Chapter II
CHAPTER – II
TWO CHARACTERISTIC MANPOWER FLOW MODEL INCLUDING DEMOTION

Earlier work on two characteristic manpower models was restricted to promotion and retention along with recruitment. The main aim of this chapter is to extend some of these results by permitting a backward transition [demotion], which is quite natural in manpower flow models.

2.1 TWO CHARACTERISTIC MANPOWER FLOW UNDER DIFFERENT RECRUITMENT POLICIES INCLUDING DEMOTION

It is obvious that for any organization, maintenance of the human resource inventory plays a very vital role regarding the success or failure of the organization. Recruitments of the required human resource play a vital part to achieve the goals of an organization. Hence things regarding the recruitment policy must be taken very seriously. The grade sizes can be determined by the pattern of loss, recruitment and transfers. Of these factors, the promotion, demotion and recruitment distributions are often amenable to control. This is the motivation of the present work.

Marshall [1975] considered the grade size distribution of two characteristic flow, where the total number of people promoted is based on either a fixed proportion of the number of people available in their state or a fixed proportion of the vacancies to be filled. The above paper deals with only one side of the problem namely promotion policies.
Chandra [1990] considered the two characteristic manpower flow model under different recruitment policies without considering the demotion case.

The aim of this section is to form a basic stock equation for a two characteristic flow by considering demotion along with promotion.

2.1.1. BASIC CHARACTERISTICS AND NOTATIONS

It is assumed that time is considered in discrete periods and that people are counted at the beginning of each period. If we consider a person for counting, he is assumed to have two characteristics $i$ & $j$ and is said to be in state $(i, j)$. Here $i$ - refers to $1^{st}$ characteristic (FC) $1 \leq i \leq N$ and $j$ represents the $2^{nd}$ characteristic (SC) $1(i) \leq j \leq u(i)$.

Assumptions

The promotion and demotion are implemented only at the end of a period. Only a single step promotion and demotion can be made in a time period. Prior experience in other institutions will also be taken into account while considering the Length of Service (LOS).

$i$ : grade index

$j$ : length of service index

$q_{ij}$ : the probability that a person in state $(i, j)$ at the start of one period will be in state $(i, j+1)$ at the start of the next period.

$p_{ij}$ : the probability that a person in state $(i, j)$ at the start of one period will be in state $(i+1, j+1)$ at the start of the next period.
\[ f_{i+1, j-1} \] : the probability that a person in state \((i+1, j-1)\) at the start of one period will be in state \((i, j)\) at the start of the next period.

\[ a_{i-1, j-1}(t) \] : the number of eligible persons in state \((i-1, j-1)\) to be promoted to state \((i, j)\) at time \(t\).

\[ b_{i+1, j-1}(t) \] : the number of persons in state \((i+1, j-1)\) to be demoted to state \((i, j)\) at time \(t\).

\[ r_{ij} \] : the fixed proportion of recruits in state \((i, j)\).

\[ m \] : the maximum time a person can spend in the system.

\[ l_{ij} \] : the fixed proportion of vacancies arising in state \((i, j)\).

\[ s_{ij}(t) \] : expected stock in state \((i, j)\) at time \(t\).

\[ k_i \] : the proportion of existing vacancies in grade \(i\) that remain unfilled during one unit of time.

\[ C_{ij} \] : the fixed proportion of recruits in state \((i, j)\) based on the fixed proportion of vacancies.

\[ R_{ij}(t) \] : number of recruits in state \((i, j)\) at time \(t\).

\[ v_i(t) \] : the expected number of vacancies existing in grade \(i\) at time \(t\).
$W_{ij}$: the probability that a person in state $(i,j)$ at the start of one period will leave before the end of that period.

Let the $m \times m$ matrices $\Gamma'$, $R_i$, $S_i(t)$, $A_i(t)$, $Q_i$, $B_i(t)$, and $F_i$ be given by

$$\Gamma' = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$R_i = \begin{bmatrix} r_{i0} & 0 & 0 & \cdots & 0 \\ 0 & r_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_{i,m-1} \end{bmatrix}$$

$$Q_i = \begin{bmatrix} q_{i0} & 0 & 0 & \cdots & 0 \\ 0 & q_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{i,m-1} \end{bmatrix}$$

$$P_i = \begin{bmatrix} p_{i0} & 0 & 0 & \cdots & 0 \\ 0 & p_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{i,m-1} \end{bmatrix}$$
\[
F_i = \begin{bmatrix}
\mathbf{f}_{i0} & 0 & - & - & - & - & - & 0 \\
0 & \mathbf{f}_{i1} & - & - & - & - & - & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{f}_{i,m-1}
\end{bmatrix}
\]

\[
S_i(t) = \begin{bmatrix}
\mathbf{s}_{i0}(t) & 0 & 0 & - & - & - & - & 0 \\
0 & \mathbf{s}_{i1}(t) & 0 & - & - & - & - & 0 \\
0 & 0 & 0 & 0 & - & - & - & \mathbf{s}_{i,(m-1)}(t)
\end{bmatrix}
\]

\[
A_i(t) = \begin{bmatrix}
\mathbf{a}_{i0}(t) & 0 & 0 & - & - & - & - & 0 \\
0 & \mathbf{a}_{i1}(t) & 0 & - & - & - & - & 0 \\
0 & 0 & 0 & 0 & - & - & - & \mathbf{a}_{i,(m-1)}(t)
\end{bmatrix}
\]

\[
B_i(t) = \begin{bmatrix}
\mathbf{b}_{i0}(t) & 0 & 0 & - & - & - & - & 0 \\
0 & \mathbf{b}_{i1}(t) & 0 & - & - & - & - & 0 \\
0 & 0 & 0 & 0 & - & - & - & \mathbf{b}_{i,(m-1)}(t)
\end{bmatrix}
\]

\[
s_i(t) = \begin{bmatrix}
\mathbf{s}_{i0}(t), \mathbf{s}_{i1}(t), \ldots, \mathbf{s}_{i,(m-1)}(t)
\end{bmatrix}
\]
2.1.2. **RECRUITS A FIXED PROPORTION OF STOCK**

Assume that the system in which every state \((i, j)\) of grade \(i\) and hence the grade as a whole is expanding at a rate \(x_i (>1)\). Thus if \(s_{ij}(t)\) is the expected stock in state \((i, j)\) at time \(t\), then

\[
s_{ij}(t) = x_i * \frac{s_{ij}(t - 1)}{x_i}
\]  

(2.1.1)

\[
R_{ij}(t) = r_{ij} * s_{ij}(t - 1)
\]  

(2.1.2)

The basic stock equation is given by the equation

\[
s_{ij}(t) = s_{i,j-1}(t-1)q_{i,j-1} + a_{i-1,j-1}(t-1)p_{i-1,j-1} + b_{i+1,j-1}(t-1)f_{i+1,j-1} + r_{ij}s_{ij}(t-1)
\]

(2.1.3)

Equation (2.1.1) can be modified as follows

\[
s_{ij}(t - 1) = \frac{s_{ij}(t)}{x_i}
\]

(2.1.4)

Replacing \(j\) by \((j-1)\) in (2.1.4), we have,

\[
s_{i,j-1}(t - 1) = \frac{s_{i,j-1}(t)}{x_i}
\]

(2.1.5)
Using (2.1.4) and (2.1.5) in (2.1.3), we have,

\[
\begin{align*}
\text{s}_{ij}(t) &= \frac{\text{s}_{i,j-1}(t)}{x_i} q_{i,j-1} + a_{i-1,j-1}(t-1) p_{i-1,j-1} + \frac{\text{s}_{ij}(t)}{x_i} r_{ij}(t-1) + b_{i+1,j-1}(t-1)f_{i+1,j-1} \\
\text{s}_{ij}(t) \left[ 1 - \frac{r_{ij}}{x_i} \right] &= \frac{\text{s}_{i,j-1}(t)}{x_i} q_{i,j-1} + a_{i-1,j-1}(t-1) p_{i-1,j-1} + b_{i+1,j-1}(t-1)f_{i+1,j-1} \\
\text{s}_{ij}(t) \left[ x_i - r_{ij} \right] &= \frac{s_{ij}(t)}{x_i} q_{i,j-1} + a_{i-1,j-1}(t-1) p_{i-1,j-1} + b_{i+1,j-1}(t-1)f_{i+1,j-1} \\
\end{align*}
\]

(2.1.6)

The general system discussed in (2.1.6) can be expressed in matrix form as follows:

\[
\begin{align*}
\text{S}_i(t) \left[ \text{r'} x_i - R_i \right] &= \text{S}_i(t) Q_i + x_i \left[ A_{i-1}(t-1) P_{i-1} + B_{i+1}(t-1) F_{i+1} \right] \\
\end{align*}
\]

(2.1.7)

Hence if \( A_i(t-1) \) and \( B_{i+1}(t-1) \) are known, from equation (2.1.7), the expected grade size vector can be evaluated.

For a general system, the basic stock equation is given by (2.1.3)

In matrix form

\[
\begin{align*}
\text{S}_i(t) \text{r'} &= \text{S}_i(t-1) \left[ Q_i + R_i \right] + A_{i-1}(t-1) P_{i-1} + B_{i+1}(t-1) F_{i+1} \\
\end{align*}
\]

(2.1.8)

When grade size vectors are constant, that is

\[
\text{S}_i(t) = \text{S}_i(t-1) \neq t
\]

We note that \( x_i = 1 \). In this case (2.1.7) reduces to the equation
S_i(t) = S_i(t)Q_i + A_{i-1}(t-1)P_{i-1} + B_{i+1}(t-1)F_{i+1} \quad (2.1.9)

Thus the, steady-state grade structure is found out when the number of recruits to any state is a fixed proportion of stock in that state.

### 2.1.3. RECRUITS A FIXED PROPORTION OF VACANCIES

The basic assumption of this model is that the vacancies are considered only after all the promotions and demotions are effected. If promotions and demotions are given only to the right persons, then there is a possibility of vacancies existing at any time.

As in section 2.1.2, an expansion of the system is considered with grade as FC and LOS as SC and grade i is expanding at a rate of \(x_i (x_i > 1)\) so that (2.1.1) holds.

Let \(l_{ij} s_{ij}(t-1)\) be the expected number of vacancies arising in state \((i,j)\) at time \(t\).

\[
v_i(t) = k_i v_i(t-1) + \sum_{j=0}^{m-1} l_{ij} s_{ij}(t-1) \quad (2.1.10)
\]

where \(m\) is the maximum time a person can spend in the system. If the management takes into consideration the vacancies in anticipation and recruits enough employees for all the vacancies then automatically \(k_i = 0\).

However, here the case \(k_i \neq 0\) is considered. This means there exists some vacancies in each grade. Note that

\[
R_{ij}(t) = C_{ij} v_i(t) \quad (2.1.11)
\]
The basic stock equation is given by

\[ s_{ij}(t) = s_{i, j-1}(t-1) q_{i, j-1} + a_{i-1, j-1}(t-1) p_{i-1, j-1} + R_{ij}(t) + b_{i+1, j-1}(t-1)f_{i+1, j-1} \]  

(2.1.12)

Using (2.1.11) in (2.1.12), we get

\[ s_{ij}(t) = s_{i, j-1}(t-1) q_{i, j-1} + a_{i-1, j-1}(t-1) p_{i-1, j-1} + C_{ij} v_{i}(t) + b_{i+1, j-1}(t-1)f_{i+1, j-1} \]  

(2.1.13)

Using (2.1.10) in (2.1.13), we get

\[ s_{ij}(t) = s_{i, j-1}(t-1) q_{i, j-1} + a_{i-1, j-1}(t-1) p_{i-1, j-1} + b_{i+1, j-1}(t-1)f_{i+1, j-1} \]

\[ + C_{ij} \left[ k_{i} v_{i}(t-1) + \sum_{j=0}^{m-1} l_{ij} s_{ij}(t-1) \right] \]  

(2.1.14)

Using (2.1.1) in (2.1.14), we get

\[ s_{ij}(t) = \frac{s_{i, j-1}(t)}{x_i} q_{i, j-1} + a_{i-1, j-1}(t-1) p_{i-1, j-1} + b_{i+1, j-1}(t-1)f_{i+1, j-1} \]

\[ + C_{ij} k_{i} v_{i}(t-1) + C_{ij} \sum_{j=0}^{m-1} l_{ij} \frac{s_{ij}(t)}{x_i} \]  

(2.1.15)
Consider the following matrices:

\[
A_i'(t) = \begin{bmatrix} a_{i0}(t), a_{i1}(t), \ldots, a_{i(m-1)}(t) \end{bmatrix}
\]

\[
B_i'(t) = \begin{bmatrix} b_{i0}(t), b_{i1}(t), \ldots, b_{i(m-1)}(t) \end{bmatrix}
\]

\[
C_i' = \begin{bmatrix} C_{i0}, C_{i1}, \ldots, C_{i(m-1)} \end{bmatrix}
\]

Then,

\[
x_i(t)S_i(t) = S_i(t)Q_i + x_i \begin{bmatrix} A_{i-1}'(t-1)P_{i-1} + B_{i+1}'(t-1)F_{i+1} + \ldots \end{bmatrix} + S_i(t)L_i
\]

When the system is of constant grade size, then \(x_i = 1\) and equation (2.1.16) reduces to this equation:

\[
S_i(t)\left(x_i(t)^{\Gamma} - Q_i - L_i\right) = x_i \begin{bmatrix} A_{i-1}'(t-1)P_{i-1} + B_{i+1}'(t-1)F_{i+1} + k_i C_i' v_i(t-1) \end{bmatrix} \tag{2.1.16}
\]
For a general system from (2.1.14) the matrix form of the basic stock equation is

\[ s_i^*(t) = T * s_i^*(t-1) + A_{i-1}^\prime(t-1)P_{i-1} + B_{i+1}^\prime(t-1)F_{i+1} \]

where \( s_i^*(t) = [v_i(t), s_{i0}(t), s_{i1}(t), \ldots, s_{im-1}(t)]^T \) and

\[
T = \begin{bmatrix}
    k_i & l_{i0} & l_{i1} & \cdots & l_{i,m-2} & l_{i,m-1} \\
    C_{i0}l_{i0} & C_{i0}l_{i0} & C_{i0}l_{i0} & \cdots & C_{i0}l_{i,m-2} & C_{i0}l_{i,m-1} \\
    C_{i1}k_i & C_{i1}l_{i0} + q_{i0} & C_{i1}l_{i1} & \cdots & C_{i1}l_{i,m-2} & C_{i1}l_{i,m-1} \\
    - & - & - & \cdots & - & - \\
    - & - & - & \cdots & - & - \\
    - & - & - & \cdots & - & - \\
    C_{i,m-1}k_i & C_{i,m-1}l_{i0} & - & - & \cdots & C_{i,m-1}l_{i,m-2} + q_{i,m-2} & C_{i,m-1}l_{i,m-1}
\end{bmatrix}
\]

2.1.4. GRADE STRUCTURE FOR GEOMETRIC RECRUITMENT

In the previous section we have discussed the geometrical expansion of the grade size. In this section, recruitment vector expands geometrically and grade structure is discussed for large \( t \). In real life, geometric recruitment may or may not exist. So here, we consider a hypothetical situation. We assume that all persons can be considered for both promotion and demotion. Here also grade is FC and LOS in SC.
For $x_i > 1$, let

$$R_i(t) = x_i^t R_i(0) \quad (2.1.18)$$

Consider the basis stock equation,

$$s_i(t) = s_i(t-1) Q_i + s_{i-1}(t-1) P_{i-1} + s_{i+1}(t-1) F_{i+1} + R_i(t)$$

$$= s_i(0) Q_i^t + \sum_{r=0}^{t-1} s_{i-1}(t-r-1) P_{i-1} Q_i^r$$

$$+ \sum_{r=0}^{t-1} s_{i+1}(t-r-1) F_{i+1} Q_i^r$$

$$+ \sum_{r=0}^{t-1} R_i(t-r) Q_i^r \quad (2.1.19)$$

$Q_i$ is the retention matrix for grade $i$ when there is no expansion in grade $i$. When there is an expansion in grade $i$, the retention matrix $G_i$ (say) will not be the same $Q_i$ and every entry in $G_i$ will be less than that of the corresponding entry in $Q_i$. Since the expansion rate in grade $i$ is $x_i$, $G_i = Q_i / x_i$. Therefore $Q_i = x_i G_i$.

Since $x_i > 1$, by the theory of non-negative matrices all the elements of matrix $G_i^t$ approach zero, as all the elements of $Q_i$ being only probabilities, are less than 1. Also, since $x_i$ greater than 1, the inverse of $(I - G_i)$ exists and is non-negative.

Write $(I - G_i)^{-1} = \sum_{t=0}^{\infty} G_i^t = E_i \quad (2.1.20)$
From (2.1.19) and (2.1.20)

\[ s_i(t) = s_i(0) x_i^t G_i^t + \sum_{r=0}^{t-1} s_{i-1}(t-r-1) P_{i-1} x_i^r G_i^r \]

\[ + \sum_{r=0}^{t-1} s_{i+1}(t-r-1) F_{i+1} x_i^r G_i^r + x_i^t \left[ \sum_{r=0}^{t-1} G_i^r \right] R_i(0) \]

Consider the term

\[ \sum_{r=0}^{t-1} s_{i-1}(t-r-1) P_{i-1} x_i^r G_i^r = \sum_{r=0}^{t-1} s_{i-1}(u) P_{i-1} x_i^{-(u+1)} G_i^t \]

\[ = \sum_{r=0}^{t-1} s_{i-1}(u) P_{i-1} \left( x_i G_i \right)^{-(u+1)} G_i^t \]

Clearly \( Q_i \) is a diagonal matrix with all elements less than one. Hence its inverse, \( Q_i^{-1} \), must be a diagonal matrix, with all elements greater than 1. The diagonal elements are

\[ \left( \frac{1}{q_i^0} \right)^{u+1} ; \left( \frac{1}{q_{i1}} \right)^{u+1} ; \ldots ; \left( \frac{1}{q_{i, m-1}} \right)^{u+1} \]

\[ \sum_{u=0}^{t-1} \left( s_{i-1,0}(u), s_{i-1,1}(u), \ldots, s_{i-1,m-1}(u) \right) P_{i-1} \left( Q_i^{-1} \right)^{u+1} G_i^t = \]

\[ = \left[ \sum_{u=0}^{t-1} s_{i-1,0}(u) \frac{P_{i-1,0} q_i^0}{q_i^0} x_i^t, \sum_{u=0}^{t-1} s_{i-1,1}(u) \frac{P_{i-1,1} q_i^1}{q_i^1} x_i^t, \ldots, \sum_{u=0}^{t-1} s_{i-1,m-1}(u) \frac{P_{i-1,m-1} q_i,m-1}{q_i,m-1} x_i^t \right] \]
Now \[ \sum_{u=0}^{t-1} \frac{s_{i-1,0}^{(u)}}{q_{i0}} \frac{q_i t_0^{u+1}}{x_i^t} = \frac{t_1}{x_i^t} \sum_{u=0}^{t-1} \frac{s_{i-1,0}^{(u)}}{q_{i0}^u} \]

\[ \frac{s_i(t)}{x_i^t} = s_i(0) G_i^t + \sum_{r=0}^{t-1} s_{i-1}^{(t-r-1)} P_{i-1} x_i^{(r-t)} G_i^r \]

\[ + \sum_{r=0}^{t-1} s_{i+1}^{(t-r-1)} F_{i+1} x_i^{r-t} G_i^r + \left( \sum_{r=0}^{t-1} G_i^r \right) R_i(0) \]

Clearly \[ \lim_{t \to \infty} \frac{P_{i-1} q_{i0}^{t-1}}{x_i^t} = 0 \]

\[ \lim_{t \to \infty} \sum_{u=0}^{t-1} \frac{s_{i-1,j}^{(u)}}{q_{ij}^u} = \infty \]

If \[ \sum_{u=0}^{t-1} \frac{s_{i-1,j}^{(u)}}{q_{ij}^u} = 0 \left[ x_i^{-(t/n)} \right], \text{where } n \text{ is a positive integer for all } j, \text{ then} \]

\[ \sum_{u=0}^{t-1} \left( \frac{s_{i-1,j}^{(u)}}{q_{ij}^u} \right) \frac{p_{i-1,j} q_{ij}^{t-1}}{x_i^t} = \frac{0 \left( x_i^{-t/n} \right) p_{i-1} q_{i}^{t-1}}{x_i^t} \]
When \( t \to \infty \), the entire sum tends to 0 in

\[
\sum_{u=0}^{t-1} s_{i-1,j} (u) p_{i-1} \left( Q_i^{-1} \right)^{u+1}
\]

and \( G_i^t \to 0 \), as \( t \to \infty \).

Similarly we can prove that

\[
\sum_{u=0}^{t-1} s_{i+1,j} (u) F_{i+1} x_i^{r-t} G_i^r \to 0 \quad \text{as} \quad t \to \infty
\]

Thus from equation (2.1.21),

\[
\frac{s_i(t)}{x_i^t} \to E_j R_j(0) \quad \text{as} \quad t \to \infty
\]

Hence, for a very large value of \( t \), the stock level can be approximated as

\[
\frac{s_i(t)}{x_i^t} = E_j R_j(0); \]

31
\[ s_i(t) = x_i^t E_i R_i(0) = x_i \left[ x_i^{t-1} E_i R_i(0) \right]; \]

\[ s_j(t) = x_j s_j(t-1) \quad (2.1.22) \]

It is clear that the recruitment vector expands geometrically and

\[
\text{when } \sum_{u=0}^{t=1} s_{i-1,j}(u) q_{ij}^{-u} = 0 \left[ x_i^{-t/n} \right], \text{ it is obvious that for a very large value of } t, \text{ the grade structure vector also expands geometrically with the same rate.}
\]

It is quite natural that whenever the recruitment vector expands, the grade structure vector will also expand. It has proved that both vectors expand geometrically, the rate of expansion being the same.

Apart from this, it is possible to discuss the behaviour of the distribution across grades, for a given recruitment policy from (2.1.22).

\[ s_{ij}(t) = x_i s_{ij}(t-1); \sum_j s_{ij}(t) = x_i \sum_j s_{ij}(t-1) \quad (2.1.23) \]

\[ \Rightarrow \frac{s_{ij}(t)}{\sum_j s_{ij}(t)} = \frac{s_{ij}(t-1)}{\sum_j s_{ij}(t-1)} \quad (2.1.24) \]

Hence the fractional state sizes found by dividing each state size by the grade size remain the same at each time point.
2.1.5. RECRUITMENT POLICY FOR THE GIVEN STRUCTURE

In the previous section, a system in which the given position in state \((i, j)\) expands linearly was considered. But here, a system in which the total size of grade \(i\) expands linearly is to be considered. Thus,

\[
\sum_{j=0}^{m-1} s_{ij}(t) = x_i \sum_{j=0}^{m-1} s_{ij}(t-1)
\]

(2.1.25)

In terms of matrix, (2.1.25) is given by

\[
s_i(t) = x_i s_i(t-1)
\]

(2.1.26)

where \(\bar{1}\) is a column vector \([1, 1, \ldots, 1]\).

Here it is assumed that all persons can be considered for both promotion and demotion. The basic stock equation in matrix form is given by

\[
s_i(t) = s_i(t-1)Q_i + s_{i-1}(t-1)P_{i-j} + s_{i+1}(t-1)F_{i+1} + R_i(t)
\]

\[
s_i(t)\bar{1} = s_i(t-1)Q_i\bar{1} + s_{i-1}(t-1)P_{i-j}\bar{1} + s_{i+1}(t-1)F_{i+1}\bar{1} + R_i(t)\bar{1}
\]

\[
x_i s_i(t-1)\bar{1} = s_i(t-1)Q_i\bar{1} + s_{i-1}(t-1)P_{i-j}\bar{1} + s_{i+1}(t-1)F_{i+1}\bar{1} + R_i(t)\bar{1}
\]

If \(W_i = [w_{i0}, w_{i1}, \ldots, w_{i(m-1)}]\) is the vector that refers to the wastage for grade \(i\), obviously

\[
W_i = \bar{1} - (Q_i + P_i + F_{i+1})\bar{1}
\]

(2.1.27)
\[ Q_i \tau = W_i - P_i \tau - F_{i+1} \tau \]
\[ x_i s_i (t-1) \tau = s_i (t-1) \left( W_i - P_i \tau - F_{i+1} \tau \right) + s_{i-1} (t-1) P_{i-1} \tau + R_i (t) \tau + s_{i+1} (t-1) F_{i+1} \tau \]
\[ R_i (t) \tau = s_i (t-1) W_i + (x_i - 1) s_i (t-1) \tau + \left( s_i (t-1) P_i - s_{i-1} (t-1) P_{i-1} \right) \tau \]
\[ + \left( s_i (t-1) - s_{i+1} (t-1) \right) F_{i+1} \tau \]

(2.1.28)

Equation (2.1.28) emphasizes the recruitment policy.

For a system of constant grade size, \( x_i = 1 \), the total size of the system remains constant.

2.2. OPTIMUM RECRUITMENT POLICY FOR CONSTANT GRADE SIZES INCLUDING DEMOTION

Recruitment is the first step in the employment of a person and hence, the methods by means of which manpower is brought into the system have a lot to do with the ultimate failure of such employment.

It is well known that any graded manpower system is controlled through the application of established policies for recruitment. Hence, one might use his results to calculate what the grade structure should be, in order to employ a desired promotion or recruitment policy. Conversely it can be decided which policy should be employed in order to attain a desired grade structure.

Andrew Young and Abodunde [1979] showed that Mathematical models of Manpower systems can be adopted to investigate the consequences of controlling
recruitment policies over fairly long periods of time. There, it is proved that if costs can be
described both under and over production, it is possible to combine the manpower models
with Mathematical programming techniques to produce optimal long-term recruitment
policies.

Mehlmann [1980] attempted to achieve optimal recruitment and transition strategies
as functions of present structure by applying dynamic programming arguments on a finite
horizon optimal control problem for a discrete time Markovian Manpower system. Leeson
[1979] employed projection and promotion models for graded Manpower systems must
consider recruitment policies and their effects on internal structures. He obtained various
policy models by illustrating a broad range of different possibilities.

All these models deal only with one character Manpower flow. However, dealing
with a two-characteristic Manpower flow model is of much importance and it is more
pertinent to combine Markov analysis for a two characteristic flow with Mathematical
Programming in order to get more insight into Manpower models.

In any organization, whenever vacancies arise in state \((i,j)\) they are filled in the
following fashion:

i. through promotion

ii. through demotion and

iii. through new recruitment

Since the grade size is taken to be constant there is no chance of keeping unfilled
vacancy.
Chandra [1990(a)] has combined different recruitment policies for a two characteristic Manpower flow and obtained an optimum recruitment policy using Integer Programming by assuming that the vacancies are filled up only through promotion and recruitment. In many private sector companies, there is a possibility of demoting persons to the next lower grade due to low rating in performance appraisal report. Based on this motivation, in this section, an optimum recruitment policy is discussed by assuming that the vacancies are filled by promotion, demotion and recruitment. The overall objective is to set up a recruitment policy in such a fashion that the total cost related to the filling up of vacancies must be minimum. This section extends the results of Chandra [1990(a)] to the present situation. For the same an Integer Programming model is developed in order to get more insight into the present Manpower model.

2.2.1. NOTATIONS AND ASSUMPTIONS

\( i \) : grade index

\( j \) : length of service; \( 1 \leq j \leq u(i) \)

\( s_{ij}(t) \) : expected stock in state \((i, j)\) at time \(t\).

\( p_{ij} \) : the probability that a person in state \((i, j)\) at the start of one period will be in state \((i+1, j+1)\) at the state of the next period.

\( q_{ij} \) : the probability that a person in state \((i, j)\) at the start of one period will be in state \((i, j+1)\) at the start of the next period.
$f_{ij}$ : the probability that a person in state $(i, j)$ at the start of one period will be in state $(i-1, j+1)$ at the start of the next period.

$r_{ij}$ : the fixed proportion of recruits at state $(i, j)$.

$r_{ij}(n)$ : the cost of recruiting a person to the state $(i, j)$ at the $n$th time point.

$a_{ij}(n)$ : the supporting cost for one person in state $(i, j)$ at time $n$.

$R_{ij}(t)$ : expected recruitment to state $(i, j)$ at time $t$.

$m$ : The maximum time a person can spend in the system.

$C_i$ : Total cost for grade $i$.

$R_i(t) : [R_{i00}(t), \ldots, R_{i0n0}(t)]$

$$
Q_i = 
\begin{bmatrix}
qu_{io} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_{i1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_{i2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_{i3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_{i, m-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_{i, m-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_{i, m-1}
\end{bmatrix}
$$

$$
P_i = 
\begin{bmatrix}
p_{io} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{i1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_{i2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_{i3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{i, m-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{i, m-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_{i, m-1}
\end{bmatrix}
$$

37
Let us assume that size values of $S_i(0) \forall i = 1, 2, \ldots, N$ are known.

The transition matrices $P_i$'s, $Q_i$'s and $F_i$'s are also known.

### 2.2.2. MATHEMATICAL MODELING

The basic stock equation obtained by Srinivasan and Mariappan [2001] is given by

\[
S_i(t) = S_i(t-1)Q_i + S_{i-1}(t-1)P_{i-1} + S_{i+1}(t-1)F_{i+1} + R_i(t)
\]  

(2.2.1)

Let us consider the stock position for a few successive time points, say $t = 1, 2, 3$.

\[
S_i(1) = S_i(0)Q_i + S_{i-1}(0)P_{i-1} + S_{i+1}(0)F_{i+1} + R_i(1)
\]  

(2.2.2)

\[
S_i(2) = S_i(1)Q_i + S_{i-1}(1)P_{i-1} + S_{i+1}(1)F_{i+1} + R_i(2)
\]  

(2.2.3)

\[
S_i(3) = S_i(2)Q_i + S_{i-1}(2)P_{i-1} + S_{i+1}(2)F_{i+1} + R_i(3)
\]  

(2.2.4)
From (2.2.2), we have

\[ S_{i-1}(1) = S_{i-1}(0) Q_{i-1} + S_{i-2}(0) P_{i-2} + S_{i}(0) F_{i} + R_{i-1} \]  \hspace{1cm} (2.2.5)

\[ S_{i+1}(1) = S_{i+1}(0) Q_{i+1} + S_{i}(0) P_{i} + S_{i+2}(0) F_{i+2} + R_{i+1} \]  \hspace{1cm} (2.2.6)

Using (2.2.2), (2.2.5) and (2.2.6) in (2.2.3), we have

\[ S_{i}(2) = S_{i}(0) \left[ Q_{i}^2 + F_{i} P_{i-1} + F_{i+1} P_{i} \right] + S_{i-1}(0) \left[ Q_{i} P_{i-1} + Q_{i-1} P_{i-1} \right] \]
\[ + S_{i-2}(0) \left[ P_{i-1} P_{i-2} \right] + S_{i+1}(0) \left[ Q_{i} F_{i+1} + Q_{i+1} F_{i+1} \right] \]
\[ + S_{i+2}(0) \left[ F_{i+1} F_{i+2} \right] + \left[ R_{i}(1) Q_{i} + R_{i-1}(1) P_{i-1} + R_{i}(2) + R_{i+1}(1) F_{i+1} \right] \]

(2.2.7)

Similarly evaluating \( S_{i-1}(2) \) and \( S_{i+1}(2) \) from (2.2.7) and substituting in (2.2.4), we have
\[ S_i(3) = S_i(0) \left\{ Q_i^3 + Q_i P_{i-1} F_1 + Q_{i-1} P_{i-1} F_1 + Q_i P_{i-1} F_1 \right\} \\
+ Q_{i+1} P_{i+1} F_{i+1} + 2 Q_i P_{i+1} F_{i+1} \}

+ S_{i-1}(0) \left\{ Q_i^2 P_{i-1} + Q_i Q_{i-1} P_{i-1} + Q_{i-1}^2 P_{i-1} + P_{i-1} P_{i-2} F_{i-1} + Q_{i-1}^2 F_i + P_{i-1} P_{i-1} F_{i+1} \right\} \\
+ S_{i-2}(0) \left\{ Q_i P_{i-1} P_{i-2} + Q_{i-1} P_{i-1} P_{i-2} + Q_{i-2} P_{i-2} P_{i-1} \right\} \\
+ S_{i-3}(0) \left[ P_{i-1} P_{i-2} P_{i-3} \right] \\
+ S_{i+1}(0) \left\{ Q_i^2 F_{i+1} + Q_i Q_{i+1} F_{i+1} + P_{i-1} P_{i+1} F_{i+1} + Q_{i+1}^2 F_{i+1} + P_{i+1} F_{i+1}^2 \right\} \\
+ S_{i+2}(0) \left\{ Q_i F_{i+1} F_{i+2} + Q_{i-1} F_{i+1} F_{i+2} + Q_{i+2} F_{i+1} F_{i+2} \right\} \\
+ S_{i+3}(0) \left[ F_{i+1} F_{i+2} F_{i+3} \right]

+ \left\lfloor R_i(1) Q_i^2 + Q_i R_{i-1}(1) P_i + Q_i R_{i+1}(1) P_{i+1} + Q_i R_i(2) \right\rfloor \\
+ Q_i P_{i-1} R_{i-1}(1) + P_{i-1} P_{i-2} R_{i-2}(1) + P_{i-1} R_{i-1}(1) F_i + P_{i-1} R_{i-1}(2) \\
+ Q_{i+1} R_{i+1}(1) F_{i+1} + P_{i+1} R_i(1) F_{i+1} + R_{i+2}(1) F_{i+1} F_{i+2} + R_{i+1}(2) F_{i+1} \\
+ R_i(3) \right\rfloor

(2.2.8)
The control is to be exercised by allocating new recruits approximately to the various states. It is assumed that the number of recruits is always non-negative so as to include the possibility of retrenchment.

Now let the condition, that the number of persons in each grade remains constant at each time point, be imposed.

Then \( \sum_{j} s_{ij}(t) = \sum_{j} s_{ij}(0) \ \forall \ t \)

Let \( a_{i}(n) = \begin{bmatrix} a_{ij}(n) \end{bmatrix} \)

Let \( r_{i}(n) = \begin{bmatrix} r_{ij}(n) \end{bmatrix} \)

Since \( S_{i}(0) \) is given, the cost of supporting it or of obtaining it initially, can be ignored for the purpose of optimization. Using the expressions for \( S_{i}(1) \), \( S_{i}(2) \) and \( S_{i}(3) \) the cost for \( S_{i}(t) \) can be expressed in terms of the cost of \( R_{i}(1) \), \( R_{i}(2) \), \ldots \( R_{i}(t-1) \).

Thus, the total cost for grade \( i \) to be minimized for the 1\textsuperscript{st} three years will be

\[
C_{i} = R_{i}(1) r_{i}(1) + R_{i}(2) r_{i}(2) + R_{i}(3) r_{i}(3) + Q_{i} R_{i-1}(1) P_{i-1} + Q_{i} R_{i+1}(1) F_{i+1}\{a_{i}(1) + R_{i}(1) Q_{i} R_{i-1}(2) + R_{i+1}(2) F_{i+1}\} a_{i}(1)
+ Q_{i} R_{i-1}(1) P_{i-1} + Q_{i} R_{i+1}(1) F_{i+1} + Q_{i} P_{i-1} R_{i-1}(1)
+ P_{i-1} P_{i-2} R_{i-2}(1) + P_{i-1} R_{i}(1) F_{i} + Q_{i+1} R_{i+1}(1) F_{i+1} + P_{i} R_{i}(1) F_{i+1}
+ R_{i+2}(1) F_{i+1} F_{i+2}\} a_{i}(2)
\]
\[
C_i = R_i(1) \left[ r_i(1) + Q_i a_i(1) + Q_i^2 a_i(2) \right] + \\
R_i(2) \left[ r_i(2) + Q_i a_i(1) \right] + R_i(3) \left[ r_i(3) \right]
\]

\[
R_{i+1}(1) \left[ p_{i-1} a_i(1) + Q_i P_{i-1} a_i(2) + Q_{i-1} P_{i-1} a_i(2) \right] + \\
R_{i+1}(2) \left[ p_{i-1} a_i(1) \right] + R_{i+2}(1) \left[ p_{i-1} P_{i-2} a_i(2) \right] + \\
R_{i+1}(1) \left[ f_{i+1} a_i(1) + Q_i F_{i+1} a_i(2) + Q_{i+1} F_{i+1} a_i(2) \right] + \\
R_{i+2}(1) \left[ f_{i+1} F_{i+2} a_i(2) \right] + R_{i+2}(2) \left[ f_{i+1} a_i(1) \right]
\]

(2.2.9)

Then \( C_i \) is the objective function, which is to be minimized. The constraints are given by this stock position for each grade.

\[
\sum_{j} s_{ij}(0) = \sum_{j} s_{ij}(1) = \sum_{j} s_{ij}(2) = \sum_{j} s_{ij}(3)
\]

(2.2.10)

\[
\sum_{j} s_{(i-1)j}(0) = \sum_{j} s_{(i-1)j}(1) = \sum_{j} s_{(i-1)j}(2)
\]

(2.2.11)

\[
\sum_{j} s_{(i-2)j}(1) = \sum_{j} s_{(i-2)j}(1)
\]

(2.2.12)

\[
R_i(t) \geq 0 \text{ and integers for all } i, j \text{ and } t.
\]

(2.2.13)

To start with, \( C_1 \) is minimized, giving the optimum values of \( R_1(1), R_1(2), R_1(3) \). Then \( C_2 \) is minimized, using the above optimum values and solving \( R_2(1), R_2(2) \) and \( R_2(3) \) and so on.
Alternate way of considering an objective function is to minimize

\[ C_1 + C_2 + \ldots + C_N, \text{ for the same constraints.} \]

2.3. ATTAINABILITY OF A TWO CHARACTERISTIC MANPOWER STRUCTURE INCLUDING DEMOTION

Bartholomew [1979], Davies [1983] and Abdallaovi [1987] analysed in detail, the problem of stochastic control of a graded manpower system and the probability of attaining a structure in one step for one dimensional manpower flow. Normally, in any organization, promotion or wastage for any grade depends on the length of service, time spent in that grade or physical location. Hence, the problem to maintain a structure was considered by Chandra [1990(a)] for a two-characteristic flow. Further the attainability of a two characteristic manpower structure also was analysed by Chandra [1990(b)] by considering promotion aspect only using Trinomial distribution. In this section the stochastic behaviour of a two characteristic manpower model including demotion is discussed with the help of multinomial distribution and the probabilities of grade structure, being attainable in one-step, under control of recruitment are calculated. Numerical results are also given to illustrate the model. In addition, the geometric description of some structures is also analysed.
2.3.1. NOTATIONS AND ASSUMPTIONS

\( i \): represents the grade [First Characteristic (FC)], \( 1 \leq i \leq N \).

\( j \): represents the length of service [Second Characteristic (SC)], \( l(i) \leq j \leq u(i) \).

\( q_{i, j-1} \): probability that a person is moving from state \((i, j-1)\) is state \((i, j)\) in one time period.

\( p_{i-1, j-1} \): probability that a person is moving from state \((i-1, j-1)\) to state \((i, j)\) in one time period.

\( f_{i+1, j-1} \): probability that a person is moving from \((i + 1, j - 1)\) to state \((i, j)\) in one time period.

\( r_{ij} \): probability of recruitment to state \((i, j)\) in one time period.

\( s_{ij}(t) \): expected stock in state \((i, j)\) at time \( t \).

\( R_i(t) \): the recruitment vector to grade \( i \) at time \( t \).

\( R_i(t) = [R_{i0}(t), \ldots, R_{iq}(t), \ldots, R_{iu}(t)] \)

\( m_{ij} \): the goal structure of the state \((i, j)\)

\( m_i = [m_{i0}, \ldots, m_{ij}, \ldots, m_{iu}] \)

\( s_{ij} \): the given structure of state \((i, j)\)

\( s_i = [s_{i0}, \ldots, s_{ij}, \ldots, s_{iu}] \)

\( x_{ij}^{1} \): number of persons who move from state \((i, j-1)\) to \((i, j)\)

\( x_{ij}^{2} \): number of persons who move from state \((i-1, j-1)\) to \((i, j)\)

\( x_{ij}^{3} \): number of persons who move from state \((i+1, j-1)\) to \((i, j)\)

\( x_{ij}^{4} \): number of recruits to state \((i, j)\)

\( P_i(m_{ij}) \): probability of attaining the structure \( m_{ij} \)

\( P_m(i) \): probability if attaining the structure \( m_i \)
We make the following assumptions,

1. Time is considered in discrete units.

2. Movement in one time period from states with FC i can only be to states with FC (i-1) or (i) or (i+1) or out of the system.

3. Service rendered in other systems before joining the present one is also counted.

4. Since only a single period (t,t+1) is considered the suffix t can be omitted.

5. Grade sizes are constant.

In the next section, the use of multinomial distribution to obtain the probability of a structure being attained in one step in a two-characteristic flow along with promotion including demotion is analysed.

### 2.3.2 MULTINOMIAL METHOD

The probability of attaining a structure in one step is of fundamental importance. For a one dimensional manpower flow Abdallaovi [1987] used trinomial distribution to evaluate such probability. Chandra [1991] evaluated the same for a two characteristic manpower flow by considering only promotion using trinomial distribution in a different fashion compared to Abdallaovi [1987]. In this section, the multinomial distribution method is used to evaluate the same for a two-characteristic manpower flow by considering promotion and demotion.

\[
 s_{i-1} = \begin{bmatrix} s_{(i-1)1}, \ldots, s_{(i-1)u(i-1)} \end{bmatrix}
\]

\[
 s_{i} = \begin{bmatrix} s_{i1}, \ldots, s_{iu(i)} \end{bmatrix}
\]
\[ m_i = \begin{bmatrix} m_{i1}, \ldots, m_{iu(i)} \end{bmatrix} \]

Since the grade sizes are taken to be constant, at least one \( m_{ij} \), say \( m_{ik} \), is greater than \( s_{ij} \). Obviously \( m_{ik} \) is attainable by promotion, demotion and recruitment, with probability one. Consider any other \( m_{ij} \) \( (m_{ij} > s_{ij}) \), then \( m_{ij} \) consists of four components namely \( x_{1ij}, x_{2ij}, x_{3ij} \) and \( x_{4ij} \).

Then
\[
x_{1ij} + x_{2ij} + x_{3ij} + x_{4ij} = m_{ij}, \forall j \quad (2.3.1)\]

If any person in state \((i, j)\) is considered, he / she has been either continuing in the same grade or promoted to grade \( i \) or demoted to grade \( i \) or recruited to grade \( i \).

Let,
\[
a = s_{i, j-1} * q_{i, j-1} / m_{ij} \\
b = s_{i-1, j-1} * p_{i-1, j-1} / m_{ij} \\
c = s_{i+1, j-1} * f_{i+1, j-1} / m_{ij} \text{ and} \\
d = 1 - a - b - c \quad (2.3.2)
\]

Here \( x_{1ij}, x_{2ij}, x_{3ij} \) and \( x_{4ij} \) are random variables occurring with the probabilities \( a, b, c \) and \( d \) respectively. Then the point probability mass function of the random variables is called multinomial distribution and is given by,
\[
P\left[ x_{1ij}, x_{2ij}, x_{3ij}, x_{4ij} \right] = \frac{m_{ij}!}{x_{1ij}! * x_{2ij}! * x_{3ij}! * x_{4ij}!} \cdot a^{x_{1ij}} \cdot b^{x_{2ij}} \cdot c^{x_{3ij}} \cdot d^{x_{4ij}} \quad (2.3.3)
\]

As \( x_{ij} \) takes the value from 0 to \( s_{i, j-1} * q_{i, j-1} \), \( x_{2ij} \) takes the value from 0 to \( s_{i-1, j-1} * p_{i-1, j-1} \), \( x_{3ij} \) takes the value from 0 to \( s_{i+1, j-1} * f_{i+1, j-1} \) and \( x_{4ij} \) takes the value from 0 to \( m_{ij} \) subject to the condition of equation (2.3.1),

46
subject to the condition that the above sum is less than or equal to 1.

The structure $m_i$ is attained only when all $m_{ij}$'s ($j = 1 \ldots, u(i)$) are attained.

\[
\therefore P(m_i) = \prod_{j=1}^{u(i)} P_j(m_{ij}) \tag{2.3.5}
\]

When $s_{ij-1}$, $s_{i-1,j-1}$ and $s_{i+1,j-1}$ are less than $m_{ij}$ for all $j$, it is very clear that the structure $m_i$ is attainable with probability 1.

For a better understanding of the method an illustration is given in the next section.

2.3.3. NUMERICAL ILLUSTRATION

Here, $P(m_i)$ is evaluated for different sets of values of $s_{i,j-1}$, $s_{i-1,j-1}, s_{i+1,j-1}$, $q_{ij}$, $p_{i-1,j-1}$ and $f_{i+1,j-1}$; ($j = 1,2,3$).

Let $m_i = (40, 25, 25)$; that is $m_{i1} = 40$, $m_{i2} = 25$ and $m_{i3} = 25$.

Taking $P(m_{i1}) = 1$, the computed values of $P(m_{i2})$, $P(m_{i3})$ and $P(m_i)$ by considering different stock level and different transition probabilities are tabulated in the following tables.
<table>
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<th>$s_j$</th>
<th>$s_{i-1,1}$</th>
<th>$s_{i+1,1}$</th>
<th>$q_{i,1}$</th>
<th>$p_{i-1,1}$</th>
<th>$f_{i+1,1}$</th>
<th>$P(m_{12})$</th>
<th>$s_{i,2}$</th>
<th>$s_{i-1,2}$</th>
<th>$s_{i+1,2}$</th>
<th>$q_{i,2}$</th>
<th>$p_{i-1,2}$</th>
<th>$f_{i+1,2}$</th>
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<td>23</td>
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<td>0.0001272440</td>
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</tbody>
</table>

From the table it is clear that $P(m_{12})$ decreases when $s_{i,1}$, $s_{i-1,1}$, and $s_{i+1,1}$ increases and also when the transition probabilities increase. In a very similar manner $P(m_{13})$ decreases when $s_{i,2}$, $s_{i-1,2}$ and $s_{i+1,2}$ increase and also when the transition probabilities increase. In other words, the probability of attaining a structure is more when the goal structure is nearer to the given structure or when the probability of a person being recruited to state $(i, j)$ is more. So, given a goal structure for grade $i$, it is possible to find the probability of attaining it from the given structure.
Table 2

<table>
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<th>$p_{i-1,2}$</th>
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From the above table we observe that keeping the retention and demotion probabilities fixed, the probability of attaining the goal structure for grade $i$ increases when the probability of promotion from grade $(i - 1)$ to grade $i$ decreases.
**Table - 3**

<table>
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</table>

From the above table, we observe that keeping the promotion and demotion probabilities fixed, the probability of attaining the goal structure for grade $i$ increases when the probability of retention from grade $i$ to grade $i$ decreases.
Table - 4

<table>
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</table>

From the above table, we observe that keeping the promotion and retention probabilities fixed, the probability of attaining the goal structure for grade $i$ increases when the probability of demotion from grade $(i + 1)$ to grade $i$ decreases.
2.3.4 GEOMETRIC RELATIONSHIP

Throughout this section, the geometric behaviour of the set of goal structures, attainable from \((s_{i-1}, s_i, s_{i+1})\) is discussed.

Let the grade be FC and the physical location be SC,

Let \(1 \leq j \leq b(i)\). Then

\[
s_{ij} = \sum_{k=1}^{b(i)} s_{ik} q_i(k, j) + \sum_{k=1}^{b(i-1)} s_{i-1,k} p_{i-1}(1, j) + \sum_{k=1}^{b(i+1)} s_{i+1,k} f_{i+1}(m, j) + R_{ij}
\]

(2.3.6)

Consider the structure,

\[
b_i = s_i Q_i + s_{i-1} P_{i-1} + s_{i+1} F_{i+1} + R_i
\]

(2.3.7)

Let \(S_i(s_i)\) be the set of structures for grade \(i\) attainable with non-zero probability of one-step, from the structure \(s\). Then

\[
m_i(t) = [m_{il}, \ldots , m_{ij}, \ldots , m_{im}]\]

is a member of the set \(S_i(s_{i-1}, s_i, s_{i+1})\) if and only if

\[
\sum_{k=1}^{b(i)} s_{ik} q_i(k, j) + \sum_{k=1}^{b(i-1)} s_{i-1,k} p_{i-1}(1, j) + \sum_{k=1}^{b(i+1)} s_{i+1,k} f_{i+1}(m, j) \leq m_{ij}
\]

i.e.,

\[
s_i Q_i + s_{i-1} P_{i-1} + s_{i+1} F_{i+1} \leq m_i
\]

(2.3.8)

The boundary of the region is determined by

\[
Y_i = s_i Q_i + s_{i-1} P_{i-1} + s_{i+1} F_{i+1}
\]

(2.3.9)

Thus \(Y_i\) is the structure obtained for grade \(i\), in one-step from \(s\), when leavers from each state are replaced with equal number of recruits. Therefore, \(Y_i\) is attainable with probability one.
Consider the points

\[ X_{ir} = Y_i + \begin{bmatrix} R_{i0}, \ldots, R_{ij}, \ldots, R_{i(m)} \end{bmatrix}' E_r \]

where \( r = 1, 2, \ldots, b(i) \); \( \begin{bmatrix} 1, 1, \ldots, 1 \end{bmatrix} \) \( b(i) \) and \( E_r \) is the \( 1 \times b(i) \) vector with 1 in the \( r \)th place and zero elsewhere. Now it can be shown that

\[ X_{ir}'s \ (r = 1, 2, \ldots, b(i)) \] are the vertices of the convex hull \( S_i(s_{i-1}, s_i, s_{i+1}) \). Obviously,

\[ X_{ir} = Y_i + A_r \]  \hspace{1cm} (2.3.10)

where \( A_r \) is the \( 1 \times b(i) \) vector with \( \{ \sum_{j=1}^{b(i)} R_{ij} \} \) in the \( r \)th place and zero elsewhere.

Using (2.3.7) and (2.3.8) we have,

\[ b_i = Y_i + R_i \]

\[ = \frac{R_{i1}}{\sum_{j} R_{ij}} x_{i1} + \frac{R_{i2}}{\sum_{j} R_{ij}} x_{i2} + \ldots + \frac{R_{i,b(i)}}{\sum_{j} R_{ij}} x_{i,b(i)} \]

\[ = \sum_{r=1}^{b(i)} C_{ir} X_{ir} \] where \( C_{ir} = \frac{R_{ir}}{\sum_{j} R_{ij}} \) and \( \sum_{r=1}^{b(i)} C_{ir} = 1 \).  \hspace{1cm} (2.3.11)

Thus, \( x_{i1}, x_{i2}, \ldots, x_{i,b(i)} \) are the vertices of a convex hull, of which \( b_i \)'s is the member.

Consider any other point

\[ b_1 = s_1 Q_i + s_{i-1} P_{i-1} + s_{i+1} F_{i+1} + R_i \]  \hspace{1cm} (2.3.12)

Since the grade sizes are assumed to be constant \( \sum_{j} R_{ij} = \sum_{j} R_{ij} ' \)
Thus \( b'_i = Y_i + R_i \).

\[
b'_i = \frac{b(i)}{R_{ij}} \frac{C_{ir}}{x_{ir}} ; \quad \text{where} \quad C_{ir} = \frac{R_{ir}}{\sum R_{ij}} \quad \text{and} \quad \sum_{r=1}^{R} C_{ir} = 1
\]

Thus \( b'_i \)'s is also expressed as a convex combination of \( x_{i1}, x_{i2}, \ldots, x_{ib} \), which are the vertices of the Convex hull \( S_i(s_{i-1}, s_i, s_{i+1}) \).

The above result explains the geometric behaviour of the attainable structures.

2.4. CONCLUSION

Throughout this chapter, it is assumed that the promotion and demotion probabilities are fixed and that only the recruitment distribution is to be controlled. Generally, a recruitment policy which satisfies some criterion of economic or social optimality is selected. A very good recruitment policy based on corporate objectives, study of the environment and corporate needs, may avoid hasty decisions and may go a long way to manage the organisation with the right type of personnel. This chapter is designed to explain the grade structure on recruitment aspects and vice versa. The results given here are not only used for prediction but also for deriving the important implications for the interpretation of crude labour turnover figures.

Next, a method is suggested here to obtain the total minimum cost for grade \( i \), along with the optimum number of recruits to each state \((i, j)\) at successive time points.

Lastly, the question of attaining a desireable structure with a stochastic environment, taking into account the aspects of promotion and demotion, is answered and thereby an extension of the results of Chandra [1991] is obtained in this direction.