CHAPTER 2

PARAMETRIC FUZZY TRANSPORTATION PROBLEM
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In this chapter a solution for solving the transportation problem having fuzzy parameters in the constraints. Utilization of Kuhn – Tucker conditions corresponding to parametric problem is discussed.

2.1. INTRODUCTION

The transportation problem is one of the sub-classes of linear programming in which the objective is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destination in such a way that the total transportation cost is kept to a minimum. The application of the general transportation problem (GTP) is not limited to transporting commodities between sources and destination. The generalized transportation problem was first posed by Ferguson and Dantzig [20] in a paper discussing the allocation of aircraft routes. A dual method for solving generalized transportation problem (GTP) was presented by Balas [3] and an operator theory of parametric programming for GTP was introduced by Balachander and Thompson [2].

A part of this chapter was published in Bulletin of Pure and Applied Sciences [37]
In this chapter, we formulate the GTP having the supply available warehouse as fuzzy parameters and we propose a solution algorithm to solve the problem of concern and numerical example is given to clarify the theory.

2.1.1. Definition

The \( \alpha \) level set of the fuzzy number \( \tilde{A} \) is defined as the ordinary set \( L_\alpha(\tilde{A}) \) for which the degree of their membership function exceeds the level \( \alpha \in [0,1] \)

\[
L_\alpha(\tilde{A}) = \{ a \in \mathbb{R}^m / \mu_{\tilde{A}}(a_i) \geq \alpha, \ i = 1, 2, \ldots m \}
\]

2.2. FUZZY GENERAL TRANSPORTATION PROBLEM (FGTP)

2.2.1. FGTP problem formulation

The problem of consideration is the following Fuzzy General Transportation Problem (FGTP)

Minimize \( z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \)

Subject to \( \sum_{i=1}^{m} x_{ij} \leq \tilde{a}_i \quad i = 1, 2, \ldots m \)
\[
\sum_{i=1}^{m} x_{ij} \geq b_j \quad j = 1, 2, \ldots n
\]
\[
x_{ij} \geq 0 \quad \forall \text{ pairs } (i, j)
\]

Where \( \tilde{a}_i, i = 1, 2, \ldots m \) represent fuzzy parameters involved in the constraints with their membership function as \( \mu_{\tilde{a}} \). For a certain degree \( \alpha \) together with the concept of \( \alpha \) level set of the fuzzy numbers \( \tilde{a}_i \), the problem (FGTP) can be understood as the following non-Fuzzy \( \alpha \)-General Transportation Problem (\( \alpha \)-FGTP)

Minimize 
\[
z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

Subject to 
\[
\sum_{i=1}^{m} x_{ij} \leq a_i \quad i = 1, 2 \ldots m
\]
\[
\sum_{i=1}^{m} x_{ij} \geq b_j \quad j = 1, 2, \ldots n
\]
\[
a_i \in \mathbb{L}_\alpha(\tilde{A}) \quad i = 1, 2, \ldots m
\]
\[
x_{ij} \geq 0 \quad \forall \text{ pairs } (i, j)
\]

Where \( \mathbb{L}_\alpha(\tilde{A}) \) are the \( \alpha \) level set of the fuzzy number \( \tilde{a}_i \).
2.2.2. Definition

Let \( x(\bar{A}) \) denotes the constraint set of problem and is supposed to be non empty. A point \( x^* \in x(\bar{A}) \) is said to be \( \alpha \) optimal solution to be \((\alpha\text{-FGTP})\) if there does not exist another \( x \in x(\bar{A}), a_i \in L_\alpha(\bar{A}) \), such that

\[
C_{ij} x_{ij} \leq C_{ij} x^*_{ij}, i = 1, 2, \ldots m, j = 1, 2, \ldots n
\]

with strict inequality holding for atleast one \( i, j \), where the corresponding values of parameters \( a_i \) are called \( \alpha \) level optimal parameters.

2.2.3. \( \alpha'\text{-FGTP problem formulation} \)

The problem can be rewritten in the following equivalent form

\((\alpha'\text{-FGTP})\)

\[
\text{Minimize } z = \sum_{j=1}^{n} \sum_{j=1}^{m} C_{ij} x_{ij}
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots m
\]

\[
\sum_{j=1}^{n} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n
\]

\[
r_i^0 \leq a_i \leq R_i^0 \quad i = 1, 2, \ldots, m
\]

\[
x_{ij} \geq 0 \quad \forall \text{ pairs } (i, j)
\]
It would be noted that the constraint \( a_i \in L_\alpha(\bar{A}) \) has been replaced by the constraint \( r_i^0 \leq a_i \leq R_i^0, \ i = 1, 2, \ldots m \), where \( r_i^0 \) and \( R_i^0 \) are lower and upper bounds on \( a_i \) and they are constants.

2.3. A PARAMETRIC STUDY ON PROBLEM

The parametric study of the above problem where \( r_i^0 \) and \( R_i^0 \), \( i = 1, 2, \ldots m \) are assumed to be parameters rather than constants renamed \( r_i \) and \( R_i \) can be understood as follows.

Let \( x(r, R) \) denotes the decision space of problem defined by

\[
\begin{align*}
\{ (x_{ij}, a_i) & \in \mathbb{R}^{m(n+1)} ; a_i - \sum_{j=1}^{n} x_{ij} \geq 0, \sum_{i=1}^{m} x_{ij} \geq 0, R_i - a_i \geq 0, a_i - r_i \geq 0, \\
x_{ij} \geq 0, & i = 1, 2, \ldots m, j = 1, 2, \ldots n \} 
\end{align*}
\]

2.4. STABILITY NOTIONS FOR THE PARAMETRIC PROBLEM

2.4.1. Stability set of the first kind of the parametric problem

Suppose that \( r^*, R^* \in \mathbb{V} \) with a corresponding \( \alpha \)-optimal solution \((x_{ij}^*, a_i^*)\) of parametric problem. The stability set of the first kind of parametric problem which is denoted by \( S(x_{ij}^*, a_i^*) \) is defined by \( S(x_{ij}^*, a_i^*) \).
\{(r, R) \in \mathbb{V} / (x_{ij}^*, a_i^*) i = 1, 2, \ldots m, j = 1, 2, \ldots n\} is \alpha- optimal solution of parametric problem

2.4.2. Utilization of Kuhn-Tucker conditions corresponding to parametric problem

For minimization fuzzy transportation problem, the Lagrange function of parametric problem can be written as follows

\begin{align*}
L &= \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} + \lambda_k \left( a_i - \sum_{j=1}^{n} x_{ij} \right) + \eta_k \left( \sum_{i=1}^{m} x_{ij} - b_j \right) + \rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \\
&\quad + \sum_{i=1}^{m} \beta (a_i - r_i) + \sum_{i=1}^{m} \gamma_i (R_i - a_i) \quad k = 1, 2, \ldots, nm
\end{align*}

then the Kuhn–Tucker necessary optimality conditions corresponding to the parametric problem at the solution \((x_{ij}^*, a_i^*), i = 1, 2, \ldots m, j = 1, 2, \ldots n\) will take the form.

\begin{align*}
\frac{\partial L}{\partial x_{ij}} &= \sum_{i=1}^{m} \sum_{j=1}^{n} \partial C_{ij} x_{ij} / \partial x_{ij} + \lambda_k \partial \left( a_i - \sum_{j=1}^{n} x_{ij} \right) / \partial x_{ij} + \eta_k \partial \left( \sum_{i=1}^{m} x_{ij} - b_j \right) / \partial x_{ij} \\
&\quad + \rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 0
\end{align*}
\[a_i - \sum_{j=1}^{n} x_{ij} \geq 0 \quad i = 1, 2, \ldots m\]

\[\sum_{i=1}^{m} x_{ij} - b_j \geq 0 \quad j = 1, 2, \ldots n\]

\[R_i - a_i \geq 0 \quad a_i - r_i \geq 0 \quad i = 1, 2, \ldots m\]

\[x_{ij} \geq 0 \quad \forall \text{ pairs } (i, j)\]

\[\lambda_i \left( a_i - \sum_{j=1}^{n} x_{ij} \right) = 0 \quad j = 1, 2, \ldots m\]

\[\eta_j \left( \sum_{i=1}^{m} x_{ij} - b_j \right) = 0 \quad i = 1, 2, \ldots n\]

\[\gamma_j (R_i - a_i) = 0 \quad i = 1, 2, \ldots m\]

\[\beta_i (a_i - r_i) = 0\]

\[\rho_k x_{ij} = 0 \quad \forall \text{ pairs } (i, j) \quad k = 1, 2, \ldots, mn\]

\[\gamma, \eta, \lambda, \beta, \rho \geq 0, \quad i = 1, 2, \ldots m, \quad j = 1, 2, \ldots n, \quad k = 1, 2, \ldots mn\]

For maximization fuzzy transportation problem, the Lagrange function of parametric problem can be rewritten as follows

\[L = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} - \lambda_k \left( a_i - \sum_{j=1}^{n} x_{ij} \right) - \eta_k \left( \sum_{i=1}^{m} x_{ij} - b_j \right) - \rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\]

\[- \sum_{i=1}^{m} \beta (a_i - r_i) - \sum_{i=1}^{m} \gamma_i (R_i - a_i) \quad k = 1, 2, \ldots nm.\]
Then, the Kuhn-Tucker necessary optimality conditions corresponding to the parametric problem at the solution \((x_{ij}^*, a_i^*)\), 
\(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\) will take the form.

\[
\frac{\partial L}{\partial x_{ij}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \partial c_{ij} x_{ij} / \partial x_{ij} - \lambda_k \partial \left( a_i - \sum_{j=1}^{n} x_{ij} \right) / \partial x_{ij} - \eta_k \partial \left( \sum_{i=1}^{m} x_{ij} - b_j \right) / \partial x_{ij} - \\
\rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 0 \\
a_i - \sum_{j=1}^{n} x_{ij} \leq 0 \quad , \quad i = 1, 2, \ldots, m \\
\sum_{i=1}^{m} x_{ij} - b_j \leq 0 \quad , \quad j = 1, 2, \ldots, n \\
R_i - a_i \leq 0 \quad , \quad a_i - r_i \leq 0 \quad , \quad i = 1, 2, \ldots, m \\
x_{ij} \geq 0 \quad , \quad \forall \text{ pairs (i, j)} \\
-\lambda_j \left( a_i - \sum_{j=1}^{n} x_{ij} \right) = 0 \quad , \quad j = 1, 2, \ldots, m \\
\eta_j \left( \sum_{i=1}^{m} x_{ij} - b_j \right) = 0 \quad , \quad i = 1, 2, \ldots, n \\
\gamma_j (R_i - a_i) = 0 \quad , \quad i = 1, 2, \ldots, m \\
\beta_i (a_i - r_i) = 0
\[ \rho_k x_{ij} = 0 \quad \forall \text{ pairs } (i, j) \quad k = 1, 2, \ldots, mn \]
\[ \gamma_i, \eta_k, \lambda_k, \beta_i, \rho_k \geq 0, i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, mn \]

Where all the relations of the above systems are evaluated at \((x_{ij}, a_i^*)\) and \(\gamma_i, \eta_k, \lambda_k, \beta_i, \rho_k\), are the Kuhn-Tucker necessary optimality conditions corresponding to parametric problem are utilized at \((x_{ij}, a_i)\).

Clearly this set can be considered a subset from the set \(S(x_{ij}^*, a_i^*)\), ie. \(T(x_{ij}^*, a_i^*) \subseteq S(x_{ij}^*, a_i^*)\).

2.5. SOLUTION ALGORITHM

We describe a solution algorithm for solving the Fuzzy General Transportation Problem (FGTP). In this algorithm, we use the Vogel’s Approximation method; Least cost method and North-west corner Rule define a penalty for each row and column.

Step 1: Determine the points \((a_1, a_2, a_2)\) for the fuzzy number in the formulation problem (FGTP)

Step 2: Convert the problem \((\alpha\text{-FGTP})\) in the form of the problem \((\alpha'\text{- FGTP})\)

Step 3: Formulate the problem \((\alpha'\text{- FGTP})\) in the parametric form
Step 4: Apply VAM, least cost, North-west corner Rule to set the feasible solution.

2.6. NUMERICAL EXAMPLES

2.6.1. Minimization problem

A company has three factories and four workhouses. The suppliers are transported from the factories to the workhouses, which are located at varying distances from the factories. The workhouse requirements are $b_1 = 6$, $b_2 = 7$, $b_3 = 7$, $b_4 = 10$. The factory capacities are given as fuzzy numbers as follows $\tilde{a}_1 = (6, 7, 8)$, $\tilde{a}_2 = (6, 9, 10)$, $\tilde{a}_3 = (8, 11, 13)$. The costs from the factory to the workhouses are shown below.

<table>
<thead>
<tr>
<th></th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$W_3$</th>
<th>$W_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>$F_2$</td>
<td>6</td>
<td>11</td>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>$F_3$</td>
<td>8</td>
<td>13</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

6  7  7  10

A triangular fuzzy number can be defined by a triplet $\tilde{a} = (a_1, a_2, a_3)$ with membership function defined as
\[
\mu_{\alpha}(x) = \begin{cases} 
0 & \text{if } x < a_1 \\
(x - a_1)/(a_2 - a_1) & \text{if } a_1 \leq x \leq a_2 \\
(x - a_3)/(a_2 - a_3) & \text{if } a_2 \leq x \leq a_3 \\
0 & \text{if } x > a_3 
\end{cases}
\]

Consider the \(\alpha\)-level set to be \(\alpha = 0.50\), then we get \(6.5 \leq a_1 \leq 8.0\), \(7.5 \leq a_2 \leq 9.5\), \(9.5 \leq a_3 \leq 13.0\).

The non-fuzzy problem can be written as follows:

Minimize \(z = 3x_{11} + 8x_{12} + 9x_{13} + 16x_{14} + 6x_{21} + 11x_{22} + 14x_{23} + 6x_{24} + 8x_{31} + 13x_{32} + 10x_{33} + 12x_{34}\)

Subject to

\(a_1 - x_{11} - x_{12} - x_{13} - x_{14} \geq 0\),

\(a_2 - x_{21} - x_{22} - x_{23} - x_{24} \geq 0\),

\(a_3 - x_{31} - x_{32} - x_{33} - x_{34} \geq 0\),

\(x_{11} + x_{21} + x_{31} \geq 6\),

\(x_{12} + x_{22} + x_{32} \geq 7\),

\(x_{13} + x_{23} + x_{33} \geq 7\),

\(x_{14} + x_{24} + x_{34} \geq 10\),

\(x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \geq 0\)

By Vogel’s approximation method
So we get the following results.

\[ x_{11} = 6, x_{12} = 2, x_{13} = 0, x_{14} = 0, x_{21} = 0, x_{22} = 0, x_{23} = 0, x_{24} = 9, x_{31} = 0, x_{32} = 5, x_{33} = 7, x_{34} = 1 \]

With the \( \alpha \)-optimal parameters \( a_1 = 8, a_2 = 9, a_3 = 13 \)

By North-west Corner Rule

\[ x_{11} = 6, x_{12} = 2, x_{13} = 0, x_{14} = 0, x_{21} = 0, x_{22} = 5, x_{23} = 4, x_{24} = 0, x_{31} = 0, x_{32} = 0, x_{33} = 3, x_{34} = 10 \]

By Least cost method

\[ x_{11} = 6, x_{12} = 2, x_{13} = 0, x_{14} = 0, x_{21} = 0, x_{22} = 0, x_{23} = 0, x_{24} = 9, x_{31} = 0, x_{32} = 5, x_{33} = 7, x_{34} = 1. \]

Where all the relations of the above system are evaluated at the solution

\[ (6, 2, 0, 0, 0, 0, 0, 9, 0, 5, 7, 1), (8, 9, 13) \]
\[ (6, 2, 0, 0, 0, 0, 5, 4, 0, 0, 0, 3, 10), (8, 9, 13) \]
\[ (6, 2, 0, 0, 0, 0, 0, 9, 0, 5, 7, 1), (8, 9, 13) \]

The parametric form of the above problem is as follows

Min \( z = 3x_{11} + 8x_{12} + 9x_{13} + 16x_{14} + 6x_{21} + 11x_{22} + 14x_{23} + 6x_{24} + 8x_{31} + 13x_{32} + 10x_{33} + 12x_{34} \)

Subject to

\[ a_1 - x_{11} - x_{12} - x_{13} - x_{14} \geq 0, \]
\[
\begin{align*}
& a_2 - x_{21} - x_{22} - x_{23} - x_{24} \geq 0, \\
& a_3 - x_{31} - x_{32} - x_{33} - x_{34} \geq 0 \\
& x_{11} + x_{21} + x_{31} \geq 6, \\
& x_{12} + x_{22} + x_{32} \geq 7, \\
& x_{13} + x_{23} + x_{33} \geq 7, \\
& x_{14} + x_{24} + x_{34} \geq 10 \\
& R_1 - a_1 \geq 0, R_2 - a_2 \geq 0, R_3 - a_3 \geq 0 \\
& r_1 - h_1 \geq 0, r_2 - h_2 \geq 0, r_3 - h_3 \geq 0. \\
& x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \geq 0.
\end{align*}
\]

The Kuhn Tucker necessary optimality conditions corresponding to the parametric problem at the solution \((x_{ij}, a_i) j = 1, 2, \ldots n, i = 1, 2, \ldots m\) will take the form.

\[
\frac{\partial L}{\partial x_{ij}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial C_{ij}}{\partial x_{ij}} x_{ij} + \lambda_k \frac{\partial}{\partial x_{ij}} \left( a_i - \sum_{j=1}^{n} x_{ij} \right) + \eta_k \frac{\partial}{\partial x_{ij}} \left( \sum_{i=1}^{m} x_{ij} - b_j \right) + \rho \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 0
\]

\[
L = 3x_{11} + 8x_{12} + 9x_{13} + 16x_{14} + 6x_{21} + 11x_{22} + 14x_{23} + 6x_{24} + 8x_{31} + 13x_{32} + 10x_{33} + 12x_{34} + \lambda_1 (a_1 - x_{11} - x_{12} - x_{13} - x_{14}) + \lambda_2 (a_1 - x_{11} - x_{12} - x_{13} - x_{14}) + \lambda_3 (a_1 - x_{11} - x_{12} - x_{13} - x_{14}) + \ldots
\]
\[
\lambda_4(a_1 - x_{11} - x_{12} - x_{13} - x_{14}) + \lambda_5(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) + \lambda_6(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) + \lambda_7(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) + \lambda_8(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) + \lambda_9(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) + \lambda_{10}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) + \lambda_{11}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) + \lambda_{12}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) + \eta_1(x_{11} + x_{21} + x_{31} - 6) + \eta_2(x_{12} + x_{22} + x_{32} - 7) + \eta_3(x_{13} + x_{23} + x_{33} - 7) + \eta_4(x_{14} + x_{24} + x_{34} - 10) + \\
\eta_5(x_{11} + x_{21} + x_{31} - 6) + \eta_6(x_{12} + x_{22} + x_{32} - 7) + \eta_7(x_{13} + x_{23} + x_{33} - 7) + \eta_8(x_{14} + x_{24} + x_{34} - 10) + \eta_9(x_{11} + x_{21} + x_{31} - 6) + \eta_{10}(x_{12} + x_{22} + x_{32} - 7) + \eta_{11}(x_{13} + x_{23} + x_{33} - 7) + \eta_{12}(x_{14} + x_{24} + x_{34} - 10) + \rho_1x_{11} + \rho_2x_{12} + \rho_3x_{13} + \rho_4x_{14} + \rho_5x_{21} + \rho_6x_{22} + \rho_7x_{23} + \rho_8x_{24} + \rho_9x_{31} + \rho_{10}x_{32} + \rho_{11}x_{33} + \rho_{12}x_{34}
\]

From this system it can be shown that

Vogel’s approximation and least cost method is

\[\rho_1 = \rho_2 = \rho_8 = \rho_{10} = \rho_{11} = \rho_{12} = 0\]

\[\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0.\]

\[\eta_3 = \eta_4 = \eta_5 = \eta_6 = \eta_7 = \eta_9 = 0\]

\[\eta_1 = -3, \eta_2 = -8, \eta_8 = -6, \eta_{10} = -13, \eta_{11} = -10, \eta_{12} = -12\]

\[\rho_3 = -9, \rho_4 = -16, \rho_5 = -6, \rho_6 = -11, \rho_7 = -14, \rho_9 = -8\]
North West Corner Rule is

\[ \rho_1 = \rho_2 = \rho_6 = \rho_7 = \rho_{11} = \rho_{12} = 0 \]
\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0 \]
\[ \eta_3 = \eta_4 = \eta_5 = \eta_8 = \eta_9 = \eta_{10} = 0 \]
\[ \eta_1 = -3, \eta_2 = -8, \eta_6 = -11, \eta_7 = -14, \eta_{11} = -10, \eta_{12} = -12 \]
\[ \rho_3 = -9, \rho_4 = -16, \rho_5 = -6, \rho_8 = -6, \rho_9 = -8, \rho_{10} = -13 \]
\[ \beta_1 = \gamma_1, \beta_2 = \gamma_2, \beta_3 = \gamma_3 \]

The stability set of the first kind of parametric problem is given by

\[ S(x_{ij}, a_i) = \{r_1, r_2, r_3, R_1, R_2, R_3 \in \mathbb{R}/r_1 = R_1 = 8, r_2 = R_2 = 9, r_3 = R_3 = 13\} \]

2.6.2. Maximization problem

A company is spending Rs.1000 on transportation of its units from three plants to four distribution centres. The centres’ requirements are \( b_1 = 5, b_2 = 8, b_3 = 7, b_4 = 11 \). The plant capacities are given as fuzzy numbers as follows \( \tilde{a}_1 = (5, 6, 8), \tilde{a}_2 = (7, 8, 11), \tilde{a}_3 = (10, 13, 15) \). The supply and demand of units with unit cost of transportation are given in the table below.

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A triangular fuzzy number can be defined by a triplet \( \tilde{a} = (a_1, a_2, a_3) \)

with membership function defined as

\[
\mu_{\tilde{a}}(x) =
\begin{cases}
0 & \text{if } x < a_1, \\
\frac{(x-a_1)}{(a_2-a_1)} & \text{if } a_1 \leq x \leq a_2, \\
\frac{(x-a_2)}{(a_3-a_2)} & \text{if } a_2 \leq x \leq a_3, \\
0 & \text{if } x > a_3,
\end{cases}
\]

Consider \( \alpha \)-level set to be 0.50, then we get 5.5 \( \leq a_1 \leq 8.5, 7.5 \leq a_2 \leq 11.5, 11.5 \leq a_3 \leq 16.5 \)

The non-fuzzy problem can be written as follows

Maximize \( z = 19 x_{11} + 30 x_{12} + 50 x_{13} + 12 x_{14} + 70 x_{21} + 50 x_{22} + 40 x_{23} + 60 x_{24} + 40 x_{31} + 10 x_{32} + 60 x_{33} + 20 x_{34} \)

Subject to

\( a_1 - x_{11} - x_{12} - x_{13} - x_{14} \geq 0, \)

\( a_2 - x_{21} - x_{22} - x_{23} - x_{24} \geq 0, \)
\[a_3 - x_{31} - x_{32} - x_{33} - x_{34} \geq 0,\]
\[x_{11} + x_{21} + x_{31} \geq 5,\]
\[x_{12} + x_{22} + x_{32} \geq 8,\]
\[x_{13} + x_{23} + x_{33} \geq 7,\]
\[x_{14} + x_{24} + x_{34} \geq 11,\]
\[5.5 \leq a_1 \leq 8.5, \quad 7.5 \leq a_2 \leq 11.5, \quad 11.5 \leq a_3 \leq 16.5\]
\[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \geq 0\]

By Vogel’s approximation method

So we get the following results

\[x_{11} = 5, \quad x_{12} = 0, \quad x_{13} = 0, \quad x_{14} = 2, \quad x_{21} = 0, \quad x_{22} = 0, \quad x_{23} = 7, \quad x_{24} = 3,\]
\[x_{31} = 0, \quad x_{32} = 8, \quad x_{33} = 0, \quad x_{34} = 6.\]

With the \(\alpha\)-optimal parameters \(a_1 = 7, \quad a_2 = 10, \quad a_3 = 14\)

By North-west Corner Rule

\[x_{11} = 5, \quad x_{12} = 2, \quad x_{13} = 0, \quad x_{14} = 0, \quad x_{21} = 0, \quad x_{22} = 6, \quad x_{23} = 4, \quad x_{24} = 0,\]
\[x_{31} = 0, \quad x_{32} = 0, \quad x_{33} = 3, \quad x_{34} = 11.\]

By Least Cost Method

\[x_{11} = 0, \quad x_{12} = 0, \quad x_{13} = 0, \quad x_{14} = 7, \quad x_{21} = 3, \quad x_{22} = 0, \quad x_{23} = 7, \quad x_{24} = 0,\]
\[x_{31} = 2, \quad x_{32} = 8, \quad x_{33} = 0, \quad x_{34} = 4.\]
Where all relations of the above system are evaluated at the solution

(5, 0, 0, 2, 0, 0, 7, 3, 0, 8, 0, 6), (7, 10, 14)

(5, 2, 0, 0, 0, 6, 4, 0, 0, 0, 3, 11), (7, 10, 14)

(0, 0, 0, 7, 3, 0, 7, 0, 2, 8, 0, 4), (7, 10, 14)

The Kuhn-Tucker necessary optimality conditions corresponding to the parametric problem at the solution \((x_{ij}, a_i), j = 1, 2, \ldots n, i = 1, 2, \ldots m\) will take the form

\[
\frac{\partial L}{\partial x_{ij}} = \sum_{i=1}^{n} \sum_{j=1}^{m} \partial C_{ij} \frac{x_{ij}}{\partial x_{ij}} - \lambda_k \frac{\partial}{\partial x_{ij}} \left( a_i - \sum_{j=1}^{n} x_{ij} \right) - \eta_k \frac{\partial}{\partial x_{ij}} \left( \sum_{i=1}^{m} x_{ij} - b_j \right) - \\
\rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 0
\]

\[
L = 19x_{11} + 30x_{12} + 50x_{13} + 12x_{14} + 70x_{21} + 50x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 10x_{32} + 60x_{33} + 20x_{24} - \lambda_1(a_1 - x_{11} - x_{12} - x_{13} - x_{14}) - \\
\lambda_2(a_1 - x_{11} - x_{12} - x_{13} - x_{14}) - \lambda_3(a_1 - x_{11} - x_{12} - x_{13} - x_{14}) - \lambda_4(a_1 - x_{11} - x_{12} - x_{13} - x_{14}) - \\
\lambda_5(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) - \lambda_6(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) - \lambda_7(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) - \\
\lambda_8(a_2 - x_{21} - x_{22} - x_{23} - x_{24}) - \lambda_9(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) - \lambda_{10}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) - \\
\lambda_{11}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) - \lambda_{12}(a_3 - x_{31} - x_{32} - x_{33} - x_{34}) - \\
\eta_1(x_{11} + x_{21} + x_{31} - 5) - \eta_2(x_{12} + x_{22} + x_{32} - 8) - \eta_3(x_{13} + x_{23} - 8)
\]
+ x_{33} - 7) - \eta_4(x_{14} + x_{24} + x_{34} - 11) - \eta_5(x_{11} + x_{21} + x_{31} - 5) - \\
\eta_6(x_{12} + x_{22} + x_{32} - 8) - \eta_7(x_{13} + x_{23} + x_{33} - 7) - \eta_8(x_{14} + x_{24} + x_{34} - 11) - \eta_9(x_{11} + x_{21} + x_{31} - 5) - \\
\eta_{10}(x_{12} + x_{22} + x_{32} - 8) - \eta_{11}(x_{13} + x_{23} + x_{33} - 7) - \eta_{12}(x_{14} + x_{24} + x_{34} - 11) - \rho_{1}x_{11} - \\
\rho_{2}x_{12} - \rho_{3}x_{13} - \rho_{4}x_{14} - \rho_{5}x_{21} - \rho_{6}x_{22} - \rho_{7}x_{23} - \rho_{8}x_{24} - \rho_{9}x_{31} - \\
\rho_{10}x_{32} - \rho_{11}x_{33} - \rho_{12}x_{34}.

From this system it can be shown that Vogel’s approximation method is

\[ \rho_1 = \rho_4 = \rho_7 = \rho_8 = \rho_{10} = \rho_{12} = 0 \]

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0 \]

\[ \eta_2 = \eta_3 = \eta_5 = \eta_6 = \eta_9 = \eta_{11} = 0 \]

\[ \eta_1 = 19, \eta_4 = 12, \eta_7 = 40, \eta_8 = 60, \eta_{10} = 10, \eta_{12} = 20 \]

\[ \rho_2 = 30, \rho_3 = 50, \rho_5 = 70, \rho_6 = 30, \rho_9 = 40, \rho_{11} = 60 \]

Northwest Corner Rule is

\[ \rho_1 = \rho_2 = \rho_6 = \rho_7 = \rho_{11} = \rho_{12} = 0 \]

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0 \]

\[ \eta_3 = \eta_4 = \eta_5 = \eta_8 = \eta_9 = \eta_{10} = 0 \]

\[ \eta_1 = 19, \eta_2 = 30, \eta_6 = 50, \eta_7 = 40, \eta_{11} = 60, \eta_{12} = 20 \]

\[ \rho_3 = 50, \rho_4 = 12, \rho_5 = 70, \rho_8 = 60, \rho_9 = 40, \rho_{10} = 10 \]
Least Cost Method is

\[ \rho_4 = \rho_5 = \rho_7 = \rho_9 = \rho_{10} = \rho_{12} = 0 \]

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0 \]

\[ \eta_1 = \eta_2 = \eta_3 = \eta_6 = \eta_8 = \eta_{11} = 0 \]

\[ \eta_4 = 12, \eta_5 = 70, \eta_7 = 40, \eta_9 = 40, \eta_{10} = 10, \eta_{12} = 20 \]

\[ \rho_1 = 19, \rho_2 = 30, \rho_3 = 50, \rho_6 = 50, \rho_8 = 60, \rho_{11} = 60 \]

\[ \beta_1 = \gamma_1, \beta_2 = \gamma_2, \beta_3 = \gamma_3 \]

The stability set of the first kind of parametric problem is given by

\[ S(x_{ij}, a_i) = \{ r_1, r_2, r_3, R_1, R_2, R_3 \in \mathbb{R} / r_1 = R_1 = 7, r_2 = R_2 = 10, r_3 = R_3 = 14 \} \]

2.6.3. Unbalanced transportation problem

Consider the following unbalanced fuzzy transportation problem (below table) involving four sources and three destinations. The cell entries represent the cost of transportation per cell.

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>O₁</td>
<td>6</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>O₂</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>O₃</td>
<td>2</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>O₄</td>
<td>4</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Fuzzy supplies and demands are
\[
\tilde{a}_1 = (9, 10, 11), \quad \tilde{a}_2 = (3, 4, 5), \quad \tilde{a}_3 = (2, 4, 6), \quad \tilde{a}_4 = (10, 12, 13)
\]
\[
\tilde{b}_1 = (11, 12, 14), \quad \tilde{b}_2 = (12, 14, 16), \quad \tilde{b}_3 = (12, 14, 15)
\]

Consider the \( \alpha \)-level set to \( \alpha = 0.75 \), then we get
\[
9.75 < a_1 < 10.25, \quad 3.75 < a_2 < 4.25, \quad 3.5 < a_3 < 4.5
\]
\[
11.5 < a_4 < 12.25,\quad 11.75 < b_1 < 12.5, \quad 13.5 < b_2 < 14.5, \quad 13.5 < b_3 < 14.25.
\]

The non-fuzzy problem can be written as follows
Minimize \( z = 6x_{11} + 7x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 6x_{23} + 2x_{31} + 7x_{32} + 8x_{33} + 4x_{41} + 6x_{42} + 6x_{43} \)
Subject to
\[
a_1 - x_{11} - x_{12} - x_{13} \geq 0, \quad a_2 - x_{21} - x_{22} - x_{23} \geq 0
\]
\[
a_3 - x_{31} - x_{32} - x_{33} \geq 0, \quad a_4 - x_{41} - x_{42} - x_{43} \geq 0
\]
\[
b_1 - x_{11} - x_{21} - x_{31} - x_{41} \geq 0, \quad b_2 - x_{12} - x_{22} - x_{32} - x_{42} \geq 0
\]
\[
b_3 - x_{13} - x_{23} - x_{33} - x_{43} \geq 0
\]
\[
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{41}, x_{42}, x_{43} \geq 0
\]

By Vogel's Approximation Method
We set the following results
\[ x_{11} = 0, x_{12} = 6, x_{13} = 4, x_{21} = 0, x_{22} = 4, x_{23} = 0, \\
\[ x_{31} = 4, x_{32} = 0, x_{33} = 0, x_{41} = 8, x_{42} = 4, x_{43} = 0, \\
\[ x_{51} = 0, x_{52} = 0, x_{53} = 10 \\
\]

with the \( \alpha \)-optimal parameters

\[ a_1 = 10, a_2 = 4, a_3 = 4, a_4 = 12, a_5 = 10, b_1 = 12, b_2 = 14, b_3 = 14 \]

The parametric form of the above problem is as follows

Minimize \[ z = 6x_{11} + 7x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 6x_{23} + 2x_{31} + 7x_{32} + 8x_{33} \\
+ 4x_{41} + 6x_{42} + 6x_{43} \]

Subject to

\[ a_1 - x_{11} - x_{12} - x_{13} \geq 0, a_2 - x_{21} - x_{22} \geq 0, \]
\[ a_3 - x_{31} - x_{32} - x_{33} \geq 0, a_4 - x_{41} - x_{42} - x_{43} \geq 0, \]
\[ a_5 - x_{51} - x_{52} - x_{53} \geq 0. \]
\[ b_1 - x_{11} - x_{21} - x_{31} - x_{41} - x_{51} \geq 0, b_2 - x_{12} - x_{22} - x_{32} - x_{42} - x_{52} \geq 0 \]
\[ b_3 - x_{13} - x_{23} - x_{33} - x_{43} - x_{53} \geq 0. \]

Then, the Kuhn-Tucker necessary optimality conditions corresponding to the parametric problem at the solution \( (x_{ij}^*, a_i^*), i = 1, 2, \ldots m, j = 1, 2, \ldots n \) will take the form
\[
\frac{\partial L}{\partial x_{ij}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \partial c_{ij} x_{ij} / \partial x_{ij} + \lambda_k \partial \left( a_i - \sum_{j=1}^{n} x_{ij} \right) / \partial x_{ij} + \eta_k \partial \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - b_j \right) / \partial x_{ij} \\
+ \rho_k \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 0
\]

\[
L = 6x_{11} + 7x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 6x_{23} + 2x_{31} + 7x_{32} + 8x_{33} + 4x_{41} + 6x_{42} + 6x_{43} + \lambda_1(a_1 - x_{11} - x_{12} - x_{13}) + \lambda_2(a_1 - x_{11} - x_{12} - x_{13}) + \lambda_3(a_1 - x_{11} - x_{12} - x_{13}) + \lambda_4(a_2 - x_{21} - x_{22} - x_{23}) + \lambda_5(a_2 - x_{21} - x_{22} - x_{23}) + \lambda_6(a_2 - x_{21} - x_{22} - x_{23}) + \lambda_7(a_3 - x_{31} - x_{32} - x_{33}) + \lambda_8(a_3 - x_{31} - x_{32} - x_{33}) + \lambda_9(a_3 - x_{31} - x_{32} - x_{33}) + \lambda_{10}(a_4 - x_{41} - x_{42} - x_{43}) + \lambda_{11}(a_4 - x_{41} - x_{42} - x_{43}) + \lambda_{12}(a_4 - x_{41} - x_{42} - x_{43}) + \lambda_{13}(a_5 - x_{51} - x_{52} - x_{53}) + \lambda_{14}(a_5 - x_{51} - x_{52} - x_{53}) + \lambda_{15}(a_5 - x_{51} - x_{52} - x_{53}) + \eta_1(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_2(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_3(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_4(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_5(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_6(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_7(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_8(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_9(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{10}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{11}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{12}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{13}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{14}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{15}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{16}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{17}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{18}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{19}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{20}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{21}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{22}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{23}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{24}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{25}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{26}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{27}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + \eta_{28}(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} - 12) + \eta_{29}(x_{12} + x_{22} + x_{32} + x_{42} + x_{52} - 14) + \eta_{30}(x_{13} + x_{23} + x_{33} + x_{43} + x_{53} - 14) + 53
\[ p_{1x_{11}} + p_{2x_{12}} + p_{3x_{13}} + p_{4x_{21}} + p_{5x_{22}} + p_{6x_{23}} + p_{7x_{31}} + p_{8x_{32}} + p_{9x_{33}} \\
+ p_{10x_{41}} + p_{11x_{42}} + p_{12x_{43}} + p_{13x_{51}} + p_{14x_{52}} + p_{15x_{53}} \]

From this system it can be shown that Vogel’s Approximation Method is

\[ p_2 = p_3 = p_5 = p_7 = p_{10} = p_{11} = p_{13} = p_{14} = p_{15} = 0 \]

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{15} = 0 \]

\[ \eta_1 = \eta_4 = \eta_6 = \eta_8 = \eta_9 = \eta_{12} = \eta_{13} = \eta_{14} = \eta_{15} = 0 \]

\[ \eta_2 = -7, \eta_3 = -3, \eta_5 = -4, \eta_7 = -6, \eta_{10} = -4, \eta_{11} = -6 \]

\[ \rho_1 = -6, \rho_4 = -6, \rho_6 = -6, \rho_8 = -7, \rho_9 = -8, \rho_{12} = -6 \]

\[ \beta_1 = \gamma_1, \beta_2 = \gamma_2, \beta_3 = \gamma_3, \beta_4 = \gamma_4, \beta_5 = \gamma_5, \beta_6 = \gamma_6, \beta_7 = \gamma_7, \beta_8 = \gamma_8 \]

The stability set of the first kind of parametric problem is given by

\[ S(x_{ij}, a_i) = \{ r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8 \in \mathbb{R} / r_1 = R_1 = 10, r_2 = R_2 = 4, r_3 = R_3 = 4, r_4 = R_4 = 12, r_5 = R_5 = 10, r_6 = R_6 = 12, r_7 = R_7 = 14, r_8 = R_8 = 14 \} \]