Chapter 4

Fuzzy finite state subautomaton and Homomorphism

4.1 Introduction

This chapter begins with the definition of sub uffsa’s of a given uffsa \( M = (Q, \Sigma, \mu, i, f) \), which are uffsa’s and as set of states they are subsets of \( Q \) with restriction mappings \( \mu', i' \) and \( f' \). This is closed under intersection and union. It is interesting to note that if \( M_1 \) is a sub uffsa of the uffsa \( M \), then \( F(M) \) is a homomorphic image of \( F(M_1) \).

This chapter then deals with homomorphism of uffsa’s. We prove that if \( (\alpha, \beta) : M_1 \rightarrow M_2 \) is a homomorphism then \( L_1(x) \leq L_2(\beta^*(x)) \forall x \in \Sigma_1^* \). However if \( \beta \) is the identity map then \( L_1 \subseteq L_2 \). Moreover if \( (\alpha, \beta) \) is a strong homomorphism and \( \alpha \) is bijective then \( L_1(x) = L_2(\beta^*(x)) \forall x \in \Sigma_1^* \). We also
prove that if $M_1$ and $M_2$ are strong isomorphic as uffsas, then $F(M_1)$ and $F(M_2)$ are isomorphic as monoids. Admissible relation on the set of states of fism is discussed in [28]. Admissible relation on the set of states of uffsa is suitably defined and characterized in this chapter. Corresponding to each admissible relation on an uffsa $M$, there exists a fuzzy function $\mu_1$ and it constructs an uffsa $M_1$, such that $M$ and $M_1$ are strong homomorphic.

4.2 Sub uffsa

In this section we define sub uffsa and prove some results on it. For the basis we refer to [35].

Definition 4.2.1.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Let $p, q \in Q$, $p$ is called an immediate successor of $q$ if there exists an $a \in \Sigma$ such that $\mu(q, a, p) > 0$. If $p$ is an immediate successor of $q$ and $q$ is an immediate successor or $r$, then $p$ is a successor of $r$.

Proposition 4.2.2. Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Let $q, p, r \in Q$. Then the following assertions hold:

(i) $q$ is a successor of $q$.

(ii) If $p$ is a successor of $q$ and $r$ is a successor $p$, then $r$ is a successor of $q$.

Definition 4.2.3.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa and let $q \in Q$. We denote by $S(q)$ the set of all successors of $q$. 


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Definition 4.2.4.
Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa and let $T \subseteq Q$. The set of successors of $T$, denoted by $S(T)$ in $Q$, is defined to be the set $S(T) = \bigcup \{ S(q) \mid q \in T \}$.

Definition 4.2.5.
Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa and $T \subseteq Q$. Let $N = (T, \Sigma, \mu', i', f')$.
$N$ is called a sub uffsa of $M$ if

(i) $S(T) \subseteq T$,

(ii) $\mu'$ is a restriction of $\mu$ on $T \times \Sigma \times T$,

(iii) $i'$ is a restriction of $i$ on $T$ and

(iv) $f'$ is a restriction of $f$ on $T$.

Note 4.2.6. If $K$ is a sub uffsa of $N$ and $N$ is a sub uffsa of $M$, then $K$ is a sub uffsa of $M$.

Theorem 4.2.7.
Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa. Let $M_j = (Q_j, \Sigma, \mu_j, i_j, f_j), j \in I$, be a family of sub uffsa's of $M$. Then $\bigcap M_j$ is a sub uffsa of $M$.

Proof. Define the uffsa $\bigcap M_j = (\bigcap Q_j, \Sigma, \bigcap \mu_j, \bigcap i_j, \bigcap f_j)$

Let $(p, a, q) \in \bigcap Q_j \times \Sigma \times \bigcap Q_j, p, q \in Q, a \in \Sigma$

$(\bigcap \mu_j)(p, a, q) = \bigwedge_{j \in I} \{ \mu_j(p, a, q) \} = \mu(p, a, q)$

Thus $\bigcap \mu_j$ is a restriction of $\mu$.

Let $p \in \bigcap Q_j$. Now $(\bigcap i_j)(p) = \bigwedge_{j \in I} \{ i_j(p) \} = i(p)$
Therefore $\bigcap i_j$ is a restriction of $i$.  

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\[(\bigcap_{j \in I} f_j)(p) = \bigwedge_{j \in I} \{f_j(p)\} = f(p)\]

Therefore \(\bigcap_{j \in I} f_j\) is a restriction of \(f\).

Clearly \(S(\bigcap_{j \in I} Q_j) \subseteq \bigcap_{j \in I} S(Q_j) \subseteq \bigcap_{j \in I} Q_j\)

Therefore \(\bigcap_{j \in I} M_j\) is a sub uffsa of \(M\).

Theorem 4.2.8.

Let \(M = (Q, \Sigma, \mu, i, f)\) be an uffsa. Let \(M_j = (Q_j, \Sigma, \mu_j, i_j, f_j), j \in I,\) be a family of sub uffsa’s of \(M\). Then \(\bigcup_{j \in I} M_j\) is a sub uffsa of \(M\).

**Proof.** Define \(\bigcup_{j \in I} M_j = (\bigcup_{j \in I} Q_j, \Sigma, \mu', i', f')\), where

\(\mu'\) is a restriction of \(\mu\) on \(\bigcup_{j \in I} Q_j \times \Sigma \times \bigcup_{j \in I} Q_j\)

\(i'\) is a restriction of \(i\) on \(\bigcup_{j \in I} Q_j\)

\(j'\) is a restriction of \(j\) on \(\bigcup_{j \in I} Q_j\)

Now \(S(\bigcup_{j \in I} Q_j) = \bigcup_{j \in I} S(Q_j) \subseteq \bigcup_{j \in I} Q_j\)

Thus \(\bigcup_{j \in I} M_j\) is a sub uffsa of \(M\).

Theorem 4.2.9.

The fuzzy regular language accepted by a sub uffsa \(M_1\) of an uffsa \(M\) is contained in the fuzzy regular language accepted by \(M\).

**Proof.** Let \(M = (Q, \Sigma, \mu, i, f)\) be an uffsa with fuzzy regular language \(L\).

Let \(M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1)\) be a sub uffsa of \(M\) with fuzzy regular language \(L_1\).

Now we prove, \(L_1 \subseteq L\).

Let \(x \in \Sigma^*\), \(L_1(x) = \bigvee\{i_1(p) \wedge \mu_1^* (p, x, q) \wedge f_1(q) \mid q \in Q_1 \mid p \in Q_1\}\)

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Since $Q_1 \subseteq Q$, $\sigma_1 = \sigma$, $i_1 = i$ and $f_1 = f$ on $Q_1$,

$$L_1(x) \leq \vee \{i(p) \wedge \mu^*(p, x, q) \wedge f(q) \mid q \in Q \mid p \in Q\} = L(x).$$

Therefore $L_1 \subseteq L$. Hence the theorem.

Theorem 4.2.10.

If $M_1$ is a sub uffsa of the uffsa $M$, then the monoid $F(M)$ is a homomorphic image of the monoid $F(M_1)$.

Proof. Let $M = (Q, \Sigma, \sigma, i, f)$ be the uffsa and $M_1 = (Q_1, \Sigma, \sigma_1, i_1, f_1)$ be a sub uffsa of $M$.

$F(M) = \{f_x \mid x \in \Sigma^*\}$ and $F(M_1) = \{f_x^{(1)} \mid x \in \Sigma^*\}$ where

$f_x : Q \to Q$ and $f_x^{(1)} : Q_1 \to Q_1$, $f_x$, $f_x^{(1)}$ are as in Definition 3.2.1.

Define $\phi : F(M_1) \to F(M)$ such that $\phi(f_x^{(1)}) = f_x$, $x \in \Sigma^*$.

We show that $\phi$ is well defined.

Let $f_x^{(1)}$, $f_y^{(1)} \in F(M_1)$ and $f_x^{(1)} = f_y^{(1)}$.

Therefore $f_x^{(1)}(p) = f_y^{(1)}(p)$ $\forall p \in Q_1$. From the definition of sub uffsa, for each symbol $a$ in the string $x$, if there is a transition with nonzero membership value then all transitions on $a$ which are in $M$ are also in $M_1$ and

$\sigma_1(p, a, q) = \sigma(p, a, q)$ $\forall p, q \in Q_1$, $a \in \Sigma$.

Therefore, $f_x(p) = f_y(p)$ $\forall p \in Q$, implies that $f_x = f_y$.

Thus $\phi$ is well defined.

Next we show that, $\phi$ is a homomorphism of monoids.

Let $f_x^{(1)}$, $f_y^{(1)} \in F(M_1)$.

Now $\phi(f_x^{(1)} \circ f_y^{(1)}) = \phi(f_{yx}) = f_x \circ f_y = \phi(f_x^{(1)}) \circ \phi(f_y^{(1)})$. 

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Also \( \phi(f_x^{(1)}) = f_\lambda, f_\lambda^{(1)} \in F(M_1) \) and \( f_\lambda \in F(M) \) are the identity elements.
Therefore \( \phi \) is a homomorphism.

Finally we prove \( \phi \) is onto. Let \( y \in F(M) \). Then \( y = f_x \), for some \( x \in \Sigma^* \).
Therefore \( f_x^{(1)} \in F(M_1) \) and we have \( \phi(f_x^{(1)}) = f_x \).
Hence \( F(M) \) is a homomorphic image of \( F(M_1) \).

\[\begin{align*}
\text{Note 4.2.11.} \text{ Number of elements in } F(M) \text{ is less than or equal to the number of elements in } F(M_1). 
\end{align*}\]

### 4.3 Homomorphism

This section includes the definition of homomorphism (strong) of uffsa’s followed by examples and some significant results. For the definition of homomorphism of ffsa’s, we refer to [28, 38].

**Definition 4.3.1.**

Let \( M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2) \) be two uffsa’s.

A pair \( (\alpha, \beta) \) of mappings \( \alpha : Q_1 \rightarrow Q_2 \) and \( \beta : \Sigma_1 \rightarrow \Sigma_2 \), is called a **homomorphism**, written \( (\alpha, \beta) : M_1 \rightarrow M_2 \), if

\[\begin{align*}
(i) \quad &\mu_1(p, a, q) \leq \mu_2(\alpha(p), \beta(a), \alpha(q)) \forall p, q \in Q_1 \text{ and } \forall a \in \Sigma_1 \\
(ii) \quad &i_1(p) \leq i_2(\alpha(p)) \forall p \in Q_1 \\
(iii) \quad &f_1(p) \leq f_2(\alpha(p)) \forall p \in Q_1 \\
\end{align*}\]

The pair \( (\alpha, \beta) \) is called a **strong homomorphism** if

\[\begin{align*}
(iv) \quad &\mu_2(\alpha(p), \beta(a), \alpha(q)) = \vee \{ \mu_1(p, a, t) \mid t \in Q_1, \alpha(t) = \alpha(q) \} \\
&\forall p, q \in Q_1 \text{ and } \forall a \in \Sigma_1 
\end{align*}\]
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Further, if \( \alpha(p) = \alpha(q) \) then

\[
\mu_2(\alpha(p), \beta(a), \alpha(q)) = \vee \{ \mu(s, a, t) \mid \alpha(t) = \alpha(q), \alpha(s) = \alpha(p) \}
\]

(v) \( i_2(\alpha(p)) = \vee \{ i_1(t) \mid t \in Q_1, \alpha(t) = \alpha(p) \} \)

(vi) \( f_2(\alpha(p)) = \vee \{ f_1(t) \mid t \in Q_1, \alpha(t) = \alpha(p) \} \)

A homomorphism (strong homomorphism) \((\alpha, \beta) : M_1 \rightarrow M_2\) is called an isomorphism (strong isomorphism) if \(\alpha\) and \(\beta\) are both one-one and onto.

Definition 4.3.2.

If \(\Sigma_1 = \Sigma_2\) and \(\beta\) is the identity map, then we write simply \(\alpha : M_1 \rightarrow M_2\) and say that \(\alpha\) is a homomorphism or strong homomorphism accordingly.

Further, if \((\alpha, \beta)\) is a strong homomorphism with \(\alpha\) one-one, then

\[
\mu_2(\alpha(p), \beta(a), \alpha(q)) = \mu_1(p, a, q), \forall p, q \in Q_1\text{ and } \forall a \in \Sigma_1.
\]

Example 4.3.3.

Let \(M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1)\) and \(M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2)\) be usfa’s, where

\(Q_1 = \{ q_0, q_1, q_2 \}, \Sigma_1 = \{ a, b \}, Q_2 = \{ q_0', q_1', q_2' \}, \Sigma_2 = \{ a, b \},\) and

\(\mu_1, \mu_2, i_1, i_2, f_1\) and \(f_2\) are defined as follows:

\[
\begin{align*}
\mu_1 : Q_1 \times \Sigma_1 \times Q_1 & \rightarrow [0,1] \text{ by } \\
\mu_1(q_0, a, q_0) &= 0.3 \\
\mu_1(q_0, a, q_1) &= 0.5 \\
\mu_1(q_0, a, q_1) &= 0.7 \\
\mu_1(q_1, a, q_0) &= 0.6 \\
\mu_2 : Q_2 \times \Sigma_2 \times Q_2 & \rightarrow [0,1] \text{ by } \\
\mu_2(q_0', a, q_0') &= 0.4 \\
\mu_2(q_0', a, q_1') &= 0.5 \\
\mu_2(q_0', b, q_1') &= 0.7 \\
\mu_2(q_1', a, q_0') &= 0.7
\end{align*}
\]

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\[
\begin{align*}
\mu_1(g_1, b, q_1) &= 0.4 & \mu_2(g'_1, b, q'_1) &= 0.5 \\
\mu_1(g_1, b, q_2) &= 0.6 & \mu_2(g'_1, b, q'_2) &= 0.8 \\
\mu_1(q_2, a, q_1) &= 0.5 & \mu_2(q'_2, a, q'_1) &= 0.5 \\
\mu_1(q_2, b, q_2) &= 0.6 & \mu_2(q'_2, b, q'_2) &= 0.7
\end{align*}
\]

\(i_1 : Q_1 \rightarrow [0, 1]\) such that \(i_1(q_0) = 0.8\), \(i_1(q_1) = 0.5\).

\(i_2 : Q_2 \rightarrow [0, 1]\) such that \(i_2(q'_0) = 0.9\), \(i_2(q'_1) = 0.5\), \(i_2(q'_2) = 0.1\).

\(f_1 : Q_1 \rightarrow [0, 1]\) by \(f_1(q_2) = 0.8\).

\(f_2 : Q_2 \rightarrow [0, 1]\) by \(f_2(q'_2) = 0.8\).

Define \(\alpha : Q_1 \rightarrow Q_2\) by \(\alpha(q_0) = q'_0\), \(\alpha(q_1) = q'_1\), \(\alpha(q_2) = q'_2\)

and \(\beta : \Sigma_1 \rightarrow \Sigma_2\) by \(\beta(a) = a\), \(\beta(b) = b\).

Clearly \((\alpha, \beta)\) is a homomorphism from \(M_1\) into \(M_2\). Since \(\alpha\) and \(\beta\) are bijective, \((\alpha, \beta)\) is an isomorphism of uffsa’s.

Example 4.3.4.

Let \(M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1)\) and \(M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2)\) be uffsa’s. \(M_1\) is defined as in Example 4.3.3, \(Q_2 = \{q'_0, q'_2\}\), \(\Sigma_2 = \{0, 1\}\).

\(\mu_2 : Q_2 \times \Sigma_2 \times Q_2 \rightarrow [0, 1]\) is defined as follows:

\[
\begin{align*}
\mu_2(q'_0, 0, q'_0) &= 0.6 & \mu_2(q'_0, 0, q'_6) &= 0.5 \\
\mu_2(q'_0, 1, q'_0) &= 0.7 & \mu_2(q'_2, 1, q'_2) &= 0.6 \\
\mu_2(q'_0, 1, q'_2) &= 0.6 &
\end{align*}
\]

\(i_2 : Q_2 \rightarrow [0, 1]\) is defined by \(i_2(q'_0) = 0.8\).

\(f_2 : Q_2 \rightarrow [0, 1]\) is defined by \(f_2(q'_2) = 0.8\).

Define \(\alpha : Q_1 \rightarrow Q_2\) by \(\alpha(q_0) = \alpha(q_1) = q'_0\), \(\alpha(q_2) = q'_2\).
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and $\beta : \Sigma_1 \to \Sigma_2$ by $\beta(a) = 0, \beta(b) = 1$.

Clearly $(\alpha, \beta)$ is a strong homomorphism of uffsa's.

Lemma 4.3.5.

Let $M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2)$ be uffsa's. Let $(\alpha, \beta) : M_1 \to M_2$ be a strong homomorphism. Then $\forall q, r \in Q_1, \forall a \in \Sigma_1$ if $\mu_2 (\alpha(p), \beta(a), \alpha(r)) > 0$, then there exists $t \in Q_1$ such that $\mu_1 (p, a, t) > 0$ and $\alpha(t) = \alpha(r)$. Further more, $\forall p \in Q_1$ if $\alpha(p) = \alpha(q)$, then $\mu_1 (p, a, t) \geq \mu_1 (q, a, r)$.

Proof. Let $p, q, r \in Q_1, a \in \Sigma_1$ and $\mu_2 (\alpha(p), \beta(a), \alpha(r)) > 0$. But $\mu_2 (\alpha(p), \beta(a), \alpha(r)) = \cup \{ \mu_1 (p, a, t) | \alpha(t) = \alpha(r) \}.$

Since $Q_1$ is finite, there exists $t \in Q_1$ such that $\alpha(t) = \alpha(r)$ and $\mu_2 (\alpha(p), \beta(a), \alpha(r)) = \mu_1 (p, a, t) > 0$.

Suppose $\alpha(p) = \alpha(q)$, then

$$
\mu_1 (p, a, t) = \mu_2 (\alpha(p), \beta(a), \alpha(r)) = \mu_2 (\alpha(q), \beta(a), \alpha(r)) \geq \mu_1 (q, a, r)
$$

Definition 4.3.6.

Let $M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2)$ be uffsa's and let $\beta : \Sigma_1 \to \Sigma_2$ be a map. Define $\beta^* : \Sigma_1^* \to \Sigma_2^*$ by

(i) $\beta^*(\lambda) = \beta(\lambda) = \lambda$

(ii) $\beta^*(a_1a_2 \ldots a_n) = \beta(a_1) \beta(a_2) \ldots \beta(a_n), n \geq 0, a_1, a_2, \ldots, a_n \in \Sigma_1$. 

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Lemma 4.3.7.

\[ \beta^*(xy) = \beta^*(x)\beta^*(y) \quad \forall x, y \in \Sigma_1^* \]

Proof. Let \( x, y \in \Sigma_1^* \), \( x = a_1a_2 \cdots a_n, \ y = b_1b_2 \cdots b_m, \ n, m \geq 1 \).

Now \( \beta^*(xy) = \beta^*(a_1a_2 \cdots a_nb_1b_2 \cdots b_m) \)
\[ = \beta(a_1)\beta(a_2) \cdots \beta(a_n)\beta(b_1)\beta(b_2) \cdots \beta(b_m) \]
\[ = \beta^*(a_1a_2 \cdots a_n)\beta^*(b_1b_2 \cdots b_m) \]
\[ = \beta^*(x)\beta^*(y) \]

Lemma 4.3.8.

Let \( M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2) \) be ufsa’s and let \( (\alpha, \beta) : M_1 \rightarrow M_2 \) be a homomorphism. Then

(i) \( \mu_1^*(p, x, q) \leq \mu_2^*(\alpha(p), \beta^*(x), \alpha(q)) \quad \forall x \in \Sigma_1^*, \ p, q \in Q_1 \)

(ii) \( i_1(p) \leq i_2(\alpha(p)) \quad \forall p \in Q_1 \)

(iii) \( f_1(p) \leq f_2(\alpha(p)) \quad \forall p \in Q_1 \)

Proof. First we prove (i). Let \( p, q \in Q_1 \) and \( x \in \Sigma_1^* \). We prove the result by induction on \( |x| = n \).

Let \( n = 0 \), then \( x = \lambda, \ \beta^*(x) = \beta^*(\lambda) = \lambda \).
If \( p = q, \ \mu_1(p, \lambda, q) = 1 = \mu_2(\alpha(p), \lambda, \alpha(q)) \)
If \( p \neq q, \ \mu_1(p, \lambda, q) = 0 \leq \mu_2(\alpha(p), \lambda, \alpha(q)) \)

Suppose the result is true for all \( x \in \Sigma_1^* \) such that \(|x| \leq n - 1, n > 0 \).
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Let $|x| = n$, $x = ya$, where $y \in \Sigma_1^*$, $a \in \Sigma_1$ and $|y| = n - 1$.

Now $\mu^*_1(p, x, q) = \mu^*_1(p, ya, q)$

$= \vee \{\mu^*_1(p, y, r) \land \mu_1(r, a, q) \mid r \in Q_1\}$

$\leq \vee \{\mu^*_2(\alpha(p), \beta^*(y), \alpha(r)) \land \mu_2(\alpha(r), \beta(a), \alpha(q)) \mid r \in Q_1\}$

$\leq \vee \{\mu^*_2(\alpha(p), \beta^*(y), r') \land \mu_2(r', \beta(a), \alpha(q)) \mid r' \in Q_2\}$

$= \mu_2(\alpha(p), \beta^*(y), \alpha(q))$

$= \mu^*_2(\alpha(p), \beta^*(x), \alpha(q))$

Thus the result is true for $|x| = n$. Hence (i).

(ii) and (iii) immediately follow from the definition.

Theorem 4.3.9.

Let $M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2)$ be uffa's with $L_1$ and $L_2$ be the fuzzy languages accepted by $M_1$ and $M_2$ respectively. Let $(\alpha, \beta) : M_1 \to M_2$ be a homomorphism. Then $L_1(x) \leq L_2(\beta^*(x)) \forall x \in \Sigma_1^*$.

Proof. Let $x \in \Sigma_1^*$.

Now $L_1(x) = \vee\{i_1(p) \land \mu^*_1(p, x, q) \land f_1(q) \mid q \in Q_1\} \mid p \in Q_1\}$

Since $Q_1$ is finite, there exists $r, s \in Q_1$, such that

$L_1(x) = i_1(r) \land \mu^*_1(r, x, s) \land f_1(s)$

$\leq i_2(\alpha(r)) \land \mu^*_2(\alpha(r), \beta^*(x), \alpha(s)) \land f_2(\alpha(s))$

$\leq \vee\{i_2(r') \land \mu^*_2(r', \beta^*(x), s') \land f_2(s') \mid s' \in Q_2\} \mid r' \in Q_2\}$

$= L_2(\beta^*(x))$
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Thus \( L_1(x) \leq L_2(\beta^*(x)) \forall x \in \Sigma_1^* \).

Corollary 4.3.10.
If \( \beta \) is the identity map then \( L_1 \subseteq L_2 \).

Lemma 4.3.11.

Let \( M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2) \) be ufsa's and
\( (\alpha, \beta) : M_1 \to M_2 \) be a strong homomorphism. If \( \alpha \) is bijective then

(i) \( \mu_1^*(p, x, q) = \mu_2^*(\alpha(p), \beta^*(x), \alpha(q)) \forall p, q \in Q_1, x \in \Sigma_1^* \)

(ii) \( i_1(p) = i_2(\alpha(p)) \forall p \in Q_1 \)

(iii) \( f_1(p) = f_2(\alpha(p)) \forall p \in Q_1 \)

Proof. Suppose \( \alpha \) is one–one and onto.

(i) Let \( p, q \in Q_1, x \in \Sigma_1^* \).

We prove the result by induction on \( |x| = n \).

Let \( n = 0, x = \lambda, \beta^*(\lambda) = \lambda \).

If \( p = q \) then \( \alpha(p) = \alpha(q) \) and \( \mu_1(p, \lambda, q) = 1 = \mu_2(\alpha(p), \lambda, \alpha(q)) \).

If \( p \neq q \) then \( \alpha(p) \neq \alpha(q) \) and \( \mu_1(p, \lambda, q) = 0 = \mu_2(\alpha(p), \lambda, \alpha(q)) \).

Therefore the result is true for \( n = 0 \).

Suppose the result is true \( \forall x \in \Sigma_1^*, |x| \leq n - 1 \).

Let \( |x| = n, x = ya, y \in \Sigma_1^*, a \in \Sigma_1, |y| \approx n - 1 \).
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Then \( \mu_2^* (\alpha(p), \beta^*(x), \alpha(q)) \)

\[ = \mu_2^* (\alpha(p), \beta^*(ya), \alpha(q)) \]
\[ = \mu_2 (\alpha(p), \beta^*(ya)\beta(a), \alpha(q)) \]
\[ = \bigvee \{ \mu_2^* (\alpha(p), \beta^*(ya), r') \land \mu_2 (r', \beta(a), \alpha(q)) \mid r' \in Q_2 \} \]

Since \( \alpha \) is onto, for \( r' \in Q_2 \), there exists an \( r \in Q_1 \) such that \( \alpha(r) = r' \).

Therefore \( \mu_2^* (\alpha(p), \beta^*(x), \alpha(q)) \)

\[ = \bigvee \{ \mu_2^* (\alpha(p), \beta^*(y), \alpha(r)) \land \mu_2 (\alpha(r), \beta(a), \alpha(q)) \mid r \in Q_1 \} \]
\[ = \bigvee \{ \mu_2^* (p, y, r) \land \mu_1 (r, a, q) \mid r \in Q_1 \} \quad \text{[by induction]} \]
\[ = \mu_1^* (p, ya, q) \]
\[ = \mu_1^* (p, x, q) \]

Thus the result is true for \( |x| = n \). Hence the result.

(ii) Let \( p \in Q_1 \),

\[ i_2(\alpha(p)) = \bigvee \{ i_1(r) \mid r \in Q_1, \alpha(r) = \alpha(p) \} . \]

Since \( \alpha \) is one–one, \( i_2(\alpha(p)) = i_1(p) \).

(iii) Let \( p \in Q_1 \),

\[ f_2(\alpha(p)) = \bigvee \{ f_1(r) \mid r \in Q_1, \alpha(r) = \alpha(p) \} . \]

Since \( \alpha \) is one–one, \( f_2(\alpha(p)) = f_1(p) \).

Theorem 4.3.12.

Let \( M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2) \) be uffsa’s such that \( L_1 \) and \( L_2 \) are the fuzzy regular languages accepted by \( M_1 \) and \( M_2 \) respectively. Let \( (\alpha, \beta) : M_1 \to M_2 \) be a strong homomorphism and if \( \alpha \) is bijective then \( L_1(x) = L_2(\beta^*(x)) \forall x \in \Sigma_1^* \).
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**Proof.** By theorem 4.3.9,

\[ L_1(x) \leq L_2(\beta^*(x)) \forall x \in \Sigma_1^* \]  

(4.1)

Let \( x \in \Sigma_1^* \),

\[ L_2(\beta^*(x)) = \bigvee \left\{ i_2(p') \land \mu_2^*(p', \beta^*(x), q') \land f_2(q') \mid q' \in Q_2 \right\} \]

Since \( Q_2 \) is finite, there exists \( r', s' \in Q_2 \) such that \( L_2(\beta^*(x)) = i_2(r') \land \mu_2^*(r', \beta^*(x), s') \land f_2(s') \)

Since \( \alpha \) is onto, there exists \( r, s \in Q_1 \) such that \( \alpha(r) = r' \) and \( \alpha(s) = s' \).

Therefore, \( L_2(\beta^*(x)) = i_2(\alpha(r)) \land \mu_2^*(\alpha(r), \beta^*(x), \alpha(s)) \land f_2(\alpha(s)) \)

By Lemma 4.3.11,

\[ L_2(\beta^*(x)) = i_1(r) \land \mu_1^*(r, x, s) \land f_1(s) \]

\[ \leq \bigvee \left\{ i_1(p) \land \mu_1^*(p, x, q) \land f_1(q) \mid q \in Q_1 \right\} \]

\[ = L_1(x) \]

Thus \( L_2(\beta^*(x)) \leq L_1(x) \forall x \in \Sigma_1^* \)  

(4.2)

From (4.1) and (4.2), \( L_1(x) = L_2(\beta^*(x)) \). Hence the theorem. \( \blacksquare \)

**Corollary 4.3.13.**

If \( \beta \) is the identity map then \( L_1(x) = L_2(x) \forall x \in \Sigma_1^* \). i.e., \( L_1 = L_2 \).

**Theorem 4.3.14.**

If \( M_1 \) and \( M_2 \) are strong isomorphic as uffsa's, then \( F(M_1) \) and \( F(M_2) \) are isomorphic as monoids.

**Proof.** Let \( M_1 = (Q_1, \Sigma_1, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma_2, \mu_2, i_2, f_2) \) be uffsa’s.

Let \((\alpha, \beta) : M_1 \rightarrow M_2\) be a strong isomorphism.
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\[ F(M_1) = \{ f_x | x \in \Sigma_1^* \}, \quad F(M_2) = \{ g_x | x \in \Sigma_2^* \} \]

Define \( \phi : F(M_1) \rightarrow F(M_2) \) by \( \phi(f_x) = g_{\beta^*(x)} \forall f_x \in F(M_1) \)

**Claim**: \( \phi \) is well defined.

Let \( f_x, f_y \in F(M_1) \) and \( f_x = f_y \).

This implies that \( f_x(p) = f_y(p) \ \forall p \in Q_1 \), therefore

\[ \alpha(f_x(p)) = \alpha(f_y(p)) \tag{4.3} \]

To prove, \( \alpha(f_x(p)) = g_{\beta^*(x)}(\alpha(p)) \ \forall x \in \Sigma_1^*, \ p \in Q_1 \).

We prove this result by induction on \( |x| = n \).

Let \( n = 0, \ x = \lambda, \ p \in Q_1 \)

Now \( \alpha(f_x(p)) = \alpha(f_\lambda(p)) = \alpha(p) \)

\( g_{\beta^*(x)}(\alpha(p)) = g_{\beta^*(\lambda)}(\alpha(p)) = g_\lambda(\alpha(p)) = \alpha(p) \)

Therefore \( \alpha(f_x(p)) = g_{\beta^*(x)}(\alpha(p)) \)

Thus the result is true for \( n = 0 \).

Suppose the result is true for all \( x \in \Sigma_1^*, \ |x| < n \).

Let \( |x| = n, \ x = ya, \ y \in \Sigma_1^*, \ a \in \Sigma_1, \ |y| = n - 1 \).

Now \( \alpha(f_x(p)) = \alpha(f_ya(p)) \)

\[ = \alpha(f_a \circ f_y(p)) = \alpha(f_a(f_y(p))) \]

\[ = \alpha(f_a(q)), \text{ where } q \text{ is such that } f_y(p) = q \in Q_1 \]

\[ = \alpha(q'), \text{ where } q' \text{ is such that } f_a(q) = q' \in Q_1 \tag{4.4} \]

\( g_{\beta^*(x)}(\alpha(p)) = g_{\beta^*(ya)}(\alpha(p)) \)

\[ = g_{\beta^*(y)a}(\alpha(p)) = g_{\beta^*(y)b(a)}(\alpha(p)) = (g_{\beta^*(y)} \circ g_{\beta^*(y)})(\alpha(p)) \]

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\[ g_{\beta(y)}(\alpha(p)) = g_{\beta(y)}(\alpha(p)) \]  

(4.5)

By induction, \( g_{\beta(y)}(\alpha(p)) = \alpha(f_y(p)) \)

Therefore,

\[ g_{\beta(a)}(\alpha(q)) = g_{\beta(a)}(\alpha(q)) \]

= \( s \),

(4.6)

where \( s \) is such that \( \mu_2(\alpha(q), \beta(a), s) = \vee\{\mu_2(\alpha(q), \beta(a), \tau) \mid \tau \in Q_2\} \).

But \( f_a(q) = q' \), therefore \( \mu_1(q, a, q') = \vee\{\mu_1(q, a, \tau') \mid \tau' \in Q_1\} \).

Since \( \alpha \) and \( \beta \) are bijective,

\[ \mu_1(q, a, q') = \mu_2(\alpha(q), \beta(a), \alpha(q')) \]

= \( \vee\{\mu_2(\alpha(q), \beta(a), \tau) \mid \tau \in Q_2\} \);

otherwise \( \mu_1(q, a, q') \) will not be maximum.

= \( \mu_2(\alpha(q), \beta(a), s) \)

We have \( \mu_2(\alpha(q), \beta(a), \alpha(q')) = \mu_2(\alpha(q), \beta(a), s) \).

Since \( M_2 \) is an uffsa, \( \alpha(q') = s \).

From (4.5) and (4.6), \( \alpha(f_a(p)) = g_{\beta(a)}(\alpha(p)) \)

Thus the result is true for \( |x| = n \).

Hence the result is true for any \( x \in \Sigma_1 \).

Applying this result to (4.4), we get

\[ g_{\beta(y)}(\alpha(p)) = g_{\beta(y)}(\alpha(p)) \forall p \in Q_1 \]

Since \( \alpha \) is onto, \( \forall q \in Q_2, g_{\beta(a)}(q) = g_{\beta(y)}(q) \)

Therefore \( g_{\beta(y)} = g_{\beta(y)} \) and hence \( \phi(f_x) = \phi(f_y) \)
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Therefore $\phi$ is well defined.

**Claim:** $\phi$ is a homomorphism.

Let $f_x, f_y \in F(M_1)$.

Now $\phi(f_x \circ f_y) = \phi(f_{yx})$

$$= g_{\beta^*(yx)} = g_{\beta^*(y)\beta^*(x)} = g_{\beta^*(x)} \circ g_{\beta^*(y)} = \phi(f_x) \circ \phi(f_y)$$

Also $\phi(f_\lambda) = g_{\beta^*(\lambda)} = g_\lambda$. Therefore $\phi$ is a homomorphism.

**Claim:** $\phi$ is one–one.

Let $f_x, f_y \in F(M_1)$ and $\phi(f_x) = \phi(f_y)$, implies that $g_{\beta^*(x)} = g_{\beta^*(y)}$ and so $g_{\beta^*(x)}(q) = g_{\beta^*(y)}(q) \forall q \in Q_2$.

Therefore $g_{\beta^*(x)}(\alpha(p)) = g_{\beta^*(y)}(\alpha(p)) \forall p \in Q_1$, that is $\alpha(f_x(p)) = \alpha(f_y(p))$.

We have $\alpha$ is one–one. Therefore $f_x(p) = f_y(p) \forall p \in Q_1$.

That is $f_x = f_y$. Therefore $\phi$ is one–one.

To prove, $\phi$ is onto, first we prove $\beta^*: \Sigma_1^* \rightarrow \Sigma_2^*$ is onto.

Let $y = b_1b_2 \cdots b_n \in \Sigma_2^*$, $b_i \in \Sigma_2$, $i = 1, 2, \ldots, n$.

$\beta$ is onto, therefore there exists $a_i \in \Sigma_1$ such that $\beta(a_i) = b_i$, $i = 1, 2, \ldots, n$.

Therefore $y = \beta(a_1)\beta(a_2) \cdots \beta(a_n)$

$$= \beta^*(a_1a_2 \cdots a_n)$$

$$= \beta^*(x), x = a_1a_2 \cdots a_n \in \Sigma_1^*$$

Hence $\beta^*$ is onto.

Let $z \in F(M_2)$. Then $z = g_y$, for some $y \in \Sigma_2^*$.

Since $\beta^*$ is onto, there exists an $x \in \Sigma_1^*$ such that $\beta^*(x) = y$. Therefore $g_y = g_{\beta^*(x)}$, $x \in \Sigma_1^*$. Hence $f_x \in F(M_1)$ and we have $\phi(f_x) = g_{\beta^*(x)} = g_y = z$.

Hence $\phi$ is onto. Therefore $\phi$ is an isomorphism of monoids. 

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4.4 Admissible Relation

In this section, we define admissible relation in uffsa and prove some results.

For the basic concepts in admissible relation, we refer to [38].

Definition 4.4.1.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa and let $\sim$ be an equivalence relation on $Q$. Then $\sim$ is called an admissible relation if and only if for all $p, q, r \in Q, \forall a \in \Sigma$, if $p \sim q$ and $\mu(p, a, r) > 0$, then there exists a $t \in Q$ such that $\mu(p, a, r) = \mu(q, a, t)$ and $t \sim r$.

Theorem 4.4.2.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffsa and let $\sim$ be an equivalence relation on $Q$. Then $\sim$ is an admissible relation if and only if for all $p, q, r \in Q, \forall x \in \Sigma^*$, if $p \sim q$ and $\mu^*(p, x, r) > 0$ then there exists a $t \in Q$ such that $\mu^*(p, x, r) = \mu^*(q, x, t)$ and $t \sim r$.

Proof. Suppose $\sim$ is an admissible relation on $Q$.

Let $p, q \in Q$ be such that $p \sim q$.

Let $x \in \Sigma^*, r \in Q$ be such that $\mu^*(p, x, r) > 0$.

We prove the result by induction on $|x| = n$.

Let $n = 0$, $x = \lambda$, $p \sim q$ and

$\mu^*(p, x, r) > 0$, implies that $p = r$ and $\mu^*(p, x, p) = 1$.

Now $\mu^*(q, x, q) = 1$ and $q \sim p$.

Thus the result is true for $n = 0$.

Suppose the result is true $\forall x \in \Sigma^*, |x| < n$. 

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Let $|x| = n$, $x = ya$, where $y \in \Sigma^*$, $a \in \Sigma$, $|y| = n - 1$.

Let $p, q \in Q$, $p \sim q$ and $\mu^* (p, x, r) > 0$

Therefore $\mu^* (p, ya, r) = \forall \{\mu^* (p, y, q_1) \land \mu (q_1, a, r) \mid q_1 \in Q\} > 0$

Since $Q$ is finite, there exists an $s \in Q$ such that

$\mu^* (p, ya, r) = \mu^* (p, y, s) \land \mu (s, a, r) > 0$

Therefore $\mu^* (p, y, s) > 0$ and $\mu (s, a, r) > 0$

By induction, there exists $t_s \in Q$ such that

$\mu^* (p, y, s) = \mu^* (q, y, t_s)$ and $t_s \sim s$.

Now $\mu (s, a, r) > 0$ and $s \sim t_s$ then there exists $t \in Q$

such that $\mu (s, a, r) = \mu (t_s, a, t), r \sim t$

Therefore $\mu^* (p, ya, r) = \mu^* (q, y, t_s) \land \mu (t_s, a, t)$

Since $M$ is an uffa, the maximum will be arrived for any $r'$ such that $r' \sim t_s$ only.

Therefore, $\mu^* (p, ya, r) = \forall \{\mu^* (q, y, r') \land \mu^* (r', a, t) \mid r' \in Q\}$

$= \mu^* (q, ya, t), r \sim t$

i.e., $\mu^* (p, x, r) = \mu^* (q, x, t), r \sim t$

Thus the result is true for $|x| = n$. Hence the result. □

Lemma 4.4.3.

Let $M = (Q, \Sigma, \mu, i, f)$ be an uffa and let $\sim$ be an admissible relation on $Q$.

Then there exists a fuzzy subset $\mu_1 : Q_1 \times \Sigma \times Q_1 \rightarrow [0, 1]$, where $Q_1 = Q/ \sim$.

Moreover, $\mu_1$ is a fuzzy function of $Q_1 \times \Sigma \times [0, 1]$ into $Q_1$.

Proof. Let $q \in Q$ and $[q]$ be the equivalence class of $q$.

i.e., $[q] = \{p \in Q \mid q \sim p\}$. Let $Q_1 = Q/ \sim = \{[q] \mid q \in Q\}$. 95
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Define $\mu_1 : Q_1 \times \Sigma \times Q_1 \to [0, 1]$ by

$$\mu_1 ([p], a, [q]) = \mu (p, a, r), \quad r \in [q] \quad \forall p \in Q, \quad a \in \Sigma$$

(4.7)

Suppose $([p], a, [q]) = ([p'], b, [q'])$

Therefore $[p] = [p'], \quad a = b, \quad [q] = [q']$

Implies that $p \sim p'$ and $q \sim q'$

Let $\mu (p, a, r) > 0, \quad r \in [q]$ and $p \sim p'$, since $\sim$ an admissible relation on $Q$, there exists $t \in Q$ such that

$$\mu (p, a, r) = \mu (p', a, t) \quad \text{and} \quad t \sim r$$

(4.8)

$r \in [q]$, implies that $r \in [q']$ and so $t \in [q']$.

By the definition of $\mu_1$,

$$\mu (p', a, t) = \mu_1 ([p'], a, [q'])$$

(4.9)

From (4.7), (4.8) and (4.9), $\mu_1 ([p], a, [q]) = \mu_1 ([p'], b, [q'])$

Therefore $\mu_1$ is well defined.

We shall prove $\mu_1$ is a fuzzy function.

Let $\mu_1 ([p], a, [q]) = \mu_1 ([p], a, [q']) > 0$

Therefore there exists $r, r' \in Q$ such that

$$\mu_1 ([p], a, [q]) = \mu (p, a, r) \quad \text{and} \quad r \sim q$$

$$\mu_1 ([p], a, [q']) = \mu (p, a, r') \quad \text{and} \quad r' \sim q'$$

Therefore $\mu (p, a, r) = \mu (p, a, r')$

Since $M$ is an ufsa, $r = r'$, therefore $r \sim q$ and $r \sim q'$.

Hence $q \sim q'$ and so, $[q] = [q']$

Therefore $\mu_1$ is a fuzzy function from $Q_1 \times \Sigma \times [0, 1]$ into $Q_1$.
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Definition 4.4.4.
Let \( M = (Q, \Sigma, \mu, i, f) \) be an uffsa and \( \sim \) be an admissible relation on \( Q \). Let \( Q_1 = Q/\sim \). Define the uffsa \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \), where \( \mu_1 \) is the fuzzy subset \( \mu_1 : Q_1 \times \Sigma \times Q_1 \rightarrow [0, 1] \) such that \( \forall [p], [q] \in Q_1 \)
\[
\mu_1([p], a, [q]) = \mu(p, a, r), \quad r \in [q]
\]
\( i_1 : Q_1 \rightarrow [0, 1] \) such that \( i_1([p]) = \vee\{i(q) \mid q \in [p]\} \)
\( f_1 : Q_1 \rightarrow [0, 1] \) such that \( f_1([p]) = \vee\{f(q) \mid q \in [p]\} \)

Theorem 4.4.5.
Let the uffsa \( M = (Q, \Sigma, \mu, i, f) \) and the uffsa \( M_1 \) be as in Definition 4.4.4.
Then there exists a strong homomorphism from \( M \) onto \( M_1 \).

Proof. Let \( (\alpha, \beta) : M \rightarrow M_1 \) be a mapping, where \( \alpha : Q \rightarrow Q_1 \) such that \( \alpha(q) = [q] \) \( \forall q \in Q \), \( \beta : \Sigma \rightarrow \Sigma \), the identity map.
Let \( p, q \in Q_1 \).

(i) \[
\mu_1(\alpha(p), \beta(a), \alpha(q)) = \mu_1(\alpha(p), a, \alpha(q))
= \mu_1([p], a, [q])
= \mu(p, a, r), r \in [q]
= \mu(p, a, r), [r] = [q]
= \mu(p, a, r), \alpha(r) = \alpha(q)
= \vee\{\mu(p, a, r) \mid \alpha(r) = \alpha(q)\}
\]

(ii) \[
i_1(\alpha(p)) = i_1([p])
= \vee\{i(q) \mid q \in [p]\}
= \vee\{i(q) \mid \alpha(q) = \alpha(p)\}
\]

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\[(iii) \quad f_1(\alpha(p)) = f_1([p])
\]

\[= \vee \{ f(q) \mid q \in [p] \}\]

\[= \vee \{ f(q) \mid \alpha(q) = \alpha(p) \}\]

Thus \((\alpha, \beta)\) is a strong homomorphism.

Clearly \(\alpha\) is onto, which completes the theorem.