Chapter III
CHAPTER – III
THREE CHARACTERISTIC MARKOV TYPE MANPOWER FLOW MODEL
INCLUDING DEMOTION

The personnel movements in an organization have been broadly analyzed by Bartholomew [1973], Grinold and Marshall [1977], Smith [1971] and Chandra [1990(a)] amongst others by considering only the forward transitions [promotion]. Srinivasan and Mariappan [2001] have extended the results of Chandra [1990(a)] by considering promotion and demotion.

An extension of the Markov model to one with a three-dimensional state space was presented by Chandra [1989], considering promotion aspect only. Besides the two possibilities of filling the vacancies by promotion and recruitment, it is worthwhile to consider the third aspect namely demotion. Inclusion of the demotion concept may create a moral fear among the employees to work with loyalty to the organization. At the same time, the employer should use this provision only with caution, as an arbitrary use of this provision leads to a critical situation. Keeping this point in mind, in this chapter the results of Chandra [1989] are extended, by introducing the concept of demotion in addition to promotion, in order to get more insight into manpower planning. Based on this stock equation for a three characteristic flow is derived.
3.1. NOTATIONS AND FRACTIONAL FLOW MATRICES

Let us assume that, an organization considers time in discrete periods and the total number of people are counted at the end of each period. Each person in an organization is assumed to have three characteristics, i, j, k and is said to be in state (i, j, k). Here,

\[ i \rightarrow \text{Grade (First Character - FC)} ; \ 1 \leq i \leq N \]

\[ j \rightarrow \text{Length of Service (Second Character - SC)} ; \ a(i) \leq j \leq b(i) \]

\[ k \rightarrow \text{Educational Qualification (Third Character - TC)} ; \ A(j) \leq k \leq B(j) \]

Let \( v_{ij} \) be the number of elements in \( J(j) = \{k / A(j) \leq k \leq B(j), a(i) \leq j \leq b(i)\} \).

Let \( w_i \) be the number of elements in \( K(i) = \{j / a(i) \leq j \leq b(i)\} \).

Let \( q_{ij}(k,l) \), \( k, l \in J(j) \) be the fraction of people in state \((i, j, k)\) in a time period who move to \((i, j, l)\) in the next time period.

\( Q_{ij} \) be the \( v_{ij} \times v_{ij} \) matrix \( (q_{ij}(k,l)) \). Thus \( Q_{ij} \) describes flow from one TC to another, both having the same FC and SC.

Let \( p_i(j,k ; m,l) \), \( j, m \in K(i), k \in J(j), l \in J(m) \) be the fraction of people in state \((i, j, k)\) in a time period who move to \((i, m, l)\) in the next time period.

For a given \( i, j, m \) consider the \( v_{ij} \times v_{im} \) matrix \( P_{ijm} \), which has \( p_i(j,k ; m,l) \) as the element in the \((k - A(j) + 1)^{th}\) row and \((l - A(m) + 1)^{th}\) column. Thus \( P_{ijm} \) describes the flow from \((i, j, J(j))\) to \((i, m, J(m))\).
Let the movement at one time from state \( i \) can only be to \((i - 1)\) or \((i)\) or \((i + 1)\) or out of the system.

Let \( h_i (j, k ; n, l) \), \( j \in K(i) \), \( n \in K(i + 1) \), \( k \in J(j) \), \( l \in J(n) \) be the fraction of people in state \((i, j, k)\) in a time period who move to \((i + 1, n, l)\) in the next time period. For a given \( i, j \) and \( n \) consider the \( v_{ij} \times v_{(i+1)n} \) matrix \( H_{ijn} \), having \( h_i (j, k ; n, l) \) as the element in \((k - A(j) + 1)^{th}\) row and \((1 - A(n) + 1)^{th}\) column. This matrix describes the flow from \((i, j, J(j))\) to \((i + 1, n, J(n))\).

Let \( f_i (j, k ; n, l) \), \( j \in K(i) \), \( n \in K(i - 1) \), \( k \in J(j) \), \( l \in J(n) \) be the fraction of people in state \((i, j, k)\) in a time period who move to \((i - 1, n, l)\) in the next time period. For given \( i, j \) and \( n \) consider the \( v_{ij} \times v_{(i-1)n} \) matrix \( F_{ijn} \), having \( f_i (j, k ; n, l) \) as the element in \((k - A(j) + 1)^{th}\) row and \((1 - A(n) + 1)^{th}\) column. This matrix describes the flow from \((i, j, J(j))\) to \((i - 1, n, J(n))\).

### 3.2. MATRICES OF \(t\)-STEP TRANSITION PROBABILITIES

Consider the probability of being in state \((i, j, k)\) \(t\)-steps after being in state \((p, l, m)\). However, we shall assume that backward movement is allowed for the FC and not allowed for the SC and TC. The matrices of these probabilities are called the \(t\)-step transition matrices.

Let \( m \{t : (p, l, m) ; (i, j, k)\} \) represent the probability being in state \((i, j, k)\) \(t\)-steps after being in state \((p, l, m)\).
For a given \( i, j, p, l \), let \( M_{piij}(t) \) be a \( v_{pl} \times v_{lj} \) matrix having \( m \{ t : (p, l, m) ; (i, j, k) \} \) as the element in 
\((m - A(l) + 1)\)th row and \((k - A(j) + 1)\)th column.

Obviously,

\[
M_{iiij}(0) = I, \text{ where } I \text{ is the unit matrix of order } v_{ij} \times v_{ij}.
\]

Further,

\[
M_{piij}(t) = 0
\]

(a) \( p > (i + 1) \)

(b) \( p = (i + 1) \) and \( t < (i + 1) - p \)

(c) \( p = i - 1 \) or \( i + 1 \) and \( l > j \)

(d) \( p < i - 1. \)

Here \( 0 \) stands for a null matrix of order \( v_{pl} \times v_{lj} \).

If the process is to be in state \((i, j, k)\) exactly \( t \)-steps after being in state \((p, l, m)\), then it must be either in \((i, j, J(j))\) or \((i, u, J(u)), u \in K(i)\) or \((i - 1, x, J(x)), x \in K(i - 1)\) or \((i + 1, y, J(y)), y \in K(i + 1)\), exactly \((t - 1)\) steps after being in \((p, l, m)\).

Conditioning on this fact leads to the recursive equation,

\[
M_{piij}(t) = M_{piij}(t-1)Q_{ij} + \sum_{n \in K(i)} M_{pniu}(t-1)P_{iuj} + \sum_{x \in K(i-1)} M_{p(i-1)x}(t-1)H_{(i-1)xj}^{(i-1)}
\]

\[
\sum_{u \neq j} \sum_{y \in K(i+1)} M_{pl(i+1)y}(t-1)F_{(i+1)yj}
\]

(3.2.1)
For any \((p, 1, m)\) and \((i, j, k)\), the sum over \(t\) of the probability matrices \(M_{pij}(t)\) gives the matrix of the expected number of visits to states \(i\) in FC, \(j\) in SC, starting from states \(p\) in FC and \(1\) in SC. Since the states are transient, it is easy to see that,

\[
\lim_{t \to \infty} M_{pij}(t) = 0 \quad (3.2.2)
\]

Also by an inductive argument, equation (3.2.1) reduces itself to,

\[
M_{pij}(t) = \sum_{r=0}^{t-1} \left[ \sum_{u \in K(i)} M_{pij}(t-(r+1)) P_{iuj} Q_i^r + \sum_{x \in K(i-1)} \left[ M_{pl(i-1)x}(t-(r+1)) H_{xj} Q_j^r \right] \right] 
+ \sum_{y \in K(i+1)} \left[ M_{pl(i+1)y}(t-(r+1)) F_{ij} Q_j^r \right] 
\quad (3.2.3)
\]

Thus, (3.2.3) expresses the \(t\)-step transition matrices \(M_{pij}(t)\) in terms of the flow matrices \(Q_{ij}, P_{ij}, H_{ij},\) and \(F_{ij}\). Later, equation (3.2.3) will be used to represent the stock vectors as a sum of steady state and transient components.

### 3.3. EQUATIONS OF STOCKS AND FLOWS

In this section, we present stocks and flows, along with a timing convention.

- **Stock** refers to the number of people in a state at the end of a period.
- **Flow** refers to the number of people who change their status in the system from one state to another during any period.
We make the following notations:

Let $S_{ijk}(t)$ : expected stock in state $(i, j, k)$ at time $t$.

$$s_{ij}(t) = [s_{ijA0}(t), \ldots, s_{ijB0}(t)]$$

$r_{ijk}(t)$ : expected direct recruitment to $(i, j, k)$ at time $t$.

$$r_{ij}(t) = [r_{ijA0}(t), \ldots, r_{ijB0}(t)]$$

Then the basic stock equation, similar to that of the two-characteristic model including demotion in Srinivasan and Mariappan [2001] can be expressed as

$$s_{ij}(t) = s_{ij}(t-1) Q_{ij} + \sum_{u \in k(i), u \neq j} s_{iu}(t-1) P_{iu} j + \sum_{x \in k(i-1)} s^{(i-1)x}(t-1) H^{(i-1)x} j$$

$$+ \sum_{y \in k(i+1)} s^{(i+1)y}(t-1) F^{(i+1)y} j + r_{ij}(t)$$

(3.3.1)

The cumulative stock equation can be derived by recursively applying the above equation, we get the cumulative stock equation as

$$s_{ij}(t) = s_{ij}(0) Q_{ij}^t + \sum_{r=0}^{t-1} r_{ij}(t-r) Q_{ij}^r + \sum_{r=0}^{t-1} \sum_{u \in k(i), u \neq j} s_{iu}(t-(r+1)) P_{iu} j Q_{ij}^r$$

$$+ \sum_{r=0}^{t-1} \sum_{x \in k(i-1)} s^{(i-1)x}(t-(r+1)) H^{(i-1)x} j Q_{ij}^r$$

$$+ \sum_{r=0}^{t-1} \sum_{y \in k(i+1)} s^{(i+1)y}(t-(r+1)) F^{(i+1)y} j Q_{ij}^r$$

(3.3.2)
Equation (3.3.2) expresses the stock matrix in terms of flow matrices. Also, it is playing a very vital role in this model, as the recruiting, demotion and promotion policies depend in short term on present stocks and in the long term on, how one can model future stocks and flows.

3.4. TRANSIENT PROPERTIES OF THE STOCKS

The vector function $\mathbf{s}_t(t)$ can be called a steady state component of $s_{ij}(t)$ if

$$\lim_{t \to \infty} \{ s_{ij}(0) - \mathbf{\bar{s}}_{ij}(0) \} = 0$$

where $\mathbf{0}$ is a zero vector. (3.4.1)

Then $\left( s_{ij}(t) - \mathbf{\bar{s}}_{ij}(t) \right)$ is said to be a transient component. Thus a transient system is one that is on its way to steady state.

Now, we state a theorem which will express the stock vector as a sum of a steady-state component and a transient component is given by:

THEOREM 3.4.1.

Consider a vector function $\mathbf{\bar{s}}_{ij}(t)$ which satisfies the basic stock equation (3.3.1).

Then, (i) the actual stocks at time $t$ are given by

$$s_{ij}(t) = \mathbf{\bar{s}}_{ij}(t) + \sum_{p=1}^{\mathbf{a}(p)} \sum_{l=1}^{\mathbf{b}(p)} \left( s_{pl}(0) - \mathbf{\bar{s}}_{pl}(0) \right) \mathbf{M}_{plij}(t)$$

and (ii) $\mathbf{\bar{s}}_{ij}(t)$ is a steady state component of $s_{ij}(t)$. 

\[ Bv \leq \mathbf{\bar{N}}_{ij} \times L \]

\[ \mathbf{1} \geq \mathbf{0} \]
Omit the proof for part (i) and part (ii), as it is similar to the one given in Chandra [1989].

One can evaluate the difference between actual stocks and their steady state component in any state \((i, j, J(j))\) and at any period. Here \(s_{ij}(t)\) indicates the level of stocks at time \(t\), if we start with present stock \(s_{ij}(0)\) and implement the policies of the steady state model.

3.5. CONCLUSION

It is obvious that in an organization, costs like salary, transportation, training etc., are highly related to stocks and flows. Hence, the planner must be very careful to model stocks and flows. The present model provides useful information regarding stocks and flows in a three characteristic model including demotion. Along with that the concept of steady state plays a vital role in the study of manpower flow systems. The outcome is highly useful when policy decision has been taken using a steady state model as given in Marshall [1977].

The three characteristic model including demotion is a boon to the organizations in making policy decisions related to promotion and recruitment decisions in maintaining the size of the organization at a desired level.