Chapter IV
CHAPTER – IV

FOUR CHARACTERISTIC MARKOV TYPE MANPOWER FLOW MODEL

The human resource flow in an organisation was studied and analysed as a simple fractional model with one characteristic by Bartholomew [1973], Grinold and Marshall [1977], Smith [1971] amongst others. Hayne and Marshall [1977] presented the extended two dimensional state space model. Chandra [1989] extended the same with three characteristics. The new fourth characteristic namely performance appraisal is more important and relevant like the other three characteristics. Whenever the recruitment is made in batches there is a possibility of many persons having all the three characteristics namely grade, length of service and educational level same. When promotions are made from one grade to another, apart from the three characteristics, the fourth characteristic like performance appraisal must be considered. The purpose of fourth characteristic is going to play a very vital role in framing the correct recruitment and promotional policies. No attempt has so far been made in this direction taking into account the importance of the aforesaid fourth characteristic.

The purpose of this chapter is to extend the models of Hayne and Marshall [1977] and Chandra [1989] to one with a four-dimensional state space in order to get more insight into the planning. More specifically, the fractional flow matrix and matrices of t-step transition probabilities are described for the four characteristic model, and equations of stocks and flows are derived with their transient properties.
4.1. FRACTIONAL FLOW MATRIX

Let the time factor for planning be considered as discrete periods in an organisation and the total number of people in an organisation is counted at the end of each period. At the time of counting, a person is assumed to possess four characteristics, \(i, j, k\) and \(l\) and is said to be in state \((i, j, k, l)\).

<table>
<thead>
<tr>
<th>Character</th>
<th>Meaning</th>
<th>Notation</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Grade</td>
<td>1C</td>
<td>(1 \leq i \leq N)</td>
</tr>
<tr>
<td>(j)</td>
<td>Length of Service</td>
<td>2C</td>
<td>(l_1(i) \leq j \leq u_1(i))</td>
</tr>
<tr>
<td>(k)</td>
<td>Educational Level</td>
<td>3C</td>
<td>(l_2(j) \leq k \leq u_2(j))</td>
</tr>
<tr>
<td>(l)</td>
<td>Performance Appraisal</td>
<td>4C</td>
<td>(l_3(k) \leq l \leq u_3(k))</td>
</tr>
</tbody>
</table>

Clearly, the range of 2C depends on 1C, the range of 3C depends on 2C and the range of 4C depends on 3C. In this paper the symbols \(i\) and \(a\) denote the first character 1C; \(j, b, e, h, n_2\) and \(n_5\) denote the second character 2C; \(k, m, c, f, y, n_1, n_3, n_4, n_6, \alpha_1\) and \(\alpha_2\) denote the third character 3C; and \(l, n, g\) and \(z\) denote the fourth character 4C.

Let \(v_{ijk}\) be the number of elements in

\[ K(k) = \{\{i, j, k\} : l_1(k) \leq i \leq u_1(k), l_2(i) \leq j \leq u_2(j), l_3(k) \leq l \leq u_3(k)\} \]

Let \(w_{ij}\) be the number of elements in

\[ J(j) = \{\{k, j\} : l_1(i) \leq j \leq u_1(i)\} \]

Let \(x_i\) be the number of elements in

\[ I(i) = \{\{j, i\} : l_1(i) \leq j \leq u_1(i)\} \]

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Let us now discuss the various possible transitions among the states.

<table>
<thead>
<tr>
<th>Number</th>
<th>Characters to be fixed</th>
<th>Characters to be varied</th>
<th>Matrix Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>i, j, k</td>
<td>L</td>
<td>Q</td>
</tr>
<tr>
<td>2</td>
<td>i, j</td>
<td>K, l</td>
<td>(iP)</td>
</tr>
<tr>
<td>3</td>
<td>i, k</td>
<td>j, l</td>
<td>(\mathbf{2P})</td>
</tr>
<tr>
<td>4</td>
<td>i, l</td>
<td>K, j</td>
<td>(\mathbf{3P})</td>
</tr>
<tr>
<td>5</td>
<td>I</td>
<td>j, k, l</td>
<td>(\mathbf{U})</td>
</tr>
<tr>
<td>6</td>
<td>--</td>
<td>i, j, k, l</td>
<td>(\mathbf{R})</td>
</tr>
</tbody>
</table>

Let \(q_{ijk}(l,n); \ l,n \in K(k)\) be the fraction of people in state \((i, j, k, l)\) in a time period, who move to \((i, j, k, n)\) in the next time period.

Let \(Q_{ijk}\) be the matrix \((q_{ijk}(l,n))\) which is of order \(v_{jk} \times v_{ik}\). Thus, \(Q_{ijk}\) describes the flow from one 4C to another 4C both having the same 1C, 2C and 3C.

Let \(p_{ijkl}(k,l; m,n), \ k,m \in J(j), \ l \in K(k), \ n \in K(m)\) be the fraction of people in state \((i, j, k, l)\) in a time period who move to \((i, j, m, n)\) in the next time period. For a given \(i, j, k\) and \(m\), consider the matrix \(p_{ijkl}\) of order \(v_{jk} \times v_{jm}\) which has \(p_{ijkl}(k,l;m,n)\) as the element in the \((1-l_3(k)+1)^{th}\) row and \((n-l_3(m)+1)^{th}\) column. Thus \(p_{ijkl}\) describes the flow from \((i, j, k, K(k))\) to \((i, j, m, K(m))\).
Let $z_{P_{ikj}}(j, l; b, d)$, $j, b \in I(i)$, $l, d \in K(k)$ be the fraction of people in state $(i, j, k, l)$ in a time period, who move to $(i, b, k, d)$ in the next time period. For a given $i, j, k$ and $a$, consider the matrix $2P_{ikjb}$ of order $v_{ikj} \times v_{ikb}$ which has $2P_{ikb}(j, l; b, d)$ as the element in the $(1 - l_3(k) + 1)^{th}$ row and $(d - l_3(k) + 1)^{th}$ column. Thus $2P_{ikjb}$ describes the flow from $(i, j, k, K(k))$ to $(i, b, k, K(k))$.

Let $3P_{ii(j, k; b, c)}$, $j, b \in I(i)$, $k \in I(j)$, $c \in I(b)$ be the fraction of people in state $(i, j, k, l)$ in a time period, who move to $(i, b, c, l)$ in the next time period. For a given $i, j, k$ and $a$ consider the matrix $3P_{iijb}$ of order $v_{iij} \times v_{iab}$ which has $3P_{iijb}(j, l; b, c)$ as the element in $(k - l_3(j) + 1)^{th}$ row and $(d - l_2(b) + 1)^{th}$ column. Thus $3P_{iijb}$ describes the flow from $(i, j, k, K(k))$ to $(i, b, j(b), l)$.

Let $U_{ij(j, k, l; e, f, g)}$, $j, e \in I(i)$, $k \in I(j)$, $l \in K(k)$, $f \in I(e)$ and $g \in K(f)$ be the fraction of people in state $(i, j, k, l)$ in a time period who move to $(i, e, f, g)$ in the next time period. For a given $i, j, k, e$ and $f$ consider the matrix $U_{ijef}$ of order $v_{ijk} \times v_{ief}$ having $U_{ijef}(j, k, l; e, f, g)$ as the element in $(1 - l_3(k) + 1)^{th}$ row and $(g - l_3(f) + 1)^{th}$ column. Thus $U_{ijef}$ describes the flow from $(i, j, k, K(k))$ to $(i, e, f, K(f))$.

Let the movement in one time period from state $i$ in $I_C$ can be only to states $i$ or $(i + 1)$ in $I_C$ or out of the system.

Let $r_{ij(j, k, l; h, y, z)}$, $j \in I(i)$, $h \in I(i + 1)$, $k \in I(j)$, $l \in K(k)$, $y \in I(h)$ and $z \in K(y)$ be the fraction of people in state $(i, j, k, l)$ in a time period, who move to $(i+1, h, y, z)$ in the next time period. For a given $i, j, k, h$ and $y$ the matrix $R_{ijkhy}$ of order $v_{ijk} \times v_{(i+1)hy}$ has $r_{ij(j, k, l; h, y, z)}$ as the element in $(1 - l_3(k) + 1)^{th}$ row and $(z - l_3(y) + 1)^{th}$ column. Thus $R_{ijkhy}$ describes the flow from $(i, j, k, K(k))$ to $(i+1, h, y, K(y))$. 

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4.2. MATRIX OF t-STEP TRANSITION PROBABILITIES

Consider the probability of moving from state (a, b, c, d) to (i, j, k, l) in t-steps, and only forward movement is allowed for any characteristic. The resultant matrix with probabilities are called t-step transition matrix.

Let \( \Pr \{t : (a, b, c, d); (i, j, k, l)\} \) be the probability of being in state (i, j, k, l) t-steps after being in state (a, b, c, d).

For a given i, j, k, a, b and c, let \( Z_{abcijkl}(t) \) be a \( V_{abc} \times V_{ijk} \) matrix having \( \Pr \{t : (a, b, c, d); (i, j, k, l)\} \) as an element in row \( (d - l_3(c) + 1) \) and column \( (1 - l_3(k) + 1) \).

Clearly, \( Z_{ijkijk}(0) = I \)

where I is the unit matrix of order \( V_{ijk} \times V_{ijk} \).

Further, \( Z_{abcijkl}(t) = 0 \)

if

(i) \( a > i \) or \( b > j \) or \( c > k \) since there is no backward motion.

(ii) \( a \leq i \) and \( t < a - i \) or \( b \leq j \) and \( t < b - j \) or \( c \leq k \) and \( t < c - k \).

since \( t < 0 \) in all the three cases. Here \( 0 \) is the null matrix of order \( V_{abc} \times V_{ijk} \).

If the process is to be in state (i, j, k, l) exactly t-steps after being in state (a, b, c, d) it must be either in (i, j, k, K(k)) or (i, j, n_1, K(n_1)), \( n_1 \in J(j) \) or (i, n_2, k, K(k)), \( n_2 \in I(i) \) or (i, n_2, n_3, l), \( n_2 \in I(i) \), \( n_3 \in J(n_2) \) or (i, n_2, n_4, K(n_4)), \( n_2 \in I(i) \), \( n_4 \in J(n_2) \) or (i-1, n_5, n_6, K(n_6)), \( n_5 \in I(i - 1) \) and \( n_6 \in J(n_5) \), exactly (t-1) steps after being in state (a, b, c, d). Conditioning on this fact leads to the recursive equation
For any \((a, b, c, d)\) and \((i, j, k, l)\) the sum over \('t'\) of the probability matrix \(Z_{abcijk}(t)\) gives the matrix of the expected number of visits to states \(i\) in 1C, \(j\) in 2C and \(k\) in 3C, starting from states \(a\) in 1C, \(b\) in 2C and \(c\) in 3C. Since the states are transient, it is easy to see that

\[
\lim_{t \to \infty} Z_{abcijk}(t) = 0
\]
Also, by an inductive argument equation (4.2.1) reduces to

\[
Z_{abcd}(t) = \sum_{r=0}^{t-1} \sum_{n_1 \in J(i)} \left[ Z_{abc}(t-(r+1)) P_{ijn,ik} Q_{ijk} \right] \\
+ \sum_{r=0}^{t-1} \sum_{n_2 \in J(n_2)} \left[ Z_{abc}(t-(r+1)) 2P_{ikn_2} Q_{ijk} \right] \\
+ \sum_{r=0}^{t-1} \sum_{n_3 \in J(n_3)} \left[ Z_{abc}(t-(r+1)) \sum_{n_2 \in J(i)} \left[ Z_{abc}(t-(r+1)) U_{in_4} P_{ijn_4} Q_{ijk} \right] \\
+ \sum_{r=0}^{t-1} \sum_{n_5 \in J(n_5)} \sum_{n_6 \in J(n_6)} \sum_{n_7 \in J(n_7)} \left[ Z_{abc}(t-(r+1)) R_{(i-1)n_5n_6} Q_{ijk} \right] \\
(4.2.3)
\]

The above equation states that the t-step transition matrices can be expressed by the flow matrices \( Q_{ijk}, Q_{ijl}, P_{ijn,k}, P_{ikn,j}, P_{ihn,j}, U_{in_4,n_4}, U_{in_5,n_5}, P_{ijn_4,k}, P_{ijn_5,k}, P_{ijn_6,k}, R_{(i-1)n_5n_6}, \) and \( R_{(i-1)n_5n_6, i} \).

The above equation will be used to represent the stock vectors as a sum of steady state and transient components.

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4.3. EQUATIONS OF STOCKS AND FLOWS

The key terms – stock and flow for the system can be defined as,

**Stock**: It refers to the number of people in a state at the end of one period.

**Flow**: It refers to the number of people who change their status in the system from one status in the system to another during any period.

In this section, we shall determine the stock and flows for a given state at a given time. We make the following notations

\[ s_{ijkl}(t) : \text{Expected stock in state } (i, j, k, l) \text{ at time } t. \]

\[ s_{ijk}(t) = \begin{bmatrix} s_{ijkl}(k)(t), \ldots, s_{ijk}(k)(t) \end{bmatrix} \]

\[ f_{ijkl}(t) : \text{Expected direct recruitment to } (i, j, k, l) \text{ at time } t, \text{ and} \]

\[ f_{ijk}(t) = \begin{bmatrix} f_{ijkl}(k)(t), \ldots, f_{ijk}(k)(t) \end{bmatrix} \]

Thus the basic stock equation, similar to that of the three characteristic model given in Chandra [1989] can be expressed as
\[ s_{ijk}(t) = s_{ijk}(t-1)Q_{ijk} + f_{ijk}(t-1) \]

\[ + \sum_{n_1 \in I(j)} s_{ijn_1}(t-1)P_{ijn_1k} \quad n_1 \neq k \]

\[ + \sum_{n_2 \in I(i)} s_{in_2k}(t-1)P_{inkn_2j} \quad n_2 \neq j \]

\[ + \sum_{n_3 \in I(n_2)} \sum_{n_2 \in I(i)} s_{in_2n_3}(t-1)P_{in_2n_3j} \quad n_3 \neq k \quad n_2 \neq j \]

\[ + \sum_{n_2 \in I(i)} \sum_{n_4 \in I(n_2)} \sum_{\alpha_4 \in I(j)} s_{in_2n_4}(t-1)P_{ija_4k} \quad n_2 \neq j \quad n_4 \neq k \quad \alpha_4 \neq k \]

\[ + \sum_{n_5 \in I(i-1)} \sum_{n_6 \in I(n_5)} \sum_{\alpha_2 \in I(j)} s_{(i-1)n_5n_6}(t-1)P_{ija_2k} \quad n_5 \neq j \quad n_5 \neq j \quad \alpha_2 \neq k \]  

By recursively applying the above equation, we get the cumulative stock equation as
\[
s_{ijk}(t) = s_{ijk}(0)Q_{ijk}^t + \sum_{r=0}^{t-1} f_{ijk}(t-r)Q_{ijk}^r
\]

\[
+ \sum_{r=0}^{t-1} \sum_{n_1 \in I(i)} \sum_{n_1 \neq k} s_{ijn_1}(t-(r+1))P_{ijn_1k}Q_{ijk}^r
\]

\[
+ \sum_{r=0}^{t-1} \sum_{n_2 \in I(j)} \sum_{n_2 \neq j} s_{in_2k}(t-(r+1))P_{ikn_2j}Q_{ijk}^r
\]

\[
+ \sum_{r=0}^{t-1} \sum_{n_3 \in I(n_2)} \sum_{n_3 \neq k} \sum_{n_2 \neq j} \left[ s_{in_2n_3}(t-(r+1))P_{iln_2j}Q_{ijl}^r \right]
\]

\[
+ \sum_{r=0}^{t-1} \sum_{n_4 \in I(n_2)} \sum_{n_4 \neq k} \sum_{n_2 \neq j} \sum_{n_1 \in I(i)} \left[ s_{in_2n_4}(t-(r+1))U_{in_2n_4i}a_1P_{ija_1k}^r \right]
\]

\[
+ \sum_{r=0}^{t-1} \sum_{n_5 \in I(i-1)} \sum_{n_5 \neq j} \sum_{n_6 \in I(n_3)} \sum_{n_6 \neq k} \sum_{n_2 \neq j} \sum_{n_4 \neq k} \left[ s_{i(i-1)n_5n_6}(t-(r+1))R_{i(i-1)n_5n_6a_2}P_{ika_2k}^r \right]
\]

\[(4.3.2)\]

This equation is of primary importance in this model, as recruiting policy and promotion policy depend on the short term on present stocks and in the long term as how we model future stocks and flows.
4.4. TRANSIENT PROPERTIES OF STOCKS

A vector function \( \tilde{s}_{ijk}(t) \) is a steady state component of \( s_{ijk}(t) \) if

\[
\lim_{t \to \infty} \left[ s_{ijk}(t) - \tilde{s}_{ijk}(t) \right] = 0,
\]

where 0 is the zero vector.

Then \( s_{ijk}(t) - \tilde{s}_{ijk}(t) \) is said to be a transient component. Thus, a transient system is one that is on its way to steady state.

The following result expresses the stock vector as a sum of a steady-state component and a transient component.

**THEOREM 4.4.1.**

Consider a vector function \( \tilde{s}_{ijk}(t) \) which satisfies the basic stock equation (4.2.1).

Then the actual stocks at time \( t \) are given by,

1. \( s_{ijk}(t) = \tilde{s}_{ijk}(t) + \sum_{a=1}^{i} \sum_{b=I(a)}^{j} \sum_{c=J(b)}^{k} \left[ \tilde{s}_{abc}(0) - \tilde{s}_{abc}(t) \right] z_{abcijk}(t) \) and

2. \( \tilde{s}_{ijk}(t) \) is a steady state component of \( s_{ijk}(t) \)
PROOF:

Since $\tilde{s}_{ijk}(t)$ satisfies the basic stock equation, it must also satisfy the cumulative stock equation

$$
\tilde{s}_{ijk}(t) = \tilde{s}_{ijk}(0) Q_{ijk}^t + \sum_{r=0}^{t-1} \tilde{s}_{ijk}(t-r) Q_{ijk}^r + \sum_{r=0}^{t-1} \sum_{n_1 \in J(j)} \tilde{s}_{ijn1}(t-(r+1)) P_{ijn1k} Q_{ijk}^r \wedge k
$$

$$
+ \sum_{r=0}^{t-1} \sum_{n_2 \in l(i)} \sum_{n_2 \neq j} \tilde{s}_{in2k}(t-(r+1)) 2 P_{in2j} Q_{ijk}^r
$$

$$
+ \sum_{r=0}^{t-1} \sum_{n_3 \in J(n_2)} \sum_{n_2 \in l(i)} \sum_{n_3 \neq k} \sum_{n_4 \neq j} \tilde{s}_{in2n3}(t-(r+1)) 3 P_{in2j} Q_{ijl}^r
$$

$$
+ \sum_{r=0}^{t-1} \sum_{n_2 \in l(i)} \sum_{n_4 \in J(n_2)} \sum_{n_4 \neq k} \sum_{a_1 \in J(i)} \sum_{a_1 \neq k} \tilde{s}_{in2n4}(t-(r+1)) U_{in2n4ja_1} 1 P_{ija_1k}^r
$$

$$
+ \sum_{r=0}^{t-1} \sum_{n_5 \in l(i-1)} \sum_{n_6 \in J(n_5)} \sum_{n_5 \neq j} \sum_{n_6 \neq k} \sum_{a_2 \in J(j)} \sum_{a_2 \neq k} \tilde{s}_{(i-1)n5n6}(t-(r+1)) R_{(i-1)n5n6ja_2} 1 P_{ika_2k}^r
$$

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Taking $i = 1$, $j = I(1) = 1$, $k = J(1) = 1$ in (4.4.2) and using (4.2.1)

\[
s_{111}(t) - \bar{s}_{111}(t) = \left( s_{111}(0) - \bar{s}_{111}(0) \right) t_{111} + \sum_{a = 1}^{I(1)} \sum_{b = 1}^{I(1)} \sum_{c = 1}^{I(1)} \left( s_{111}(0) - \bar{s}_{111}(0) \right) Z_{abcijk}(t)
\]

Thus, we have shown that part (i) of the theorem is true when $i = j = k = 1$.

Suppose that the part (i) is true for

\[
\left( i, j, k, K(k) \right), \left( i, n_1, k(n_1) \right), \left( i, n_2, k, K(k) \right), \left( n_2 \in I(1), \left( i, n_2, 1 \right) \right), \left( n_3 \in J(n_2) \right), \left( i, n_4, K(n_4) \right), \left( n_5 \in I(1) \right)
\]

and $i, j, k, n_2, n_3, n_4, n_5, K(n_4), K(n_5)$.
As the state \((i, j, k, K(k))\) can be reached only from the above states.

\[
s_{ij_1n_1}(t-(r+1)) - \tilde{s}_{ij_1n_1}(t-(r+1)) = \sum_{a=1}^{i} \sum_{b=I(a)}^{j} \sum_{c=J(b)}^{n_1} \left[ s_{abc}(0) - \tilde{s}_{abc}(0) \right] Z_{abcijn_1}(t-(r+1))
\]

\[
s_{in_2k}(t-(r+1)) - \tilde{s}_{in_2k}(t-(r+1)) = \sum_{a=1}^{i} \sum_{b=I(a)}^{n_2} \sum_{c=J(b)}^{k} \left[ s_{abc}(0) - \tilde{s}_{abc}(0) \right] Z_{abcin_2k}(t-(r+1))
\]

\[
s_{in_2n_3}(t-(r+1)) - \tilde{s}_{in_2n_3}(t-(r+1)) = \sum_{a=1}^{i} \sum_{b=I(a)}^{n_2} \sum_{c=J(b)}^{n_3} \left[ s_{abc}(0) - \tilde{s}_{abc}(0) \right] Z_{abcin_2n_3}(t-(r+1))
\]

\[
s_{in_2n_4}(t-(r+1)) - \tilde{s}_{in_2n_4}(t-(r+1)) = \sum_{a=1}^{i} \sum_{b=I(a)}^{n_2} \sum_{c=J(b)}^{n_4} \left[ s_{abc}(0) - \tilde{s}_{abc}(0) \right] Z_{abcin_2n_4}(t-(r+1))
\]

and

\[
s_{(i-1)n_5n_6}(t-(r+1)) - \tilde{s}_{(i-1)n_5n_6}(t-(r+1)) = \sum_{a=1}^{i-1} \sum_{b=I(a)}^{n_5} \sum_{c=J(b)}^{n_6} \left[ s_{abc}(0) - \tilde{s}_{abc}(0) \right] Z_{abc(i-1)n_5n_6}(t-(r+1))
\]
Substituting in (4.4.2), we get

\[ s_{ijk}(t) - \tilde{s}_{ijk}(t) = \left( s_{ijk}(0) - \tilde{s}_{ijk}(0) \right) q_{ijk}^r \]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_1} \left\{ i \quad j \quad n_1 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{j} \sum_{c=1}^{k} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_1 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_1} \left\{ i \quad j \quad n_1 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{j} \sum_{c=1}^{k} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_1 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
\begin{align*}
t^{-1} & \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\} \\
& \quad \quad + \sum_{r=0}^{n_2} \sum_{n_3 \in I(i)} \sum_{n_4 \in I(j)} \sum_{n_5 \in I(a)} \left\{ i \quad n_2 \quad n_3 \right\} \left\{ \sum_{a=1}^{i} \sum_{b=1}^{n_3} \sum_{c=1}^{n_3} \left[ \delta_{abc} - \tilde{\delta}_{abc} \right] Z_{abc(i-1)} (t-(r+1)) p_{ijn_2 k} q_{ijk}^r \right\}
\end{align*}
\]

\[
Z_{abc(i-1)} n_5 n_6 (t-(r+1)) = 0
\]
\[ s_{ijk}(t) - \tilde{s}_{ijk}(t) = \left[ s_{ijk}(0) - \tilde{s}_{ijk}(0) \right] Q_{ijk}^t + \]

\[
\sum_{n_1 \in J(j)} \left[ \sum_{b \in I(a)} \sum_{c \in J(b)} \left( s_{abc}(0) - \tilde{s}_{abc}(0) \right) Z_{abcijn_1} \left( t - (r + 1) \right) P_{ijn_1k} Q_{ijk}^r \right] +
\]

\[
\sum_{n_2 \in I(i)} \left[ \sum_{b \in I(a)} \sum_{c \in J(b)} \left( s_{abc}(0) - \tilde{s}_{abc}(0) \right) Z_{abcin_2k} \left( t - (r + 1) \right) P_{lkn_2j} Q_{ijk}^r \right] +
\]

\[
\sum_{n_3 \in J(n_2)} \sum_{n_2 \in I(i)} \left[ \sum_{b \in I(a)} \sum_{c \in J(b)} \left( s_{abc}(0) - \tilde{s}_{abc}(0) \right) Z_{abcin_2n_3} \left( t - (r + 1) \right) 3P_{ln_2j} Q_{ijl}^r \right] +
\]

\[
\sum_{n_2 \in I(i)} \sum_{n_4 \in J(n_2)} \sum_{\alpha_i \in J(i)} \left[ \sum_{b \in I(a)} \sum_{c \in J(b)} \left( s_{abc}(0) - \tilde{s}_{abc}(0) \right) Z_{abcin_2n_4} \left( t - (r + 1) \right) U_{ijn_2n_4j} P_{ij\alpha_1k}^r \right] +
\]

\[
\sum_{n_5 \in I(i-1)} \sum_{n_6 \in J(n_5)} \sum_{\alpha_2 \in J(i)} \left[ \sum_{b \in I(a)} \sum_{c \in J(b)} \left( s_{abc}(0) - \tilde{s}_{abc}(0) \right) Z_{abc(i-1)n_5n_6} \left( t - (r + 1) \right) \right] +
\]

\[
R(i-1)n_5n_6j_2 P_{ij\alpha_2k}^r \]
Using (4.2.3) we have,

\[
\bar{s}_{ijk}(t) - \bar{\bar{s}}_{ijk}(t) = \left(s_{ijk}(0) - \bar{\bar{s}}_{ijk}(0)\right)Q_{ijk}^t + \sum_{a=1}^{i} \sum_{b=I(a)} j \sum_{c=J(b)} k \left(s_{abc}(0) - \bar{\bar{s}}_{abc}(0)\right)Z_{abcijk}(t)
\]

But \( M_{ijk}(t) = Q_{ijk}^t \)

\[
\bar{s}_{ijk}(t) - \bar{\bar{s}}_{ijk}(t) = \sum_{a=1}^{i} \sum_{b=I(a)} j \sum_{c=J(b)} k \left(s_{abc}(0) - \bar{\bar{s}}_{abc}(0)\right)Z_{abcijk}(t)
\]

This proves part (i) of the theorem. We shall omit the proof for part (ii), as it is similar to the one given in Hayne and Marshall [1977] for the two characteristic model.

The difference between the actual stock and their steady state components in any state \((i, j, k, K(k))\) and at any period can be evaluated. In this case \(s_{ijk}(t)\) refers to what
stocks will be at time t, if we start with the present stocks $s_{ijk}(0)$ and implement the policies of the steady state model.

4.5. CONCLUSION

In any organisation, the stocks and flows are very closely related to the costs like transportation, training, etc. Hence it is highly essential that the planner is able to model stocks and flows. The model that presented here provides valuable information regarding stocks and flows for the four characteristic model.

Further, the notion of steady state plays a central role in the study of manpower flow systems. The outcome that result in analysing a steady state system is a useful approximation to the actual system. The outcome of the end result is very much useful when long-range planning is done using steady state model, as given in Marshall [1979].

This model can be effectively utilized by manpower planner to develop policies that are optimal in some sense, in maintaining the size of the organisation at a desired level, while dealing with four characteristic.