Chapter VII
CHAPTER VII
EXPECTED TIME TO RECRUITMENT IN MANPOWER PLANNING
ASSOCIATED WITH CORRELATED RENEWAL SEQUENCES – A CUMULATIVE SHOCK MODEL APPROACH

In this chapter the author considered the general Manpower model governed by the correlated pairs \([X_n, Y_n]\) of renewal sequences where \(X\) is the man hours lost due to a random number of staff depletion which takes place at a decision making epoch and \(Y\) is the time between two consecutive decision making epochs. We vindicated that the organization reaches an uneconomic status which may otherwise be called a breakdown point if the maximum of the magnitude of \(X\) corresponding to different decisions taken in a given interval exceeds a pre specified threshold level \(z > 0\) for the first time.

More specifically different models are discussed. In Model – I, the system is assumed to be new at time \(t = 0\). The magnitude \(X_n\) of the nth decision is correlated to the time interval \(Y_n\) such the \((n - 1)\)th decision and does not affect the future events. In Model – II, the magnitude, \(X_n\), of the nth shock affects \(Y_n\), the time interval until \((n+1)\)th shock.

For these two models, the expected duration to the break down of the organization or the expected time to recruitment due to the staff depletion and its variance using correlated cumulative shock model approach of Ushio Sumita and George Shanthi Kumar [1985] are derived.
7.1. ASSUMPTION AND NOTATIONS OF THE MODEL

1. The organization takes decision at \( n \) random epochs, and at every decision making epoch a random number of person quit the organization.

2. There is an associated loss of man hour to the organization if a person quits.

3. If the maximum of the loss of man hours exceeds a pre specified threshold level, for the first time, the organization faces a break down and so recruitment is necessary.

4. The magnitude of the loss of man hour at the \( n \)th decision making epoch is correlated to the time between two consecutive decision making epochs, corresponding to the \( n \)th decision.

\( X_j \) : a continuous random variable denoting the man hours lost due to the random number of persons who leave the organization at the \( j \)th decision making epoch.

\( Y_j \) : a continuous random variable denoting the time between the two consecutive decision making epochs, corresponding to the \( j \)th decision

\( N(t) \) : number of decisions made in \((0, t]\)

\( z \) : pre specified threshold level which is positive

\( T_z \) : time to recruitment when, the threshold level is \( z \).

\( F_{XY}(x, y) \) : joint distribution of \( X \) and \( Y \).

\( f_{XY}(x, y) \) : joint density function of \( X \) and \( Y \).

\( F_X(x) \) : marginal distribution of \( X \).

\( f_X(x) \) : marginal density function of \( X \)

\( Z(t) \) : total loss of man hours in the first \( N(t) \) decision.

\( E(T_z) \) : Expected time to recruitment
\begin{align*}
V(T_z) & : \text{Variance of time to recruitment} \\
\eta_x & : \text{common mean with respect to } X \\
\eta_y & : \text{common mean with respect to } Y \\
\eta_{x^2} & : \text{second moment with respect to } X \\
\eta_{y^2} & : \text{second moment with respect to } Y \\
\sigma_x^2 & : \text{variance of } X \\
\sigma_y^2 & : \text{variance of } Y \\
M(z) & : \text{the counting process associated with the renewal process } \sum_{j=1}^{\infty} X_j \\
\phi_X(w) & = \int_0^\infty e^{-wz} f_X(x)dx \\
\eta_{XY} & : E(XY) \\
H_x(z) & : E(M(z)) \\
v(w) & : 2 \int_0^\infty e^{-wz} E(Y/X = z) dF_X(z) \\
u(w) & : 2 \int_0^\infty e^{-wz} E(Y^2/X = z) dF_X(z) \\
\text{We note that,} \\
Z(t) & = \sum_{j=0}^{N(t)} X_j \\
T_z & : \inf\{t : Z(t) > z\}
\end{align*}
7.2. RESULTS

Model – I

The organization starts at time $t = 0$. $X_n$ the man hours lost due to the random number of persons leaving the organization at the $n$th decision making epoch is correlated to the time interval $Y_n$, since the $[n-1]$th decision making epoch does not affect the future events. For this model, from Ushio Sumita and George Shanthi Kumar [1985], the following results are obtained.

\[
E(T_z) = \eta_Y \{1 + H_X(z)\} \tag{7.2.1}
\]

\[
\int_0^\infty e^{-wz}E(T_z^2)dz = \frac{\eta_Y^2}{w[1-\phi_X(w)]} + \eta_Y \frac{v(w)}{w[1-\phi_X(w)]^2} \tag{7.2.2}
\]

Model – II

It differs significantly from Model – I in that $X_n$, the man hours lost due to the random number of persons leaving the organization at the $n$th decision making epoch affects the future events. $Y_n$ the time interval until the $[n+1]$th decision making epoch. For this model, the following results are obtained from Ushio Sumita and George Shanthi Kumar [1985].

\[
E(T_z) = \{1 + H_X(z-x)\}E(Y/X = x)dF_X(x) \tag{7.2.3}
\]

\[
\int_0^\infty e^{-wz}E(T_z^2)dz = \frac{u(w)}{w[1-\phi_X(w)]} + \frac{1}{2w} \left[\frac{v(w)}{[1-\phi_X(w)]}\right]^2 \tag{7.2.4}
\]
7.3. SPECIAL CASE

In this section we assume that \( X \) and \( Y \) are correlated and follow a bivariate exponential distribution. More specifically, for \( n = 1, 2, \ldots \), the joint distribution function of \( X_n \) and \( Y_n \) is given by Johnson Samuel Kotz [1972].

\[
F_{XY}(x, y) = e^{-(x + y)} \{1 + \alpha(2e^{-x} - 1)(2e^{-y} - 1) \}
\]

(7.3.1)

where \( \alpha \) is the parameter of the bivariate exponential distribution with \( |\alpha| \leq 1 \) and \( x, y > 0 \).

Model – I

For the bivariate exponential distribution case, from (7.2.1) and (7.2.2) the following are derived.

\[
E(T^z) = 1 + z \quad (7.3.2)
\]

\[
E(T^2_z) = \frac{15}{8} + \left(\alpha + \frac{13}{4}\right)z + \left(\frac{3 + 2\alpha}{4}\right)z^2 + \frac{e^{-2z}}{8} \quad (7.3.3)
\]

\[
V(T^z) = \frac{7}{8} + \left(\frac{4\alpha + 5}{4}\right)z + \left(\frac{2\alpha - 1}{4}\right)z^2 + \frac{e^{-2z}}{8} \quad (7.3.4)
\]

Model – II

For the bivariate exponential distribution case, (7.2.3) and (7.2.4) are reduced to

\[
E(T^z) = z + \frac{\alpha}{4}(e^{-2z} - 1) \quad (7.3.5)
\]

\[
E(T^2_z) = \frac{\alpha(\alpha - 2)}{8} + (2 - \alpha)z + z^2 \left(\frac{\alpha(\alpha - 2)}{8}\right)e^{-2z} - \frac{\alpha^2}{4}ze^{-2z} \quad (7.3.6)
\]
\[ V(T_z) = \frac{\alpha^2 - 4\alpha}{16} + \left( \frac{4 - \alpha}{2} \right) z + \frac{\alpha}{4} e^{-2z} - \left( \frac{\alpha^2 + 2\alpha}{4} \right) z e^{-2z} - \frac{\alpha^2}{16} e^{-4z} \] (7.3.7)

### 7.4. NUMERICAL ILLUSTRATION

**Model - I**

**Case (1)**

The parameter \( \alpha \) is kept fixed and variations are with respect to \( z \). As the value of \( z \) increases the mean and the variance of the time to recruitment also increase. This is represented in Table - 13 and Graph - 9. It is quite reasonable that as the value of \( z \), (the threshold value) increases, the time to recruitment also increases and hence the result that the mean time to next recruitment also increases.

**Case (2)**

The parameter \( z \) is kept fixed and variations are with respect to \( \alpha \). Since the mean time to recruitment, \( E(T_z) \), is independent of \( \alpha \) in this model, it is constant. But when the value of \( \alpha \) increases, the variance of the time to recruitment also increases. This is represented in Table - 14 and Graph - 10.
Table – I3

$\alpha$ is kept fixed. Variations are with respect to $z$.

$\alpha = 1$ and $z = 0.1(0.1)1$

<table>
<thead>
<tr>
<th>Value of $z$</th>
<th>$E(T_2)$</th>
<th>$V(T_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.1</td>
<td>1.204841</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2</td>
<td>1.41879</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3</td>
<td>1.641101</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4</td>
<td>1.871166</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>2.108485</td>
</tr>
<tr>
<td>0.6</td>
<td>1.6</td>
<td>2.352649</td>
</tr>
<tr>
<td>0.7</td>
<td>1.7</td>
<td>2.603325</td>
</tr>
<tr>
<td>0.8</td>
<td>1.8</td>
<td>2.860237</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9</td>
<td>3.123162</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>3.391917</td>
</tr>
</tbody>
</table>
**Table - 14**

\(z\) is kept fixed. Variations are with respect to \(\alpha\).

\(z = 1\) and \(\alpha = -1(0.2)1\)

<table>
<thead>
<tr>
<th>Value of (z)</th>
<th>(E(T_z))</th>
<th>(V(T_z))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
<td>0.3919169</td>
</tr>
<tr>
<td>-0.8</td>
<td>2</td>
<td>0.6919169</td>
</tr>
<tr>
<td>-0.6</td>
<td>2</td>
<td>0.9919169</td>
</tr>
<tr>
<td>-0.4</td>
<td>2</td>
<td>1.291917</td>
</tr>
<tr>
<td>-0.2</td>
<td>2</td>
<td>1.591917</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>1.891917</td>
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<tr>
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<td>2</td>
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<tr>
<td>0.4</td>
<td>2</td>
<td>2.491917</td>
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<td>2</td>
<td>2.791917</td>
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<td>0.8</td>
<td>2</td>
<td>3.091917</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3.391917</td>
</tr>
</tbody>
</table>
Model – II

Case (1)

The parameter $\alpha$ is kept fixed and variations are with respect to $z$ as the value of $z$ increases the mean and the variance of the time to recruitment also increase. This is represented in Table - 15 and graph - 11. It is quite reasonable that as the value of $z$ increases, the time to recruitment also increases.

Case (2)

The parameter $z$ is kept fixed and the variations are with respect to $\alpha$. As the value of $\alpha$ increases the mean and the variance of the time to recruitment decreases. This is represented in Table – 16 and graph – 12.

**Table – 15**

$\alpha$ is kept fixed. Variations are with respect to $z$

$\alpha = 1$ and $z = 0.1(0.1)1$

<table>
<thead>
<tr>
<th>Value of $z$</th>
<th>$E(T_z)$</th>
<th>$V(T_z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.054683</td>
<td>0.0638289</td>
</tr>
<tr>
<td>0.2</td>
<td>0.11758</td>
<td>0.151449</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1872029</td>
<td>0.2573957</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2623322</td>
<td>0.377415</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3419698</td>
<td>0.5080566</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4252986</td>
<td>0.6465913</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5116493</td>
<td>0.7908852</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6004741</td>
<td>0.9392886</td>
</tr>
<tr>
<td>0.9</td>
<td>0.6913247</td>
<td>1.09054</td>
</tr>
<tr>
<td>1</td>
<td>0.7838338</td>
<td>1.243688</td>
</tr>
</tbody>
</table>
Table – 16

z is kept fixed. Variations are with respect to $\alpha$

$z = 1$ and $\alpha = -1(0.2)1$

<table>
<thead>
<tr>
<th>Value of $z$</th>
<th>$E(T_z)$</th>
<th>$V(T_z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1.216166</td>
<td>2.811355</td>
</tr>
<tr>
<td>-0.8</td>
<td>1.172933</td>
<td>2.644681</td>
</tr>
<tr>
<td>-0.6</td>
<td>1.1297</td>
<td>2.480208</td>
</tr>
<tr>
<td>-0.4</td>
<td>1.086466</td>
<td>2.317937</td>
</tr>
<tr>
<td>-0.2</td>
<td>1.043233</td>
<td>2.157868</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9567688</td>
<td>1.844334</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9135335</td>
<td>1.69087</td>
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<tr>
<td>0.6</td>
<td>0.8703003</td>
<td>1.539607</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8270671</td>
<td>1.390547</td>
</tr>
<tr>
<td>1</td>
<td>0.7838339</td>
<td>1.243688</td>
</tr>
</tbody>
</table>
Graph - 12

- $E(Tz)$
- $V(Tz)$

Values:
- 2.811355
- 2.644681
- 2.480208
- 2.317937
- 2.157868
- 2.000000
- 1.844334
- 1.690870
- 1.539607
- 1.390547
- 1.243688
- 1.172933
- 1.086466
- 1.043233
- 1.000000
- 0.956768
- 0.870303
- 0.783839
7.5. CONCLUSION

Based upon the hypothetical data the following conclusions are made

1. The mean and variance of the time to recruitment for Model – I and II increases when $\alpha$ increases, when $\alpha$ is fixed and $z$ is allowed to increase.

2. The mean and variance of the time to recruitment for Model – II decreases when $z$ is fixed and $\alpha$ is allowed to increase.

3. For Model – I, while the mean time to recruitment is constant, the variance of the time to recruitment increases, when $z$ is fixed and $\alpha$ is allowed to increase.

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