Chapter Three

Surface Geometry
3.1 Differential Geometry of Space Curves
3.1 Differential Geometry of Space Curves

3.1.1 Introduction
A space curve is the track of a point that moves in three dimensional space, called by Mathematicians $E^3$ space where $E$ stands for Euclidean. A plane curve is a space curve that lies in $E^2$ space i.e. in any single fixed plane. In intrinsic coordinates the independent variable is the arc length $s$, and points on the curve are described in terms of their arc length from a starting point $s = 0$. The point that is at a distance $s$ along the curve from the starting point is denoted by $r(s)$. These position vectors are measured from a fixed origin $O$ that in general does not lie on the curve (fig.3.1).

![Fig. 3.1](image)

A Space Curve

3.1.2 Tangent to a Curve
The curve in fig.3.2 is a general (possibly twisted) space curve. The vector $\delta r = r(s+\delta s) - r(s)$ represents the chord PQ joining the two points P and Q with parameters $s$ and $s+\delta s$ on the curve $r(s)$.

![Fig. 3.2](image)

A Twisted Space Curve
As $\delta s \to 0$ the vector $\delta \mathbf{r}/\delta s$ has a direction which approaches the direction of the tangent at $P$, if the curve has a well defined tangent there. The chord length $|\delta \mathbf{r}|$ and the arc length $\delta s$ become equal in the limit.

$$\frac{d\mathbf{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = T \quad \ldots \text{Eq. 3.1}$$

is a vector of unit length in the direction of the tangent to the curve known as the **unit tangent vector**.

For a general parameter $u$, $d\mathbf{r}/du$ is proportional to $T$ so that

$$T = \frac{d\mathbf{r}}{du} \quad \text{provided} \quad \frac{d\mathbf{r}}{du} \neq 0 \quad \ldots \text{Eq. 3.2}$$

Moreover, since

$$T = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du} \quad \text{then} \quad \frac{ds}{du} = \left| \frac{d\mathbf{r}}{ds} \right|$$

If one denotes differentiation by a prime then

$$\mathbf{r}' = s'T$$

The parameterization of a curve should if possible be chosen such that $d\mathbf{r}/du \neq 0$ at any point. This means that $s$ should increase monotonically with $u$ so that $u'>u$ implies $s(u') > s(u)$.

### 3.1.3 The Principal Normal and Binormal to a Curve

For a two dimensional curve, a well-defined normal exists. However, for a three dimensional curve any vector perpendicular to the tangent vector $T$ is a normal vector. In particular, because $T$ is a unit vector, the derivative of it is normal to $T$. The unit vector $N$ perpendicular to $T$ is known as the **principal normal vector**. When the parameter is the arc length, one can write $dT/ds = kN$ where $k$ is known as the **curvature** of the curve. By convention $k>0$, the sense of $T$ being defined by that of the vector $N$. Thus $T = dT/du = s kN$. There is an infinity of vectors that are orthogonal to the tangent vector lying in a normal plane. The vector product $T \times N$ defines a third
unit vector perpendicular to both $T$ and $N$, known as the **Binormal Vector** $B$. The three vectors $T, N$ and $B$ form a right handed set of mutually orthogonal vectors so that $B = T \times N$, $T = N \times B$, and $N = B \times T$. The planes through a given point on the curve which contain the vectors $T$ and $N$, $N$ and $B$ and $B$ and $T$ are known as the **Osculating Plane**, the **Normal Plane** and the **Rectifying Plane** respectively (Fig 3.3).
The word Osculating comes from the Latin word Osculum, meaning a Kiss. Locally a curve lies in its osculating plane. If one considers three successive points on the curve - the point one is at and a point just before and just after it - these are sufficient to define a plane. The osculating plane is the limit plane as the two outside points approach a central line. As a point moves along a space curve the osculating plane will change unless the curve is locally a plane curve.

The normal vector to the osculating plane is the curve binormal B; so the motion of the osculating plane is described by the motion of B. A plane curve has just one osculating plane and all its binormal are parallel.

The word rectifying is much harder to explain. One starts with the definition that a rectifiable curve is a curve whose arc length can be calculated or measured exactly. The complete sequence of rectifying planes as the space curve progresses defines a surface known as the envelope surface of the rectifying planes. Each individual rectifying plane in the sequence is a tangent plane to this surface and the space curve lies on this surface. Now this surface is developable, that is to say it can be rolled out on a flat surface. When this is done the space curve becomes a straight line and is then rectifiable because its arc length can be measured along this straight line.

### 3.1.4 Curvature of the Curve

The radius of curvature $\rho$ of a plane curve $y = y(x)$ is given by the well known formula

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}$$

where the prime denotes differentiation with respect to $x$. Because the radius of curvature becomes infinite at points of inflexion it is usually better to use the curvature $k = 1/\rho$ which is finite unless there are cusps in the curve. Thus

$$k = \frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}}$$

Eq. 3.3

For a space curve the curvature is defined as follows.
The tangent vector $\mathbf{T}(s)$ is a unit vector and hence $\mathbf{T}'(s)$ is orthogonal to $\mathbf{T}(s)$. Here $s$ denotes the arc length. The magnitude of $\mathbf{T}'(s)$ will depend upon how rapidly the curve is bending. If the curve is straight at the point in question the magnitude will be zero but if the curve is not straight one may write

$$\mathbf{T}'(s) = k(s) \mathbf{N}(s)$$

where $\mathbf{N}(s)$ is the unit vector (Principal Normal Vector) in the direction of $\mathbf{T}'(s)$ and $k(s)$ is the magnitude of $\mathbf{T}'(s)$ and $k(s)$ is called the curvature of the curve.

Points on the curve at which the curvature $k(s)$ is zero are called points of inflexion. If there is an S bend in the space curve the vector $\mathbf{T}'(s)$ changes sign as the curve passes through the point of inflexion. Thus the vector $\mathbf{N}(s)$ is flung from one side of the curve to the other at points of inflexion; although elsewhere $\mathbf{n}(s)$ is continuous. The curvature on both sides of the point of inflexion is positive (Fig.3.4).

![Fig. 3.4 Points of Inflexion on the Parent Curve](image)

This is a most unsatisfactory state of affairs. The sudden reversal of $\mathbf{N}$ implies that the binormal vector also reverses. Suppose now that the curve is a plane curve, does it mean that the binormal (which is the normal vector of the plane) reverses at every point of inflexion?

To extricate from this dilemma, one can change the definition of $\mathbf{N}(s)$. Instead of defining it as the direction of $\mathbf{T}'(s)$, one denotes it at will to be either in the same direction as $\mathbf{T}'(s)$ or in the reverse direction at some key point and thereafter maintain its presence even at a point of inflexion without any sudden reversals of direction. This
need not destroy the governing relation $T'(s) = kN$ as one now permits the curvature to be either positive or negative as required [Nutbourne, A.W. et.al. 1988].

The advantage of this redefinition is that the curvature profile- the graph of $k$ as a function of $s$- may now cross the $s$ axis smoothly at all points of inflexion. These are simply seen as zeros of the curvature profile (Fig 3.5).

![Fig. 3.5 Zeros of the Curvature Profile](image)

### 3.1.5 Torsion of a Curve

In the case of a plane curve, the osculating plane is the plane of the curve and the binormal vector $B$ is fixed. Thus $dB/ds = 0$ for a plane curve. On the other hand, when the curve is not plane, the vector $B$ is no longer constant and an understanding of the twisted nature of the curve requires the evaluation of $dB/ds$. One knows that $B.T = 0$ so that

$$\frac{dB}{ds} T + B \frac{dT}{ds} = 0$$

Also

$$B \frac{dT}{ds} = B k N = k (B.N) = 0$$

since $B$ and $N$ are orthogonal vectors. Hence $(dB/ds) \ T = 0$.

Moreover, because $B$ is a unit vector $(dB/ds) \ T = 0$. 22
Thus $dB/ds$ is perpendicular to $T$ and $B$ and is therefore in the direction of $N$. By convention one can write

$$\frac{dB}{ds} = -\tau N \quad \ldots \text{Eq. 3.5}$$

where $\tau$ is called the **Torsion of the curve**. Generally torsion is denoted as $\tau(s)$ which measures how a space curve is twisting as a point moves along it.

$$B'(s) = -\tau(s) N(s)$$

Now

$$\frac{dN}{ds} = \frac{d}{ds}(B \times T) = \left(\frac{dB}{ds} \times T\right) + \left(B \times \frac{dT}{ds}\right) = \left(\frac{dB}{ds} \times T\right) + (B \times kN) = -\tau N \times T + (B \times kN)$$

$$\frac{dN}{ds} = -\tau N \times T + (B \times kN) = \tau B - kT \quad \ldots \text{Eq. 3.6}$$

so that the torsion of a curve is positive when the curve normal turns out of the osculating plane into the positive direction of $B$.

Now, all the principal equations of the differential geometry of space curves are derived. These are collectively known as the **Frenet - Sernet formulae** after the French mathematician who first invented the method of describing the motion of space curve [Nutbourne, A.W. et al. 1988. Faux.I.D. et.al. 1979] and are restated below

$$\frac{dr}{ds} = T \quad \ldots \text{Eq. 3.7}$$

$$\frac{dT}{ds} = k N \quad \ldots \text{Eq. 3.8}$$

$$\frac{dN}{ds} = \tau B - k T \quad \ldots \text{Eq. 3.9}$$

$$\frac{dB}{ds} = -\tau N \quad \ldots \text{Eq. 3.10}$$

### 3.1.6 Examples of Simple Cases

The curvature and torsion profiles written in the form of equations $k = k(s)$ and $\tau = \tau(s)$ are known as the intrinsic equations of the space curve. It can be shown that the shape of a space curve is uniquely defined by these equations (profiles). Regarding the curve as a stiff wire of a certain shape it may be relocated by a change in $r(0)$ and reoriented by a change in $T(0)$ and $N(0)$. Space curves that have the same shape but different
starting parameters are said to be congruent. Some simple space curves will now be described by their profiles.

*Straight Line*

If both the curvature and torsion functions are zero the curve is a simple (untwisted) straight line.

*Circular Arc*

If the curvature $k(s)$ of a plane curve is a constant and non-zero and torsion is zero, the curve is a circular arc.

*Twisted straight line*

If the curvature $k(s)$ is zero for all $s$ but the torsion is not zero the space curve is a twisted straight line whose twist along its length is governed by the torsion profile. It is a useful analogue to consider a straight line in representing a piece of taut elastic. A minor amount of twist in the elastic is permissible without impairing its straightness. The analogy breaks down if the elastic is twisted through a large number of turns because knots start to appear in the elastic as every one who has wound the propeller of a model aeroplane will know. No such limitation exists for a twisted straight line, although in curve synthesis one would not normally expect the twist to be more than a few turns and often only a fraction of a turn. The concept of twisted straight line is fundamental in the synthesis of curves and surfaces because it allows the osculating plane to change along the length of the line to match in with the external curve frames at each end of the line.

*All plane curves*

If the torsion $\tau(s)$ is zero for all $s$ the space curve is a plane curve whose shape is determined by the curvature profile.

*Piece of Cornu Spiral*

If the curvature profile is a linear function of arc length "s" and the torsion is zero, the plane curve is a piece of a Cornu spiral known in American literature as a clothoid, but also called a transition spiral, or railway curve, since it is often used in rail track layout to increase or decrease curvature linearly. The curve has a history, before M.A.Cornu first presented it in pictorial form.[Nutmourne, A.W. et.al. 1988]
**Helices**

The simplest space curve discounting the twisted straight line is a curve whose curvature and torsion profiles are non-zero constants. This is a helix and it may be drawn on the cylindrical surface of a right circular cylinder. The helix is a special case of a generalized helix which has an arbitrary curvature profile but a torsion profile that is proportional to the curvature profile. For a helix, \( k(s) = A, \tau(s) = B \), where \( A \) and \( B \) are constants. For a generalized helix \( k(s) \) is arbitrary and so \( \tau(s) = c \, k(s) \) where \( c \) is a constant.

Other simple cases perhaps have no universally accepted names. For instance, the curvature profile may be constant while the torsion profile is varied. One could call such a curve as a twisted circle. Similarly, the curvature profile may be linear \( k(s) = A \, s \) while the torsion profile is varied. One could call such a curve a twisted Cornu Spiral. As a special case, if the torsion profile is constant, one could call it as a Cornu Spiral helix.
Chapter 3

3.2 Differential Geometry of Surfaces
3.2 DIFFERENTIAL GEOMETRY OF SURFACES

3.2.1 Concept of a surface

A surface can be thought of as a set of points in three-dimensional space. The neighbourhood of each of its points resembles a portion of a plane. A simple sheet of a surface is the set of points whose position vectors are the values of a one-to-one continuous vector valued function \( r = r(u,v) \) defined in a closed rectangle \( a \leq u \leq b \) and \( c \leq v \leq d \) of the plane parameters \( u \) and \( v \). The set of points which satisfy the equation \( z = F(x,y) \) where \( F \) is a continuous function of two variables is an example of a simple sheet of a surface. Indeed in this case \( x \) and \( y \) play the role of parameters and the one to one continuous is \( r = x\hat{i} + y\hat{j} + F(x,y)\hat{k} \). This function is obviously one-to-one since for two different points of the plane \( XY \), the values of the function must differ in at least one component. Intuitively one can say that a simple sheet of a surface is obtained from a rectangle by stretching, squeezing and bending but without tearing or gluing together. Instead of using a rectangle in this definition one could use any bounded, closed, simply connected domain in the plane - in particular a closed disk.

The surface of a cylinder is not a simple sheet because it cannot be obtained from a rectangle without gluing together. Similarly neither a sphere nor a flat annulus is a simple sheet. However a cylinder with an open stripe between two elements removed (fig.3.6) or a sphere with an open cap sector removed (fig.3.7) are simple sheets of a surface. The examples show that the notion of a simple sheet of a surface is too restrictive for the purpose of study of surfaces and indicates a way for an adequate generalization. Unfortunately, one cannot define a surface, in full analogy to curves, as a locally one-to-one and bicontinuous image of a plane, because already a sphere which should certainly be included among the surfaces does not have this property. This forces one to use the more involved definition. Using the notion of an abstract two-dimensional manifold one could define a surface as one-to-one continuous image of an abstract manifold and obtain a generalization admitting self intersections by considering locally one-to-one images of manifolds.
Fig. 3.6
A Cylinder with an Open Strip

Fig. 3.7
A Sphere with an Open Cap
3.2.2 The Surface Normal

In order to calculate the cutter offsets for 3D numerical programming, for example, one needs to determine the normal of the part of the surface being machined. Because the machining of a surface \( \mathbf{r} = \mathbf{r}(u,v) \) is often performed by following the parametric curves \( u = \text{constant} \) and \( v = \text{constant} \), the tangents to these curves are also of interest in some applications. Two sets of constant parameter lines can be drawn on the surface. A particular point on the surface lies at the intersection of one line of constant \( u \) and one line of constant \( v \). These two lines will not intersect orthogonally except fortuitously. The same surface may have more than one parameterization and in the synthesis of a surface one often chooses the parameters so that the constant \( u \) and constant \( v \) are lines of curvature on the surface. In contrast with the arc length parameter that one has used for curves, one cannot consistently choose \( u,v \) to be arc lengths along the lines of curvature. To see this, one has to consider a sphere (fig.3.9). Going a distance \( s_1 \) along the parallel AB and then a distance \( s_2 \) along meridian BC does not arrive at the same point as going a distance \( s_2 \) up the meridian AD and then a distance \( s_1 \) along the parallel DCE.

![Fig. 3.8](image)

The Parallels and Meridians on a Sphere
In contrast, the usual spherical polar coordinates
\[ r(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u) \]
parameterize the surface so that; lines of constant \( u \) are parallels and the lines of constant \( v \) are meridians. For all parameterizations, the tangents to the constant parameter lines are partial derivatives of the position vector. The tangent vector to a parametric curve \( r = r(u, v_0) \) where \( v_0 \) is a constant is a multiple of the vector \( dr/du \).

Similarly, the tangent vector to the curve \( r = r(u_0, v) \) is a multiple of \( dr/dv \). The tangent plane at the intersection of these curves at \( r(u_0, v_0) \) contains these two tangent vectors so that normal to the surface is a multiple of their vector product. The unit normal vector \( N \) is then given by

\[
N = \pm \frac{\begin{vmatrix}
\frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\
\frac{\partial r}{\partial v} & \frac{\partial r}{\partial u}
\end{vmatrix}}{\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|}, \quad \text{Eq. 3.11}
\]

where the derivatives are evaluated at \( u = u_0, v = v_0 \). The sense of \( N \) is chosen to suit the application. This equation is used in section 5.10. The exceptional points where the partial derivatives do not exist or where

\[
\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0
\]

corresponds either to singularities of the parameterization or to ridges or cusps in the surface.

### 3.2.3 The surface curve frame

As surface curve is a space curve that lies on a surface, it can still be described as a space curve using the methods described in the previous section, whenever it is desirable. But the existence of a surface enables one to use a surface curve frame \( F(s) \) in the place of space curve frame \( f(s) \) often to a considerable advantage. Whereas at a point on a curve there is one unit tangent vector \( t \), and an infinity of normal vectors forming the normal plane orthogonal to the tangent vector, the converse is true for a surface. At a point on a
surface there is one unit normal vector \( \mathbf{N} \), and an infinity of tangent vectors forming the tangent plane orthogonal to the normal vector. At any point on a surface, the normal vector is like a pin of unit length sticking out of the surface at right angles to the tangent plane.

At a point on a surface curve whose arc length measured from a datum point on the curve is \( s \), the tangent vector \( \mathbf{t}(s) \) is the same as the tangent vector that it has when considered as a space curve. The change in description of the curve is concerned solely with the normal vectors. The curve normal \( \mathbf{n}(s) \) can now take a secondary role and be supplanted by the surface normal \( \mathbf{N}(s) \). The third vector that makes the surface curve frame lies in the normal plane of the curve, but in the tangent plane of the surface. Indeed, it is the line of intersection of these two planes. It is denoted by \( \mathbf{T}(s) \), which suggests that it is primarily to be thought of as a tangent to the surface. It is orthogonal to both \( \mathbf{t}(s) \) and \( \mathbf{N}(s) \). One can call it the bitangent by analogy with binormal, but this is not a standard name. With regard to the planes, the plane containing \( \mathbf{t} \) and \( \mathbf{T} \) is the tangent plane, the plane containing \( \mathbf{T} \) and \( \mathbf{N} \) is the normal plane, but there is no name for the plane containing \( \mathbf{N} \) and \( \mathbf{t} \). It is called as the Cleaver plane because a butcher smiting the surface along the curve and down the surface normal will have his cleaver in this plane.

The surface curve frame is

\[
\mathbf{F}(s) = \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \end{bmatrix}
\]  

..Eq. 3.12

The vector relations for the right-handed frame are

\[
\mathbf{N}(s) = \mathbf{t}(s) \times \mathbf{T}(s) 
\]  

..Eq. 3.13

\[
\mathbf{T}(s) = \mathbf{N}(s) \times \mathbf{t}(s)  
\]  

..Eq. 3.14

\[
\mathbf{t}(s) = \mathbf{T}(s) \times \mathbf{N}(s)  
\]  

..Eq. 3.15

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3.2.4 Metrical properties of surfaces

Now, the metrical properties of curves lying on a parametric surface and the area of a region of the surface are considered. A curve on the parametric surface \( \mathbf{r} = \mathbf{r}(u,v) \) may be represented parametrically by the equations \( \mathbf{u} = \mathbf{u}(t), \mathbf{v} = \mathbf{v}(t) \), which may be summarized as

\[
\mathbf{u} = \mathbf{u}(t),
\]

where \( \mathbf{u} = [u(t), v(t)]^T \). Whereas \( \mathbf{r}(u,v) \) denotes a general point on the surface, here \( \mathbf{r}(t) \) will be used to denote a point on the curve. A tangent vector to this curve is given by \( \mathbf{r} \), which one may expand by the chain rule to give

\[
\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} u + \frac{\partial \mathbf{r}}{\partial v} v = A \mathbf{u}
\]

where

\[
A = \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{bmatrix}
\]

The length of the tangent vector is given by

\[
s^2 = |\mathbf{r}|^2 = \mathbf{r}^T \mathbf{r} = \mathbf{u}^T A \mathbf{A} \mathbf{u} = \mathbf{u}^T G \mathbf{u}
\]

where

\[
G = A^T A = \begin{bmatrix}
\frac{\partial r}{\partial u} & \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\
\frac{\partial r}{\partial u} & \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\
\frac{\partial r}{\partial v} & \frac{\partial r}{\partial v} & \frac{\partial r}{\partial v}
\end{bmatrix}
\]

This matrix \( G \) is the fundamental matrix of the surface [Faux, I.D. et.al. 1979]. Its importance is seen in the metrical formulae.
The unit tangent vector along the curve \( u = u(t) \) is given by

\[
T = \frac{\dot{r}}{|\dot{r}|} = \frac{u}{(u^T G u)^{1/2}}.
\]

The length of the curve segment \( u = u(t) \), to \( t < t_1 \) is given by

\[
s = \int_0^t \sqrt{\dot{r}^T G \dot{r}} dt = \int_0^t (u^T G u)^{1/2} dt.
\]

If two curves \( u = u(t) \) and \( u = u(t) \) lie on the surface and intersect at angle \( \theta \), then

\[
T_1 \cdot T_2 = -\frac{u_1^T A u_2}{(u_1^T G u_1)^{1/2}(u_2^T G u_2)^{1/2}}.
\]

so that the angle of intersection \( \theta \) between the two curves is given by

\[
\cos \theta = \frac{u_1^T G u_2}{(u_1^T G u_1)^{1/2}(u_2^T G u_2)^{1/2}} \quad \text{...Eq. 3.17}
\]

Finally, is considered the surface area enclosed between the parametric curves \( u = u_0 + u, \ v = v_0 + v \) Approimating this curved surface by the plane parallelogram the approximate area is given by

\[
\delta s \approx \left| \frac{\delta r}{\delta u} \times \frac{\delta r}{\delta v} \right| \delta u \delta v
\]

It is noted that

\[
\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|^2 = \left| \frac{\partial r}{\partial u} \right|^2 \left| \frac{\partial r}{\partial v} \right|^2 - \left( \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial v} \right)^2 = g_{11} g_{22} - g_{12} g_{21} = |G| \quad \text{...Eq. 3.18}
\]

where the \( g_{ij} \) are elements of the first fundamental matrix \( G \).
Thus the area of a bounded region of the surface corresponding to a region \( R \) in the \( u-v \) plane may be obtained from the integral
\[
S = \iint_R \sqrt{\det[\mathbf{G}]} \, du \, dv
\]

One can now derive a condition for the tangent vector \( T \) to be well-defined on all curves \( u = u(t) \), in terms of tangents to the curves of constant \( u \) and constant \( v \). From equation one has the requirement that
\[
u \mathbf{G} u > 0
\]
for every point on the surface. It can be shown that this will be so provided
\[
u \neq 0, g_{11} > 0 \text{ and } |\mathbf{G}| > 0.
\]

Assuming that \( dr/du \) is well-defined everywhere, one can see from equation that \( g_{11} > 0 \) automatically, and since it has been seen that
\[
|\mathbf{G}| = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^2
\]
it follows that the tangent to a general curve \( u = u(t) \) is well-defined at all points for which
\[
u \neq 0 \text{ and } \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq 0
\]
At points where the second condition holds but not the first, the tangent plane and normal to the surface are well-defined but the curve \( u = u(t) \) contains a kink so that \( T \) does not exist there. Where the first condition holds but not the second the surface has no well-defined tangent plane, either because it contains a sharp ridge or because the lines of constant \( u \) and constant \( v \) are parallel there. Then the failure of the condition may stem either from an inherent geometrical property of the surface or from an unfortunate choice of parameterization.
3.2.5 **Torsion in a surface**

One needs a knowledge of surface curves when coming to describe the boundaries of surface patches. A space curve does not lose its identity when it lies on a surface. It retains its own frame \( f(s) \) and its curvature and torsion profiles. What does happen is that it gains a new frame \( F(s) \) with the same tangent vector, but with a new pair of normal vectors at an angle \( \phi(s) \) with the old pair. One of these is the normal to the surface. The curvature profile resolves into two new curvature profiles called normal curvature profile \( n(s) \) and the geodesic curvature profile \( g(s) \). The torsion profile is modified so that it measures the torsion of the new frame and is called the geodesic torsion profile. The surface frame is illustrated in fig. 3.9. The precise definition of \( \phi(s) \) is that it is the angle of rotation that must be applied to the space curve frame \( f(s) \) to convert it to the surface curve frame \( F(s) \). It is therefore the angle between \( b(s) \) and \( N(s) \). [measured positively if this rotation is clockwise when viewed along the tangent vector \( t(s) \)] and it is also the angle between \( n(s) \) and \( T(s) \). It should be noted that the angle will be zero or 180 depending on the curve normal and the surface normal for a plane curve considered as a surface curve lying on its own plane.

![Fig. 3.9](image-url)

**The Surface Frame**
From the fig. 3.9,

\[ T(s) = \cos(\phi(s)) n(s) + \sin(\phi(s)) b(s) \]

\[ N(s) = -\sin(\phi(s)) n(s) + \cos(\phi(s)) b(s) \]

The relation between the surface curve frame \( F(s) \) and space curve frame \( f(s) \) is given by

\[ F(s) = \phi(s) f(s) \]

where

\[ \phi(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi(s)) & \sin(\phi(s)) \\ 0 & -\sin(\phi(s)) & \cos(\phi(s)) \end{bmatrix} \]

When \( \phi(s) = 0 \), \( F(s) \) and \( f(s) \) are coincident.

By analogy with the torsion in a curve, one can write

\[ F'(s) = \Gamma(s) F(s) \]

where

\[ \Gamma(s) = \begin{bmatrix} 0 & k(s)\cos(\phi(s)) & k(s)\sin(\phi(s)) \\ -k(s)\cos(\phi(s)) & 0 & \tau(s) + \phi(s) \\ -k(s)\sin(\phi(s)) & -\tau(s) + \phi(s) & 0 \end{bmatrix} \]
Denoting
\[ g = k \cos \phi \]
\[ n = k \sin \phi \]
\[ t = \tau + \phi \]

\[
\Gamma = \begin{bmatrix}
0 & g & n \\
-g & 0 & t \\
-n & -t & 0 \\
\end{bmatrix}
\]

Here \( g \) is called the geodesic curvature of the curve; \( n \) is called the normal curvature of the curve; it is also the normal curvature of the surface in the direction of the curve. \( \tau \) is called the geodesic torsion. It is also the surface torsion in the direction of the curve. All curves on the surface that pass through a particular point in the same direction must have the same value for the normal curvature and geodesic torsion but they are free to have their own geodesic curvature. While interpolating two contours torsion can be introduced as shown in Fig. 120 in Chapter 7.