Chapter 3

Monoids and Transformation
Semigroups

3.1 Introduction

Malik, Mordeson and Sen [31] have generated two distinct semigroups from the fuzzy transition function of an ffsm. We, on our part, have been successful in generating two other monoids namely $F(M)$ and $S_M$ in an uffsa $M$. These monoids (underlying semigroups) are not only different from the two semigroups mentioned above, but also that they could be generated only in uffsa’s, and are not possible in ffsm’s. The existence of such monoids has been illustrated with example. We are also able to prove in the reverse process, a result that for a given finite fuzzy monoid $S$ there exists an uffsa $M$ such that $F(M)$ is isomorphic to $S$. A new concept called fuzzy anti-transformation semigroup (fats) is defined and is linked with an uffsa in the sense that, given an uffsa $M$, we can construct a faithful fats $(Q, F(M), \rho)$. A similar result
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holds good for anti-polytransformation semigroup as well.

3.2 Fuzzy monoids

In this section we begin with proving the existence of the monid $F(M)$ in an ufsfa $M$ and then illustrate it with an example. Some interesting results are also obtained. We then prove the existence of the other monoid $S_M$.

Definition 3.2.1.

Let $M = (Q, \Sigma, \mu, i, f)$ be an ufsfa.

For $a \in \Sigma$, we define $f_a : Q \to Q$ by

$$f_a(p) = \begin{cases} q, \text{ if } \mu(p, a, q) = \vee \{ \mu(p, a, r) \mid r \in Q \} > 0 \\ p, \text{ if } \vee \{ \mu(p, a, r) \mid r \in Q \} = 0 \end{cases}$$

For $x \in \Sigma^*$, we define $f_x : Q \to Q$ by

(i) $f_x(p) = p$, and inductively

(ii) $f_{ax}(p) = f_x(q)$, where $q$ is such that $\mu(p, a, q) = \vee \{ \mu(p, a, r) \mid r \in Q \}$

$\forall p \in Q$.

Theorem 3.2.2.

Let $M = (Q, \Sigma, \mu, i, f)$ be an ufsfa and let $F(M) = \{ f_x \mid x \in \Sigma^* \}$, then $F(M)$ is a finite monoid (submonoid of $Q^Q$) under $\circ$, the composition of functions.

$^1$A part of these findings appeared in Bull. of Pure Appl. Sciences 24 E(2) (2005) 279–287.
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**Proof.** Let \( f_x, f_y \in F(M), x, y \in \Sigma^* \).

For \( p \in Q \),
\[
(f_x \circ f_y)(p) = f_x(f_y(p))
\]
\[
= f_x(q), \quad \text{where } q \text{ is such that } f_y(p) = q
\]
\[
= s, \quad \text{where } s \text{ is such that } f_x(q) = s.
\]
Now \( f_{yx}(p) = f_x(q) = s \). Therefore \( (f_x \circ f_y)(p) = f_{yx}(p) \).

Since \( p \) is arbitrary, \( f_x \circ f_y = f_{yx} \in F(M) \), a unique function. Therefore \( F(M) \) is closed under composition \( \circ \).

Let \( f_x, f_y, f_z \in F(M) \).

Now \( f_x \circ (f_y \circ f_z) = f_x \circ (f_{yz}) = f_{xyz} = (f_{yx}) \circ f_z = (f_x \circ f_y) \circ f_z \).

Thus \( \circ \) is associative.

For \( p \in Q \), \( (f_x \circ f_\lambda)(p) = f_x(f_\lambda(p)) = f_x(p) = f_\lambda(f_x(p)) = (f_\lambda \circ f_x)(p) \).

Since \( p \) is arbitrary, \( f_x \circ f_\lambda = f_\lambda \circ f_x \).

Hence \( f_\lambda \) is the identity element which is in \( F(M) \).

Therefore \( (F(M), \circ) \) is a finite monoid.

Since \( \text{Im}(\mu) \) is finite, \( F(M) \) is finite.

**Corollary 3.2.3.**

\((F(M), \circ)\) is a finite fuzzy monoid.

**Proof.** Let \( x \in \Sigma^*, p \in Q, x = a_1 a_2 \cdots a_n, a_i \in \Sigma, i = 1, 2, \ldots, n. \)

Let \( f_x(p) = q \) and the sequence of membership values in the path be
\[
\mu(p, a_1, p_1), \mu(p_1, a_2, p_2), \ldots, \mu(p_{n-1}, a_n, p_n), \text{ where } p_n = q.
\]

Define \( \mu_1 : F(M) \rightarrow [0, 1] \) by
\[
\mu_1(f_{a_1a_2\cdots a_n}) = \bigvee \{ \mu(p, a_1, p_1) \land \mu(p_1, a_2, p_2) \land \cdots \land \mu(p_{n-1}, a_n, p_n) \mid p \in Q \}
\]
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Therefore \( \mu_1(f_x \circ f_y) = \mu_1(f_{yx}) \geq \wedge(\mu_1(f_y), \mu_1(f_x)) \).

Thus \((F(M), \circ)\) is a fuzzy monoid. 

\[ \text{Theorem 3.2.4.} \]

\( F(M) \) is an anti-homomorphic image of \( \Sigma^* \).

\textbf{Proof.} Let \( \cdot \) be the concatenation operator. Then \((\Sigma^*, \cdot)\) is a semigroup with identity element \( \lambda \).

Define \( \phi : (\Sigma^*, \cdot) \rightarrow (F(M), \circ) \) by \( \phi(x) = f_x \ \forall x \in \Sigma^* \).

Clearly \( \phi \) is well-defined, since \( x = y \in \Sigma^* \), implies that \( f_x(p) = f_y(p) \) \( \forall p \in Q \), therefore \( f_x = f_y \). Hence \( \phi(x) = \phi(y) \).

Again, for \( x, y \in \Sigma^* \), we have \( \phi(xy) = f_{xy} = f_y \circ f_x = \phi(y) \circ \phi(x) \)

and \( \phi(\lambda) = f_\lambda \). Also for each \( y \in F(M) \), there exists \( x \in \Sigma^* \) such that \( y = f_x = \phi(x) \), which implies \( \phi \) is onto.

Thus \( F(M) \) is an anti-homomorphic image of \( \Sigma^* \).

\[ \text{Theorem 3.2.5.} \]

Let \( M = (Q, \Sigma, \mu, i, f) \) be an uffsa. Define a relation \( \equiv \) on \( \Sigma^* \) by \( x \equiv y \) if and only if \( f_x = f_y \), \( \forall x, y \in \Sigma^* \). Then \( \equiv \) is a congruence relation on \( \Sigma^* \).

\textbf{Proof.} Let \( x \in \Sigma^* \), \( f_x = f_x \), implies that \( x \equiv x \). Therefore \( \equiv \) is reflexive.

Let \( x, y \in \Sigma^* \) and \( x \equiv y \). Therefore \( f_x = f_y \), implies that \( f_y = f_x \).

Therefore \( y \equiv x \). Hence \( \equiv \) is symmetric.

Let \( x, y, z \in \Sigma^* \), \( x \equiv y \) and \( y \equiv z \). Therefore \( f_x = f_y \) and \( f_y = f_z \), implies that \( f_x = f_z \). Therefore \( x \equiv z \). Thus \( \equiv \) is transitive.

Hence \( \equiv \) is an equivalence relation.

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We show that $\equiv$ is a congruence relation. Let $x, y \in \Sigma^*$ and $x \equiv y$.

Then $f_x = f_y$, that is, $f_x(p) = f_y(p) \forall p \in Q$.

Now for any $z \in \Sigma^*$, $f_{zx}(p) = f_x(f_z(p)) = f_y(f_z(p)) = f_{zy}(p) \forall p \in Q$.

Therefore $f_{zx} = f_{zy}$, and so $zx \equiv zy$. Similarly $xz \equiv yz$.

Therefore $\equiv$ is a congruence relation on $\Sigma^*$.

Example 3.2.6.

Consider the ufsa $M = (Q, \Sigma, \mu, i, f)$, where $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, $\mu : Q \times \Sigma \times Q \to [0, 1]$ is defined as follows:

\[
\begin{align*}
\mu(q_1, a, q_1) &= 1 \\
\mu(q_1, a, q_2) &= 0.8 \\
\mu(q_1, b, q_1) &= 1 \\
\mu(q_1, b, q_2) &= 0.6 \\
\mu(q_1, b, q_3) &= 1 \\
\mu(q_1, b, q_2) &= 0.6 \\
\mu(q_2, a, q_3) &= 1 \\
\mu(q_2, a, q_2) &= 1 \\
\mu(q_3, a, q_2) &= 1.
\end{align*}
\]

$i : Q \to [0, 1]$ such that $i(q_1) = 1$.

$f : Q \to [0, 1]$ such that $f(q_3) = 1$.

The fuzzy transition diagram is shown below.

![Figure 3.1: Fuzzy Transition Diagram](image-url)
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The fuzzy regular language accepted by $M$ is $L_M : \Sigma^* \rightarrow [0, 1]$ such that

$$L_M(x) = \begin{cases} 
0.8, & \text{if } x \in \{a, b\}^*aa \\
0.6, & \text{if } x \in \{a, b\}^*ba \\
0, & \text{otherwise}
\end{cases}$$

For example, $f_{\{a,b\}^*}(q_1) = q_1$, $f_{\{a,b\}^*}(q_2) = q_3$, $f_{\{a,b\}^*}(q_2) = q_2$.

Now, $aba \equiv aaba$. Since

$$f_{aba}(q_1) = q_1, \quad f_{aba}(q_2) = q_3, \quad f_{aba}(q_3) = q_3,$$

$$f_{aaba}(q_1) = q_1, \quad f_{aaba}(q_2) = q_3 \quad f_{aaba}(q_3) = q_3.$$  

However, $aba$ is not congruent to $ba$, since $f_{aba}(q_2) = q_3$ while $f_{ba}(q_2) = q_2$.

Note that, any two strings beginning with $a$ are equivalent and any two strings beginning with $b$ are equivalent; and any other two strings are not equivalent.

Therefore, there are only three equivalence classes namely, $[\lambda]$, $[a\{a,b\}^*]$, $[b\{a,b\}^*]$.

Also, $\Sigma^* = [\lambda] \cup [a\{a,b\}^*] \cup [b\{a,b\}^*]$.

Next we compute $F(M) = \{f_x | x \in \Sigma^*\}$.

We have $f_{\{a,b\}^*} = f_a$ and $f_{b\{a,b\}^*} = f_b$. Therefore $F(M) = \{f_\lambda, f_a, f_b\}$.

The following is the required table for operations.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$f_\lambda$</th>
<th>$f_a$</th>
<th>$f_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_\lambda$</td>
<td>$f_\lambda$</td>
<td>$f_a$</td>
<td>$f_b$</td>
</tr>
<tr>
<td>$f_a$</td>
<td>$f_a$</td>
<td>$f_{aa}$</td>
<td>$f_{ba}$</td>
</tr>
<tr>
<td>$f_b$</td>
<td>$f_b$</td>
<td>$f_{ab}$</td>
<td>$f_{bb}$</td>
</tr>
</tbody>
</table>
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But $f_{ab} = f_a$, $f_{ba} = f_b$, $f_{ba} = f_b$, $f_{ab} = f_a$.

Therefore the operation table changes as follows:

<table>
<thead>
<tr>
<th></th>
<th>$f_{\lambda}$</th>
<th>$f_a$</th>
<th>$f_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\lambda}$</td>
<td>$f_{\lambda}$</td>
<td>$f_a$</td>
<td>$f_b$</td>
</tr>
<tr>
<td>$f_a$</td>
<td>$f_a$</td>
<td>$f_a$</td>
<td>$f_b$</td>
</tr>
<tr>
<td>$f_b$</td>
<td>$f_b$</td>
<td>$f_a$</td>
<td>$f_b$</td>
</tr>
</tbody>
</table>

Thus $(F(M), \circ)$ is a finite monoid.

The fuzzy function $\mu_1 : F(M) \rightarrow [0, 1]$ is defined by

$\mu_1(f_{\lambda}) = 1$, $\mu_1(f_a) = 1$, $\mu_1(f_b) = 1$.

**Theorem 3.2.7.**

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa, then $\Sigma^*/\equiv$ is a monoid.

**Proof.** Let $x \in \Sigma^*$, $[x] = \{y \in \Sigma^* | x \equiv y\}$, $\Sigma^*/\equiv = \{[x] | x \in \Sigma^*\}$.

Define a binary operation $*$ on $\Sigma^*/\equiv$ by

$\forall [x], [y] \in \Sigma^*/\equiv$, $[x] * [y] = [yx]$.

Clearly $*$ is well defined.

Let $[x], [y], [z] \in \Sigma^*/\equiv$.

Now $[x] * ([y] * [z]) = [x] * [zy] = [yxz] = [y * [x] * [z] = (i * [y]) * [z] * [y]$.

Therefore $*$ is associative.

For $\lambda \in \Sigma^*$, we have $[\lambda] \in \Sigma^*/\equiv$, and for any $[x] \in \Sigma^*/\equiv$,

$[x] * [\lambda] = [\lambda x] = [x] = [x \lambda] = [\lambda] * [x]$.
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Thus $[\lambda]$ is the identity element which is in $\Sigma^*/\equiv$.

Hence $\Sigma^*/\equiv$ is a monoid. ■

Theorem 3.2.8.

Let $M = (Q, \Sigma, \mu, i, f)$ be an ufsa, then $\Sigma^*/\equiv$ is isomorphic to $F(M)$.

Proof. Let $M = (Q, \Sigma, \mu, i, f)$ be the ufsa.

$\Sigma^*/\equiv = \{[x] \mid x \in \Sigma^*\}$ and $F(M) = \{f_x \mid x \in \Sigma^*\}$

$(\Sigma^*/\equiv, \ast)$ and $(F(M), \circ)$ are monoids.

Define $\phi : (\Sigma^*/\equiv, \ast) \rightarrow (F(M), \circ)$ by $\phi(x) = f_x \forall x \in \Sigma^*$.

By Theorem 3.2.4, $\phi$ is an anti-epimorphism.

Define $g : (\Sigma^*/\equiv, \ast) \rightarrow (F(M), \circ)$ by $g[x] = \phi(x) \forall x \in \Sigma^*/\equiv$.

We show that, $g$ is well defined. Let $[x], [y] \in \Sigma^*/\equiv$.

Now $[x] = [y]$, implies that $x \equiv y$. Therefore $f_x = f_y$.

Since $\phi$ is onto, there exists $x, y \in \Sigma^*$ such that $\phi(x) = f_x$ and $\phi(y) = f_y$.

Therefore $\phi(x) = \phi(y)$, implies that $g[x] = g[y]$. Thus $g$ is well defined.

Now we prove $g$ is a homomorphism. Let $[x], [y] \in \Sigma^*/\equiv$.

$g([x] \ast [y]) = g[xy] = \phi(xy) = f_x \circ f_y = g[x] \circ g[y]$ and $g([\lambda]) = \phi(\lambda) = f_\lambda$.

Therefore $g$ is a homomorphism of monoids.

To prove $g$ is one–one, take $[x], [y] \in \Sigma^*/\equiv$ and $g[x] = g[y]$.

Now $\phi(x) = \phi(y)$, implies that $f_x = f_y$. Therefore $x \equiv y$, implies that $[x] \equiv [y]$. Therefore $g$ is one–one.

Finally we prove, $g$ is onto. Let $f_x \in F(M)$.

Since $\phi$ is onto there exists an $x \in \Sigma^*$ such that $\phi(x) = f_x$, which implies that $g[x] = f_x$. Hence $g$ is an isomorphism of monoids. ■
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Definition 3.2.9.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. For $p, q \in Q$, $x \in \Sigma^*$, $x = a_1a_2 \ldots a_n$.
Let $f_x(p) = q$ and the sequence of membership values in the path (unique) be
$\mu(p, a_1, p_1), \mu(p_1, a_2, p_2), \ldots, \mu(p_{n-1}, a_n, q)$.
Define $\mu^M(p, x, q) = \mu(p, a_1, p_1) \land \mu(p_1, a_2, p_2) \land \cdots \land \mu(p_{n-1}, a_n, q)$.
Therefore $\mu^M : Q \times \Sigma^* \times Q \rightarrow [0,1]$ is defined as follows:

$$
\mu^M(p, a, q) = \begin{cases} 
\mu(p, a, q), & \text{if } \mu(p, a, q) = \lor \{\mu(p, a, r), \mid r \in Q\}, \\
0, & \text{otherwise.}
\end{cases}
$$

Inductively, $\mu^M(p, xa, q) = \mu^M(p, x, r) \land \mu^M(r, a, q)$, for some $r \in Q$.
Since $M$ is an uffsa, $r$ will be unique.

Lemma 3.2.10.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Then for any $x, y \in \Sigma^*$,

$$
\mu^M(p, xy, q) = \mu^M(p, x, r) \land \mu^M(r, y, q), \text{ some } r \in Q \forall p, q \in Q.
$$

Proof. Let $p, q \in Q$ and $x, y \in \Sigma^*$. We prove the result by induction on $|y| = n$.
Let $n = 0$, then $y = \lambda$, $xy = x\lambda = x$.

$$
\mu^M(p, xy, q) = \mu^M(p, x\lambda, q) = \mu^M(p, x, q) \land \mu^M(q, \lambda, q),
$$

since $\mu^M(q, \lambda, q) = 1$.

Therefore $\mu^M(p, xy, q) = \mu^M(p, x, r) \land \mu^M(r, y, q)$, such that $r = q \in Q$.
Thus the result is true for $n = 0$.
Suppose the result is true for all $y \in \Sigma^*$ such that $|y| \leq n - 1$. 

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Let \( y = ua \) where \( u \in \Sigma^* \) and \(|u| = n - 1, n > 0\). Now

\[
\mu^M(p, xy, q) = \mu^M(p, xua, q)
\]

\[
= \mu^M(p, xu, s) \land \mu^M(s, a, q), \text{ some } s \in Q
\]

\[
= \mu^M(p, x, r) \land \mu^M(r, u, s) \land \mu^M(s, a, q), \text{ some } r, s \in Q
\]

\[
= \mu^M(p, x, r) \land \mu^M(r, ua, q)
\]

\[
= \mu^M(p, x, r) \land \mu^M(r, y, q), \text{ } r \in Q
\]

Thus the result is true for \(|y| = n\). Hence the result.

Definition 3.2.11.

Let \( M = (Q, \Sigma, \mu, i, f) \) be an uffsa. For \( x \in \Sigma^* \), define the fuzzy subset

\( x^M : Q \times Q \to [0,1] \) by \( x^M(p, q) = \mu^M(p, x, q) \forall p, q \in Q \).

Theorem 3.2.12.

Let \( M = (Q, \Sigma, \mu, i, f) \) be an uffsa. Let \( S_M = \{ x^M | x \in \Sigma^* \} \). Then \((S_M, \circ)\) is a finite monoid.

Proof. Let \( x^M, y^M, z^M \in S_M, \forall p, q \in Q \). By Definition 1.3.4,

\[
(x^M \circ y^M)(p, q) = \vee \{ x^M(p, r) \land y^M(r, q) | r \in Q \}
\]

\[
= \mu^M(p, x, r) \land \mu^M(r, y, q), \text{ for some } r \in Q
\]

\[
= \mu^M(p, xy, q)
\]

\[
= (xy)^M(p, q)
\]

Thus \((x^M \circ y^M) = (xy)^M\). Therefore \( S_M \) is closed under \( \circ \).

Associative law is also satisfied.

In fact,
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\((x^M \circ y^M) \circ z^M = (xy)^M \circ z^M = (xyz)^M = x^M \circ (yz)^M = x^M \circ (y^M \circ z^M)\)

\(\lambda^M \in S_M\), and \((x^M \circ \lambda^M) = (x\lambda)^M = x^M = (\lambda^M \circ x^M)\).

Therefore \(\lambda^M\) is the identity element which is in \(S_M\). \(\text{Im}(\mu)\) is finite, implies that \(S_M\) is finite. Thus \((S_M, \circ)\) is a finite monoid.

Theorem 3.2.13.
Let \(M = (Q, \Sigma, \mu, i, f)\) be an uffsa. Then \(S_M\) and \(F(M)\) are anti-isomorphic as monoids.

Proof. Define \(\phi: (S_M, \circ) \rightarrow (F(M), \circ)\) by \(\phi(x^M) = f_x \ \forall x^M \in S_M\).

We show that, \(\phi\) is well-defined. Let \(x^M, y^M \in S_M\).

Now \(x^M = y^M\), implies that \(x^M(p, q) = y^M(p, q)\ \forall p, q \in Q\).

That is \(\mu^M(p, x, q) = \mu^M(p, y, q)\),

and this implies that \(f_x(p) = f_y(p) \ \forall p \in Q\).

Therefore \(f_x = f_y\), and so \(\phi(x^M) = \phi(y^M)\). Hence \(\phi\) is well-defined.

To prove, \(\phi\) is an anti-homomorphism of monoids, let \(x^M, y^M \in S_M\).

Now \(\phi(x^M \circ y^M) = \phi((xy)^M) = f_{xy} = f_y \circ f_x = \phi(y^M) \circ \phi(x^M)\).

Also \(\phi(\lambda^M) = f_\lambda, \lambda^M \in S_M\).

Therefore \(\phi\) is an anti-homomorphism of monoids.

Next we prove \(\phi\) is one–one. Let \(x^M, y^M \in S_M\) and \(\phi(x^M) = \phi(y^M)\).

Therefore \(f_x = f_y\), implies that \(f_x(p) = f_y(p) = q \in Q\).

Therefore \(\mu^M(p, x, q) = \mu^M(p, y, q)\).

In uffsa for \(r \neq q\), \(\mu^M(p, x, r) = \mu^M(p, y, r) = 0\).

Therefore, \(\mu^M(p, x, q) = \mu^M(p, y, q) \ \forall p, q \in Q\), implies that \(x^M(p, q) = y^M(p, q)\). Hence \(x^M = y^M\). Therefore \(\phi\) is one–one.
Finally we prove $\phi$ is onto. Let $f_x \in F(M)$, $x \in \Sigma^*$, $x^M \in S_M$. Therefore we have $\phi(x^M) = f_x$. Thus $\phi$ is onto.

Hence $\phi$ is an anti-isomorphism.

### 3.3 Construction of uffsa from Fuzzy Monoids

In this section we prove the existence of the uffsa from a given finite fuzzy monoid. We illustrate this with an example.

**Theorem 3.3.1.**

Let $(S, \ast)$ be a finite fuzzy monoid. Then there exists an uffsa $M$ such that the monoid $F(M)$ is isomorphic to $S$.

**Proof.** Let $(S, \ast)$ be a finite fuzzy monoid and let the fuzzy subset $\mu_S : S \to [0, 1]$ be such that $\mu_S(x \ast y) \geq \wedge(\mu_S(x), \mu_S(y)) \forall x, y \in S$.

Define $M = (Q, \Sigma, \mu, i, f)$, where $Q = \Sigma = S$,

$\mu : Q \times \Sigma \times Q \to [0, 1]$ is defined by

$$
\mu(p, s, q) = \begin{cases} 
\mu_S(q), & \text{if } q = s \ast p \\
0, & \text{otherwise}
\end{cases}
$$

$i : Q \to [0, 1]$ such that $i(s) = \mu_S(s) \forall s \in Q$.

$f : Q \to [0, 1]$ such that $f(s) = \mu_S(s) \forall s \in Q$.

We treat $s_1 \ast s_2 = s_1s_2$, the concatenation of $s_1$ and $s_2$, therefore

$S = S^* = \Sigma = \Sigma^*$.

Now $F(M) = \{f_s | s \in \Sigma^*\}$, where $f_s(p) = s \ast p$. ($f_s$ is the transition function).

Let $p, q \in Q$, $p = q$, implies that $s \ast p = s \ast q$, $\forall s \in \Sigma$. Therefore $f_s(p) = f_s(q)$. 

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Thus \( f_\alpha \) is well defined.

We show that \( F(M) \) is a monoid.

Let \( f_{s_1}, f_{s_2} \in F(M), \ p \in Q \).

\[
(f_{s_1} \circ f_{s_2})(p) = f_{s_1}(f_{s_2}(p)) = f_{s_1}(s_2 \cdot p) = s_1 \cdot (s_2 \cdot p) = (s_1 \cdot s_2) \cdot (p) = f_{(s_1 \cdot s_2)}(p)
\]

Therefore \( (f_{s_1} \circ f_{s_2}) = f_{(s_1 \cdot s_2)} \). Thus \( F(M) \) is closed under \( \circ \).

Let \( e \in S \) the identity element of \( S \), implies that \( f_e \in F(M) \).

Now \( f_e \circ f_s = f_{s \cdot e} = f_s = f_{s \cdot e} = f_e \circ f_s \ \forall f_s \in F(M) \).

Therefore \( f_e \) is the identity element which is in \( F(M) \).

Clearly \( (f_{s_1} \circ f_{s_2}) \circ f_{s_3} = f_{s_1} \circ (f_{s_2} \circ f_{s_3}) \ \forall f_{s_1}, f_{s_2}, f_{s_3} \in F(M) \).

Hence \( (F(M), \circ) \) is a monoid.

Define \( g : S \rightarrow F(M) \) by \( g(s) = f_s \forall s \in S \).

To prove \( g \) is well-defined, let us take \( s_1, s_2 \in S \) with \( s_1 = s_2 \).

Therefore \( s_1 \cdot p = s_2 \cdot p \ \forall p \in Q \), implies that \( f_{s_1}(p) = f_{s_2}(p) \).

Therefore \( f_{s_1} = f_{s_2} \). Hence \( g(s_1) = g(s_2) \). Thus \( g \) is well defined.

We show that, \( g \) is one-one. Let \( s_1, s_2 \in S \) and \( g(s_1) = g(s_2) \).

Therefore \( f_{s_1} = f_{s_2} \), implies that \( f_{s_1}(p) = f_{s_2}(p) \ \forall p \in Q \).

So \( f_{s_1}(e) = f_{s_2}(e) \), implies that \( s_1 \cdot e = s_2 \cdot e \). Therefore \( s_1 = s_2 \).

Hence \( g \) is one-one.

Let \( f_s \in F(M) \) then there exists an \( s \in S \) such that \( g(s) = f_s \).

Thus \( g \) is onto.

We prove \( g \) is a homomorphism. Let \( s_1, s_2 \in S \).

Now \( g(s_1 \cdot s_2) = f_{s_1 \cdot s_2} = f_{s_1} \circ f_{s_2} = g(s_1) \circ g(s_2) \).

Also, \( g(e) = f_e \), \( e \) is the identity element in \( S \).
3.3 Construction of uffsa from Fuzzy Monoids

Therefore $g$ is an isomorphism of monoids. 

Corollary 3.3.2.

$(F(M), \circ)$ is a fuzzy monoid.

Proof. Define the fuzzy subset $\mu^1 : F(M) \to [0,1]$ such that

$$\mu^1(f_s) = \mu_S(s) \forall f_s \in F(M).$$

Let $f_{s_1}, f_{s_2} \in F(M), f_{s_1} = f_{s_2}$, implies that $f_{s_1}(e) = f_{s_2}(e)$, i.e., $s_1 * e = s_2 * e$.
Therefore $s_1 = s_2$, implies that $\mu_S(s_1) = \mu_S(s_2)$. That is, $\mu^1(f_{s_1}) = \mu^1(f_{s_2})$.
Therefore $\mu^1$ is well-defined.

$$\mu^1(f_{s_1} \circ f_{s_2}) = \mu^1(f_{s_1} * s_2) = \mu_S(s_1 * s_2) \geq \wedge(\mu_S(s_1), \mu_S(s_2))$$

$$= \wedge(\mu^1(f_{s_1}), \mu^1(f_{s_2}))$$

Therefore $(F(M), \circ)$ is a fuzzy monoid.

Example 3.3.3.

Let $(S, \ast)$ be the fuzzy finite monoid, where $S = \{e, a, b, ab\}$
with $a \ast a = b \ast b = e$, $a \ast b = b \ast a$, i.e., $aa = bb = e$, $ab = ba$.
$\mu_S : S \to [0,1]$ is defined by

$$\mu_S(e) = 1, \quad \mu_S(a) = 0.6, \quad \mu_S(b) = 0.7, \quad \mu_S(ab) = 0.8$$

Define the uffsa, $M = (S, S, S, \mu, i, f)$, where $\mu : S \times S \times S \to [0,1]$ is defined as follows:
3.3 Construction of uffsa from Fuzzy Monoids

\[\mu(e,e,e) = 1\]  \[\mu(b,e,b) = 0.7\]

\[\mu(e,a,a) = 0.6\]  \[\mu(b,a,ba) = 0.8\]

\[\mu(e,b,b) = 0.7\]  \[\mu(b,b,e) = 1\]

\[\mu(e,ab,ab) = 0.8\]  \[\mu(b,ab,a) = 0.6\]

\[\mu(a,e,a) = 0.6\]  \[\mu(ab,e,ab) = 0.6\]

\[\mu(a,a,e) = 1\]  \[\mu(ab,a,b) = 0.7\]

\[\mu(a,b,ab) = 0.8\]  \[\mu(ab,b,a) = 0.8\]

\[\mu(a,ab,b) = 0.7\]  \[\mu(ab,ab,a) = 1\]

\(i : S \to [0,1]\) is defined as follows:

\[i(e) = 1, \quad i(a) = 0.6, \quad i(b) = 0.7 \quad \text{and} \quad i(ab) = 0.8\]

\(f : S \to [0,1]\) is defined as follows:

\[f(e) = 1, \quad f(a) = 0.6, \quad f(b) = 0.7 \quad \text{and} \quad f(ab) = 0.8\]

The Fuzzy monoid \(F(M) = \{fe, fa, fb, fab\}\) and the operations are as follows:

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<tr>
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3.4 Transformation Semigroups

The fuzzy function \( \mu^1 : F(M) \to [0,1] \) is defined by

\[
\mu^1(f_e) = 1, \quad \mu^1(f_a) = 0.6, \quad \mu^1(f_b) = 0.7 \quad \text{and} \quad \mu^1(f_{ab}) = 0.8
\]

Therefore \((F(M), \circ)\) is a fuzzy monoid.

Clearly \(S\) and \(F(M)\) are isomorphic as monoids, with the following relations:

\[
e \to f_e, \quad a \to f_a, \quad b \to f_b, \quad ab \to f_{ab}
\]

3.4 Transformation Semigroups

Fuzzy transformation semigroups and polytransformation semigroups are defined in [28]. In this section we define fuzzy anti-transformation semigroups and anti-polytransformation semigroups. We prove that for a given ufsa \(M\), using the corresponding \(F(M)\), the above two transformation semigroups can be generated.

Definition 3.4.1.

A fuzzy transformation semigroup (fts) is a triple \((Q, S, \rho)\) where \(Q\) is a finite non-empty set, \(S\) is a finite semigroup, and \(\rho\) is a fuzzy subset of \(Q \times S \times Q\) such that

(i) \(\rho(p, uv, q) = \bigvee \{\rho(p, u, r) \land \rho(r, v, q) \mid r \in Q\} \forall u, v \in S \text{ and } \forall p, q \in Q\).

(ii) If \(S\) contains the identity element \(e\), then \(\forall p, q \in Q\),

\[
\rho(p, e, q) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}
\]

If, in addition, the following property holds, then \((Q, S, \rho)\) is called faithful.
3.4 Transformation Semigroups

(iii) Let \( u, v \in S \), if \( \rho(p, u, q) = \rho(p, v, q) \forall p, q \in Q \), then \( u = v \).

Definition 3.4.2.

A fuzzy anti-transformation semigroup (fats) is a triple \((Q, S, \rho)\), where \( Q \) is a finite non-empty set, \( S \) is a finite semigroup, and \( \rho \) is a fuzzy subset of \( Q \times S \times Q \) such that

(i) \( \rho(p, uv, q) = \bigvee \{ \rho(p, v, r) \wedge \rho(r, u, q) \mid r \in Q \} \forall u, v \in S \) and \( \forall p, q \in Q \).

(ii) If \( S \) contains the identity element \( e \), then \( \forall p, q \in Q \),

\[
\rho(p, e, q) = \begin{cases} 
1, & \text{if } p = q \\
0, & \text{if } p \neq q
\end{cases}
\]

If, in addition, the following property holds, then \((Q, S, \rho)\) is called faithful.

(iii) Let \( u, v \in S \) if \( \rho(p, u, q) = \rho(p, v, q) \forall p, q \in Q \), then \( u = v \).

Theorem 3.4.3.

Let \( M = (Q, \Sigma, \mu, i, f) \) be an ufsa. Then \((Q, F(M), \rho)\) is a faithful fats.

Proof. \( F(M) \) is a finite semigroup with identity \( f_{\lambda} \).

Define \( \rho : Q \times F(M) \times Q \to [0, 1] \) by

\[
\rho(p, f_{x}, q) = \mu^{M}(p, x, q) \forall p, q \in Q, \ f_{x} \in F(M)
\]

We show that, \( \rho \) is well-defined.

Let \( p, q, p', q' \in Q, \ f_{x}, f_{y} \in F(M) \) and \( (p, f_{x}, q) = (p', f_{y}, q') \).

Therefore \( p = p', \ q = q' \) and \( f_{x} = f_{y} \), implies that \( f_{x}(r) = f_{y}(r) \forall r \in Q \).

Let \( r \in Q, \ f_{x}(r) = f_{y}(r) = r' \), for some \( r' \in Q \), implies that
3.4 Transformation Semigroups

\[ \mu^M(r, x, r') = \mu^M(r, y, r'), \] which is true for any \( r, r' \in Q \).

Therefore \( \mu^M(p, x, q) = \mu^M(p', y, q') \). Thus \( \rho \) is well-defined.

Let \( p, q \in Q \) and \( f_x, f_y \in F(M) \).

\[
\rho(p, f_x \circ f_y, q) = \rho(p, f_y, q) \\
= \mu^M(p, yx, q) \\
= \mu^M(p, y, r) \land \mu^M(r, x, q), \text{ for some } r \in Q \\
= \lor \{ \mu^M(p, y, r) \land \mu^M(r, x, q) \mid r \in Q \} \\
= \lor \{ \rho(p, f_y, r) \land \rho(r, f_x, q) \mid r \in Q \} \\
\]

Hence (i) of Definition 3.4.2.

\[ \rho(p, f_\lambda, q) = \mu^M(p, \lambda, q) = \mu(p, \lambda, q) \]

Therefore, \( \rho(p, f_\lambda, q) = \begin{cases} 
1, & \text{if } p = q \\
0, & \text{if } p \neq q 
\end{cases} \).

Hence (ii) of Definition 3.4.2.

Suppose \( \rho(p, f_x, q) = \rho(p, f_y, q) \forall p, q \in Q \). This implies that

\[ \mu^M(p, x, q) = \mu^M(p, y, q). \]

Therefore \( f_x(p) = f_y(p) \forall p \in Q \), implies that \( f_x = f_y \).

Hence (iii) of Definition 3.4.2.

Thus \( (Q, F(M), \rho) \) is a faithful fats. \( \blacksquare \)

Definition 3.4.4.

A polytransformation semigroup is a triple \( (Q, S, \gamma) \) where \( Q \) is a finite non-empty set, \( S \) is a finite semigroup and \( \gamma : Q \times S \to \mathcal{P}(Q) \setminus \{\emptyset\} \) such that

(i) \( \gamma(\gamma(p, u), v) = \gamma(p, uv) \forall p \in Q, u, v \in S \) and
3.4 Transformation Semigroups

\[ \gamma(P, u) = \bigcup \{ \gamma(p, u) \mid p \in P \}, \quad P \subseteq Q. \]

(ii) If \( S \) contains identity element \( e \), then \( \gamma(p, e) = \{p\} \forall p \in Q. \)

If, in addition, the following holds, then \( (Q, S, \gamma) \) is called \textbf{faithful}.

(iii) Let \( u, v \in S \). If \( \gamma(p, u) = \gamma(p, v), \forall p \in Q \) then \( u = v. \)

Definition 3.4.5.

An \textbf{anti-polytransformation semigroup} is a triple \( (Q, S, \gamma) \), where \( Q \) is a finite non-empty set, \( S \) is a finite semigroup and \( \gamma : Q \times S \rightarrow P(Q)\setminus\{\phi\} \) such that

(i) \( \gamma(\gamma(p, u), v) = \gamma(p, vu) \forall p \in Q, u, v \in S \) and

\[ \gamma(P, u) = \bigcup \{ \gamma(p, u) \mid p \in P \}, \quad P \subseteq Q. \]

(ii) If \( S \) contains the identity element \( e \), then \( \gamma(p, e) = \{p\} \forall p \in Q. \)

If, in addition, the following holds, then \( (Q, S, \gamma) \) is called \textbf{faithful}.

(iii) Let \( u, v \in S \). If \( \gamma(p, u) = \gamma(p, v) \forall p \in Q \), then \( u = v. \)

Theorem 3.4.6.

Let \( M = (Q, \Sigma, \mu, i, f) \), then there exists a faithful anti-polytransformation semigroup with identity, denoted by \( (Q, F(M), \gamma) \).

\textbf{Proof.} Define \( \gamma : Q \times F(M) \rightarrow P(Q)\setminus\{\phi\} \) by

\[ \gamma(p, f_x) = \{q \in Q \mid f_x(p) = q\} = \{q\} \subseteq P(Q)\setminus\{\phi\} \]

To prove \( \gamma \) is well-defined, take \( f_x, f_y \in F(M) \) and \( f_x = f_y. \)

Therefore \( f_x(p) = f_y(p) \forall p \in Q \), implies that \( \gamma(p, f_x) = \gamma(p, f_y). \)
3.4 Transformation Semigroups

Hence $\gamma$ is well-defined.

Let $f_x, f_y \in F(M), p \in Q$.

Now $\gamma(\gamma(p, f_x), f_y) = \gamma(\{q\}, f_y)$, where $q$ is such that $f_x(p) = q, q \in Q$,

\[
\begin{align*}
\cup \{ \gamma(q, f_y) \mid q \in \{q\} \} &= \{ f_y(q) \} \\
\end{align*}
\]

$= \{ r \}$, where $r$ is such that $f_y(q) = r, r \in Q$ \quad (3.1)

Also $\gamma(p, f_y \circ f_x) = \{ (f_y \circ f_x)(p) \}

\[
\begin{align*}
= \{ f_y(f_x(p)) \} \\
= \{ f_y(q) \}, \text{ since } f_x(p) = q, q \in Q \\
= \{ r \}, \text{ since } f_y(q) = r, r \in Q \\
\end{align*}
\]

From (3.1) and (3.2), we have $\gamma(\gamma(p, f_x), f_y) = \gamma(p, f_y \circ f_x)$.

Hence (i) of Definition 3.4.5.

$f_x$ is the identity element in $F(M)$.

Now $\gamma(p, f_x) = \{ f_x(p) \} = \{ p \} \forall p \in Q$.

Hence (ii) of Definition 3.4.5.

Let $f_x, f_y \in F(M), p \in Q$ and $\gamma(p, f_x) = \gamma(p, f_y)$.

Therefore $\{ f_x(p) \} = \{ f_y(p) \}$, implies that $f_x(p) = f_y(p)$. Since $p$ is arbitrary,

$f_x = f_y$.

Hence (iii) of Definition 3.4.5.

Thus $(Q, F(M), \gamma)$ is an anti-polytransformation semigroup.

Hence the theorem.