Chapter 4
CONSECUTIVE LABELINGS FOR THE SUBDIVISION
OF SWINGS AND DOVE TAILED GRAPHS

4.1 swing graphs

Consider $n$ copies of $C_k$ and join one vertex of each $C_k$ to a common vertex $c$; the new graph so formed is called a swing graph $S_n^k$. The subdivision of $S_n^k$ is then obtained by subdividing each edge of $S_n^k$. The resulting graph is denoted by $S(S_n^k)$ and it has naturally $n(2k + 1) + 1$ vertices and $2n(k + 1)$ edges.

![Subdivision of Swing graph $S_n^4$](image)

Fig. 4.1: Subdivision of Swing graph $S_n^4$

In the following theorems we prove that $S(S_n^k)$ has consecutive labeling for all integers $n \geq 2$ and $k \geq 3$. 
Theorem 1: The subdivision graph $S(S^n_k)$ has consecutive labeling for $n \geq 2$ and $k \geq 3$ when both $n$ and $k$ are even.

Proof: For $j = 1, 2, ..., n$ and $i = 1, 2, ..., k$ the vertices of $C_k$ are denoted by $u_{j,i}$; the new vertices inserted while subdividing each edge of $C_k$ are denoted by $v_{j,i}$; and the new vertices between one vertex of each $C_k$ and the common vertex $c$ are denoted by $w_j$.

The mapping $f : V(S(S^n_k)) \rightarrow \{1, 2, ..., n(2k + 1) + 1\}$ is defined as follows:

$$f(u_{j,i}) = \begin{cases} 
\frac{n(2i-1) + j}{2}; j = 1,2,...,n \\
\frac{n(2i-1) + 1 + j}{2}; j = 1,2,...,n \\
\end{cases}\quad j = 1,2,...,n$$

$$f(v_{j,i}) = \begin{cases} 
\frac{n(2i+1) - j}{2}; j = 1,2,...,n \\
\frac{n(4i+1) + 1 + j}{2}; j = \frac{n}{2} + 1,...,n \\
\frac{n(4i-1) + j}{2}; j = \frac{n}{2}; j = \frac{n}{2} + 1,...,n \\
\frac{n(2i+1) + 2 - j}{2}; j = \frac{n}{2} + 1,...,k; j = 1,2,...,n \\
\end{cases}\quad j = 1,2,...,n$$

$$f(w_j) = j, j = 1,2,...,n$$

$$f(c) = \frac{n(2k + 1) + 2}{2}$$

Clearly $\text{wt}(f_j) = \sum_{i=1}^{k} \{f(u_{j,i}) + f(v_{j,i})\}, j = 1, 2, ..., n \quad \ldots (*)$

$$\begin{cases} 
n(2k^2 + 2k - \frac{1}{2}) + 2k + 2j, j = 1,2,...,\frac{n}{2} \\
n(2k^2 + 2k - \frac{3}{2}) + 2k + 2j - 1, j = \frac{n}{2} + 1,...,n \\
\end{cases}\quad \ldots (1a)$$

$$\begin{cases} 
n(2k^2 + 2k - \frac{1}{2}) + 2k + 2j, j = 1,2,...,\frac{n}{2} \\
n(2k^2 + 2k - \frac{3}{2}) + 2k + 2j - 1, j = \frac{n}{2} + 1,...,n \\
\end{cases}\quad \ldots (1b)$$

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If equation (1a) gives all odd (or even) integers from \(n(2k^2 + 2k - \frac{1}{2}) + 2k + 1\) to \(n(2k^2 + 2k + \frac{1}{2}) + 2k\) then equation (1b) gives all even (or odd) integers from \(n(2k^2 + 2k - \frac{1}{2}) + 2k + 1\) to \(n(2k^2 + 2k + \frac{1}{2}) + 2k - 1\). Thus \(\text{wt}(l)\) constitute the set of consecutive integers from \(n(2k^2 + 2k - \frac{1}{2}) + 2k + 1\) to \(n(2k^2 + 2k + \frac{1}{2}) + 2k\) and hence \(S(S_n^k)\) for both \(n\) and \(k\) even, has consecutive labeling.

\[\]

Fig. 4.2: Consecutive labeling for \(S(S_n^k)\)

**Theorem 2:** The subdivision graph \(S(S_n^k)\) has consecutive labeling for \(n \geq 2\) odd and \(k \geq 3\) even.

**Proof:** The mapping \(f: V(S(S_n^k)) \rightarrow \{1, 2, \ldots, n(2k+1)+1\}\) is given as follows:
\[
\begin{align*}
\text{f}(u_{ij}) &= \begin{cases} 
  n(2i - 1) + j, i = 1, 2, \ldots, \frac{k}{2}; j = 1, 2, \ldots, n \\
  n(2i - 1) + 1 + j, i = \frac{k}{2} + 1, \ldots, k; j = 1, 2, \ldots, n 
\end{cases} \\
\text{f}(v_{ji}) &= \begin{cases} 
  n(2i + 1) + 1 - j, i = 1, 2, \ldots, \frac{k-2}{2}; j = 1, 2, \ldots, n \\
  n(4i + 1) + 3 + j, i = \frac{k}{2}; j = 1, 2, \ldots, \frac{n-1}{2} \\
  n(4i - 1) + 3 + j, i = \frac{k}{2}; j = \frac{n+1}{2}, \ldots, n \\
  n(2i + 1) + 2 - j, i = \frac{k+2}{2}, \ldots, k; j = 1, 2, \ldots, n 
\end{cases} \\
\text{f}(w_j) &= j; j = 1, 2, \ldots, n \\
\text{f}(c) &= nk + 1
\end{align*}
\]

Using equation (*) we get
\[
\text{wt}(f_j) = \begin{cases} 
  n(2k^2 + 2k - \frac{1}{2}) + 4k + 1 + 2j; j = 1, 2, \ldots, \frac{n-1}{2} \\
  n(2k^2 + 2k - \frac{3}{2}) + 4k + 1 + 2j; j = \frac{n+1}{2}, \ldots, n
\end{cases} \quad \text{... (2a)}
\]
\[
\begin{cases} 
  n(2k^2 + 2k + \frac{1}{2}) + 4k - 1 + 2j; j = 1, 2, \ldots, \frac{n-1}{2} \\
  n(2k^2 + 2k + \frac{3}{2}) + 4k + 1 + 2j; j = \frac{n+1}{2}, \ldots, n
\end{cases} \quad \text{... (2b)}
\]

If equation (2a) gives all odd (or even) integers from \(n(2k^2 + 2k - \frac{1}{2}) + \frac{4k+5}{2}\) to \(n(2k^2 + 2k + \frac{1}{2}) + \frac{4k-1}{2}\) then equation (2b) gives even (or odd) integers from \(n(2k^2 + 2k - \frac{1}{2}) + \frac{4k+3}{2}\) to \(n(2k^2 + 2k + \frac{1}{2}) + \frac{4k+1}{2}\). Clearly the weight of the faces vary from \(n(2k^2 + 2k - \frac{1}{2}) + \frac{4k+3}{2}\) to \(n(2k^2 + 2k + \frac{1}{2}) + \frac{4k+1}{2}\) and hence the graph \(S(S_n^k)\) for \(n\) odd and \(k\) even has consecutive labeling.
Theorem 3: The subdivision graph $S(S^k_n)$ has consecutive labeling for $n \geq 2$ odd and $k \geq 3$ odd.

Proof: The mapping $f : V(S(S^k_n)) \rightarrow \{1, 2, \ldots, n(2k + 1) + 1\}$ is defined as follows:

$$f(u_{i,j}) = \begin{cases} 
\frac{n(2i - 1) + j}{2}, & i = 1, 2, \ldots, \frac{k-1}{2}; j = 1, 2, \ldots, n \\
\frac{n(4i - 1) + 5}{2} - j, & i = \frac{k+1}{2}; j = 1, 2, \ldots, \frac{n+1}{2} \\
\frac{n(4i + 1) + 5}{2} - j, & i = \frac{k+1}{2}; j = \frac{n+3}{2}, \ldots, n \\
\frac{n(2i - 1) + 1 + j}{2}, & i = \frac{k+3}{2}, \ldots, k; j = 1, 2, \ldots, n
\end{cases}$$

$$f(v_{i,j}) = \begin{cases} 
\frac{n(2i + 1) + 1 - j}{2}, & i = 1, 2, \ldots, \frac{k-1}{2}; j = 1, 2, \ldots, n \\
\frac{n(2i + 1) + 2 - j}{2}, & i = \frac{k+1}{2}, \ldots, k; j = 1, 2, \ldots, n
\end{cases}$$

$$f(w_j) = j, j = 1, 2, \ldots, n$$

$$f(c) = nk + 1$$
Clearly \( \text{wt}(f_j) = \begin{cases} \frac{n(2k^2 + 2k + \frac{1}{2}) + \frac{5}{2} + 2(k - j)}{2}, j = 1, 2, ..., \frac{n+1}{2} & \ldots (3a) \\ \frac{n(2k^2 + 2k + \frac{3}{2}) + \frac{5}{2} + 2(k - j)}{2}, j = \frac{n+3}{2}, ..., n & \ldots (3b) \end{cases} \)

If equation (3a) gives all odd (or even) integers from \( n(2k^2 + 2k - \frac{1}{2}) + 2k + \frac{3}{2} \) to \( n(2k^2 + 2k + \frac{1}{2}) + 2k + \frac{1}{2} \) then equation (3b) gives all even (or odd) integers from \( n(2k^2 + 2k - \frac{1}{2}) + 2k + \frac{1}{2} \) to \( n(2k^2 + 2k + \frac{1}{2}) + 2k - \frac{1}{2} \).

The weights of the faces assume all the consecutive integers from \( n(2k^2 + 2k - \frac{1}{2}) + 2k + \frac{3}{2} \) to \( n(2k^2 + 2k + \frac{1}{2}) + 2k + \frac{1}{2} \) and so the graph \( S(S^4_n) \) for both \( n \) and \( k \) odd has consecutive labeling.

Fig. 4.4: Consecutive labeling for \( S(S^4_n) \)
Theorem 4: The subdivision graph $S(S_n^k)$ has consecutive labeling for $n \geq 2$ even and $k \geq 3$ odd.

Proof: The mapping $f: V(S(S_n^k)) \rightarrow \{1, 2, ..., n(2k+1)+1\}$ is given as follows:

$$f(u_{ji}) = \begin{cases} 
\frac{n(2i-1) + j}{2}; & j = 1, 2, ..., n \\
\frac{n}{2} (4i-1) + 1 - j; & j = 1, 2, ..., n \\
\frac{n}{2} (4i+1) + 2 - j; & j = 1, 2, ..., n \\
\frac{n(2i-1) + 1 + j}{2}; & j = 1, 2, ..., n \\
\frac{n(2i+1) + 1 - j}{2}; & j = 1, 2, ..., n \\
\frac{n(2i+1) + 2 - j}{2}; & j = 1, 2, ..., n \\
\frac{n(2i + 1) + 1 + j}{2}; & j = 1, 2, ..., n \\
\frac{n(2i + 1) + 2}{2} 
\end{cases}$$

Clearly $w(f_j) = \frac{n(2k^2 + 2k - 1) + 2(k - j) + 1}{2}; j = 1, 2, ..., n \ldots (4a)$

If equation (4a) gives all odd (or even) integers from $n(2k^2 + 2k - 1) + 2k + 1$ to $n(2k^2 + 2k + 1) + 2k - 1$ then equation (4b) gives all even (or odd) integers from $n(2k^2 + 2k - 1) + 2(k + 1)$ to $n(2k^2 + 2k + 1) + 2k$. Thus the weights of all faces
possess the consecutive integers from \( n \left( 2k^2 + 2k - \frac{1}{2} \right) + 2k + 1 \) to \( n \left( 2k^2 + 2k + \frac{1}{2} \right) + 2k \) and hence the graph \( S(S_1^k) \) for \( n \) even and \( k \) odd has consecutive labeling.

Combining the above four theorems, we get the general result that the subdivided swing graph \( S(S_1^k) \) has consecutive labeling for all \( n \geq 2 \) and \( k \geq 3 \).

4.2 Dove tailed graphs:

Dove tailed graph \( D_n \) is the graph \( P_n + k_1 \), \( n \geq 2 \). The subdivision of \( D_n \) is obtained by subdividing each edge of \( D_n \) and so \( S(D_n) \) contains \( 3n \) vertices and \( 4n-2 \) edges.
Now, we prove that \((D_n), n \geq 2\) has a consecutive labeling.

**Theorem 5:** The subdivision graph \(S(D_n)\) for all \(n \geq 2\) odd has a consecutive labeling.

**Proof:** We define the mapping \(f: V(S(D_n)) \to \{1, 2, \ldots, 3n\}\) in the following way,

\[
f(c) = 3n
\]

\[
f(u_i) = \begin{cases} 
2n + i, & i = 1 \\
2n + 1 - 2i, & i = 2, 3, \ldots, \left\lfloor \frac{n-3}{2} \right\rfloor \\
2n - 2 + 2i, & i = \left\lfloor \frac{n-1}{2} \right\rfloor \\
1, & i = \left\lfloor \frac{n+1}{2} \right\rfloor \\
4n + 1 - 2i, & i = \left\lfloor \frac{n+3}{2} \right\rfloor \\
2i - 2, & i = \left\lfloor \frac{n+5}{2} \right\rfloor, \left\lfloor \frac{n+7}{2} \right\rfloor, \ldots, n-1 \\
2i, & i = n.
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
2n - i, & i = 1 \\
2i + 1, & i = 2, 3, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \\
2n - 2i, & i = \left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{n+3}{2} \right\rfloor, \ldots, n - 2 \\
2i, & i = n-1
\end{cases}
\]
\[
f(w_i) = \begin{cases} 
2i, & i = 1 \\
2n-2+2i, & i = 2, 3, \ldots, \frac{n-3}{2} \\
2n+1-2i, & i = \frac{n-1}{2} \\
2n-2+2i, & i = \frac{n+1}{2} \\
2i-2, & i = \frac{n+3}{2} \\
4n+1-2i, & i = \frac{n+5}{2}, \frac{n+7}{2}, \ldots, n-1 \\
3, & i = n 
\end{cases}
\]

\[
w_i(f_i) = \begin{cases} 
11n-1+2i, & i = 1, 2, \ldots, \frac{n-3}{2} \\
12n-2-2i, & i = \frac{n-1}{2} \\
12n-1-2i, & i = \frac{n+1}{2} \\
13n-2-2i, & i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n-1 
\end{cases}
\]

It can be seen that the weights of all faces vary from $11n-2$ to $12n-4$.\[\]
Fig. 4.6: Consecutive labeling of $S(D_7)$

The labeling of $S(D_n)$ for any $n$ odd can be diagrammatically represented as in fig. 4.7.

Fig. 4.7: Consecutive labeling of $S(D_n)$, $n$ odd
**Theorem 6:** The subdivision graph $S(D_n)$ for all $n \geq 2$ even has a consecutive labeling.

**Proof:** The mapping $f: V(S(D_n)) \to \{1, 2, ..., 3n\}$ is defined as follows:

$$f(c) = 3n$$

$$f(u_i) =
\begin{cases}
2n+i, & i=1 \\
2n-3, & i=2 \\
2n+1-2i, & i=3,4,..., \frac{n}{2}-2 \\
2n-1+2i, & i=\frac{n}{2}-1, \frac{n}{2} \\
4n-2i, & i=\frac{n}{2}+1, \frac{n}{2}+2 \\
2i-2, & i=\frac{n}{2}+3, \frac{n}{2}+4,...,n-1 \\
2i, & i=n
\end{cases}$$

$$f(v_i) =
\begin{cases}
2n-i, & i=1 \\
2i+1, & i=2,3,...,\frac{n}{2}-1 \\
1, & i=\frac{n}{2} \\
2n-2i, & i=\frac{n}{2}+1, \frac{n}{2}+2,...,n-2 \\
2i, & i=n-1.
\end{cases}$$
\[
f(w_i) = \begin{cases} 
2i, i=1 \\
2n+2, i=2 \\
2n-2+2i, i=3,4,\ldots, \frac{n}{2}-2 \\
2n-2i, i=\frac{n}{2}-1,\frac{n}{2} \\
2i-1, i=\frac{n}{2}+1,\frac{n}{2}+2 \\
4n+1-2i, i=\frac{n}{2}+3,\frac{n}{2}+4,\ldots n-1 \\
2n+3, i=n-1 \\
3, i=n 
\end{cases}
\]

\[
w_i(f_i) = \begin{cases} 
11n-1+2i, i=1,2,\ldots, \frac{n-2}{2} \\
12n-1-2i, i=\frac{n}{2} \\
13n-2-2i, i=\frac{n}{2}+1,\frac{n}{2}+2,\ldots, n-1. 
\end{cases}
\]

It can be seen in this case that the weights of all faces vary from \(11n-1\) to \(12n-3\). The diagrammatic representation of the labeling of \(S(D_n)\) is given in Fig. 4.8.
Thus, from theorems 5 and 6 the subdivision of dove tailed graph has a
Consecutive labeling and so \( S(D_n) \) for all \( n \geq 2 \) is consecutive.