** It is enough to establish the theorem for \(m = n + 3\). For, if \(m > n + 3\), then the graph \(<K_{1,n}:K_{1,m}>\) will naturally contain the graph \(<K_{1,n}:K_{1,n+3}>\) as a sub graph and hence mean labeling of the graph for \(m > n + 3\) cannot exist.

Since \(m = n + 3\), the number of edges is \(2n + 5\) and so the vertices should be labeled from 0 to \(2n + 5\).
Case 1a: Let $f(w) = 2n + 5$ and $f(v) = 2n + 4$.

By Observation 4, no $v_i$ can get the label 0 and therefore the minimum edge label for any $v_i$ is $n + 3$. As there are $n + 3$ such $v_i$'s, one of these edges will get the label $2n + 5$ which is already assigned to the edge $vw$. Thus the edge labels are not distinct.

Case 1b: Let $f(w) = 2n + 4$ and $f(v) = 2n + 5$.

A similar argument for Case 1a holds good in this case.
Case 2a: Let $f(w) = 2n + 5$ and $f(u) = 2n + 4$ and $f(v) = 2$.

The maximum edge number which can be assigned to any $v_i$ is $n + 3$ which we get by assigning $f(v_i) = 2n + 3$. But this number $2n + 3$ is again needed to label one of the $u_i$'s to get $f(u_i) = 2n + 4$. Thus labeling with distinct numbers is not possible.

Case 2b: Let $f(u) = 2n + 4$ and $f(w) = 2n + 5$ and $f(v) = 0$ or $1$.

Since $f(u) = 2n + 4$ and $f(w) = 2n + 5$, the next highest number is $2n + 3$. Suppose that $2n + 3$ is assigned to any one of the $v_i$'s. Then the maximum edge
number to $v_i$ is $n + 2$. But the number of $v_i$ edges is $(n + 3)$. So the mean labeling is not possible.

A similar case holds if any one of the $u_i$ is assigned the number $2n + 3$.

Case 3a: Let $f(w) = 2n + 4$, $f(u) = 2n + 5$ and $f(v) = 2$.

The maximum possible label to any $v_i$ is $2n + 3$ which leads to $f(v_{vi}) = n + 3$. But already $f(vw) = n + 3$. Thus labeling with distinct numbers is not possible.

Case 3b: Let $f(w) = 2n + 4$, $f(u) = 2n + 5$ and $f(v) = 0$ or $1$.

A similar argument in Case 2b holds good.
Case 4a: Let $f(v) = 2n + 4$ and $f(v_i) = 2n + 5$ for some $i$.

Let $f(u) = 0$ or $1$. The maximum possible label to $u_i$ is $2n + 3$. Then the maximum edge label to $u_i$ is $n + 2$. Also the minimum possible label to $v_i$ is 2, which implies that the corresponding edge label to $v_i$ varies from $n + 3$ to $2n + 5$. Thus the numbers from 1 to $n + 2$ must be the labels to the edges $uw$, $vw$ and $u_i$ for $i = 1, 2, \ldots, n$. Therefore, any two of the numbers between 1 and $n + 2$ are the edge numbers to $uw$ and $vw$. To give one of $1, 2, \ldots, n + 2$ to $uw$, $f(w)$ must vary from 1 to $2n + 3$. But then $f(vw)$ will vary from $n + 3$ to $2n + 4$ and this has already been the edge label of some $v_i$.

The Cases 4b, 5a and 5b written below are similar to Case 4a and so the proof is omitted.
Case 4b: Let $f(v_i) = 2n + 5$ for some $i$, $f(v) = 2n + 4$ and $f(u) = 2$.

Case 5a: Let $f(v) = 2n + 5$, $f(v_i) = 2n + 4$ for some $i$ and $f(u) = 0$ or $1$.

Case 5b: Let $f(v) = 2n + 5$, $f(v_i) = 2n + 4$ for some $i$ and $f(u) = 2$. 
Case 6a: Let \( f(u) = 2n + 5 \), \( f(u_i) = 2n + 4 \) for some \( i \) and \( f(v) = 0 \) or \( 1 \).

The maximum possible label number to any edge \( vv_i \) is \( n + 2 \) with \( f(v_i) = 2n + 3 \) for some \( i \). But there are \( n + 3 \) edges. So, all \( vv_i \)'s cannot be labeled distinctly.

Case 6b: Let \( f(u) = 2n + 5 \), \( f(u_i) = 2n + 4 \) for some \( i \) and \( f(v) = 2 \).

The maximum possible number to \( v_i \) and \( w \) is \( 2n + 3 \) and hence the maximum edge label to any edge incident on \( v \) is \( n + 3 \). But there are \( n + 4 \) edges, including \( vw \). So, \( vv_i \)'s cannot be labeled distinctly.
Case 7a: Let \( f(u) = 2n + 4 \), \( f(u_i) = 2n + 5 \) for some \( i \) and \( f(v) = 0 \) or \( 1 \).

Same as Case 6a.

Case 7b: Let \( f(u) = 2n + 4 \), \( f(u_i) = 2n + 5 \) for some \( i \) and \( f(v) = 2 \).

Same as Case 6b.

Thus, by summing up all the cases, the mean labeling of the graph \( <K_{1,n}:K_{1,m}> \) for \( m = n + 3 \) is impossible. Hence this graph is not a mean graph.