DISCUSSION AND RECOMMENDATIONS

The study was undertaken to arrive at conclusions regarding the numerical strategies to solve a system of IVPs using the fourth order RK methods, based on a variety of means. The comparisons made between the methods are of course strictly valid only with respect to the particular set of problems that we have chosen. However, because of the varying nature of the problems, and the existence of consistent solutions, we believe our conclusions are a good indication of the merits of the methods for most practical purposes. We summarize our conclusions in the following paragraphs.

The initial intention of the thesis was to find out how the fourth order RK methods, based on a variety of means, suit for a system of IVPs? Though all the discussed RK methods, based on a variety of means, have yielded better results satisfying the error tolerance, it is observed from the results that the fourth order RK methods, based on Centroidal Mean, and Arithmetic Mean are more accurate and consistent to give discrete solutions. The study on the stability region presents the stability of the RK methods based on a variety of means with respect to the test equation, $y' = \lambda y$.

In any numerical method, error is an important and vital factor to be recovered. Numerical Technique and error are inseparable as flesh and blood. Hence for every numerical execution there is an error, when a method has more number of operations, then undeniably, error will be more. In Chapter - 3, during the comparison of a number of arithmetic operations involved in single step for every discussed RK method, it is found that the RKCeM has more arithmetic operations than the other methods. This is amazing because, having more number of arithmetic operations, RKCeM gives better results even with fixed step-size for non-stiff and
stiff IVPs. Also, for a few IVPs, it is found to be better than the much consistent century old RKAM method.

To solve a system of IVPs using embedded pairs of RK formulae, one has to take a long time to arrive at a discrete solution through manual calculation. But due to the modern availability of computer and software, one can obtain the results within a fraction of milliseconds (research works are growing more because of the invention of high speed processors). Hence in Chapter - 4, a new method RKACeM(4,4), with error control, has been presented for the determination of the discrete solution of IVPs. Hall and Watt [69] said “Among a wide variety of possibilities it has become clear from DETEST that the Fehlberg type method / error estimation is the most promising”. In this chapter, it is demonstrated that RKACeM(4,4) agrees very well with the famous RK Fehlberg and RK Merson methods. It is noted that RKACeM(4,4), RK(4,4), RKF(4,5) and RK Merson methods are most efficient in terms of accuracy and total execution time, though the derivative evaluations are numerous.

In Chapter - 5, we have extended the RK results for second order IVPs, which are also effectively and easily accessible via computer. It is shown that RKCeM procedure works very well to solve the second order IVPs and it occupies first position when compared with the other methods, which include the classical fourth order RK method (RKAM).

Time-varying singular system is another important and vital area in the field of systems theory. In Chapter – 6, a numerical study on time-varying singular system has been discussed via STWS, RKAM and RKCeM methods. It is found that RKAM, RKCeM and STWS suit very well with an error tolerance of $10^{-6}$. The inconvenience in the explicit RK methods is that one has to rewrite the singular system explicitly in the form of $y' = f(x, y)$ or $y'' = f(x, y, y')$ and this cannot be done for all the singular cases. For all general non-linear problems the
method STWS is not flexible to apply. It is also notable that Balachandran and Murugesan [12] confirmed that the STWS technique is not applicable to singular systems of index three.

From the one dimensional heat flow problem and RLC smoothing circuit discussed in Chapter 7, it is observed that the RKCeM suits well. In particular, with the help of the Rayleigh-Ritz and Galerkin procedures one can solve the PDE using STWS and RK Methods. In addition, the analysis of smoothing circuits reveals that when the output of a digital-to-analogue converter is a staircase function, one has to introduce the initial condition at every sub-interval for the given range to attain more accurate results.

The methods RKAM, RKCeM, RKHaM and RKCoM are found to be suitable to study the IVPs, satisfying the prescribed error tolerance. The automatic step-size selection can be introduced by the error estimates of the RK methods, using the local truncation error or by the method of interval halving procedure, which is an important factor for choosing the initial step-size ‘h’.

**Recommendations:**

The major findings of this thesis is the applicability of the fourth order RK method based on Centroidal Mean (RKCeM) for various realistic and scientific IVPs. In particular, RKCeM occupies first position to get best numerical solution for many of the discussed problems and placed second for the remaining problems. Hence one can use the concept of Centroidal Mean, wherever it is applicable.

In the light of the work done by Evans and Yaakub [148], which is a study on Runge-Kutta based on Contraharmonic mean method with extrapolation for the parallel solutions of ODEs, and from the conclusions of Chapters 2, 3, 4, we infer that further research can be carried out on RKCeM with extrapolation for the parallel solutions of ODEs.
Further, this study can be extended for a system of IVPs of delay differential equations and one can check the validity of the fourth order RK method based on Centroidal Mean. It is to be noted that Sanugi [119] has discussed the delay differential equations using RK method based on Geometric Mean.

One can investigate the stiff and non-stiff subsystems using RKCeM method by incorporating the concept of partitioned methods. Also, a study on IVPs of periodic [3, 131], IVPs with oscillating solutions [20, 134], IVPs with rapidly varying right-hand sides [29] and for rigid body Lie-Poisson equations [48], can be carried out using RKCeM method. Further, one can present an article on continuous explicit Runge-Kutta method based on CeM.

Based on the stability regions discussed in the Chapter – 2 for the RK methods and further to ascertain the applicability of the methods, one can aim to discuss the concept of stability of the methods based on the other major properties of stability like L-stable, D-stable etc.