CHAPTER 6

FUZZY SOLID TRANSPORTATION PROBLEM
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This chapter presents a fuzzy solution to the fuzzy solid transportation problem in which supplies, demands and conveyance capacities are trapezoidal fuzzy numbers. A parametric approach is used to obtain a fuzzy solution.

6.1. INTRODUCTION

The solid transportation problem (STP) is a generalization of the well-known transportation problem in which properties of three items are taken into account in the constraint set instead of two (source and destination). The STP was stated by Shell [48] is 1955. He also suggested the situations where the STP would rise, and the four causes of STP are discussed according to the data given on the properties of item (three planar sums, two planar sums, one planar and one axial sum, and three axial sums and mode of transport (conveyance) as third property.

With the above considerations, the posing of the STP is as follows:

A homogenous, product is to be transported from each of m sources to n
destinations. The sources are production facilities, warehouses or supply points characterized by available capacities $a_i$ for $i = 1, 2, \ldots , m$. The destinations are consumption facilities, warehouses or demand points, characterized by required levels of demand $b_j$ for $j = 1, 2, \ldots , n$. Let $c_k (k = 1, 2, \ldots , l)$ be the amount of this product, which can be carried by different modes of transport or conveyances. A penalty or cost $c_{ijk} \geq 0$ is associated with transportation of a unit of the product from source $i$ to destination $j$ by means of the conveyance $k$. One must determine the amounts $x_{ijk}$ of the product to be transported from all sources $i$ to all destinations $j$ by means of each conveyance $k$ such that the total transportation cost is minimized. The STP can be formulated as a standard linear programming problem in the following.

Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} C_{ij} x_{ijk}$

Subject to

$$\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = a_i , \quad i = 1, 2, \ldots , m$$

$$\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = b_j , \quad j = 1, 2, \ldots , n \quad \text{................ (6.1)}$$
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = e_k, \quad k = 1, 2, \ldots, l
\]

\[x_{ijk} \geq 0 \quad \text{for all } i, j, k.\]

where \(a_i \geq 0, b_j \geq 0, e_k \geq 0, c_{ijk} \geq 0, i = 1, 2, \ldots m, j = 1, 2, \ldots n, k = 1, 2, \ldots l\) and \(\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{l} e_k\) (balanced condition). It is easy to see that formulation (6.1) generalizes classical TP (two item properties) if we consider only one conveyance (\(l = 1\)).

The existence of the feasible selection to problem (6.1) is guaranteed [48], and a non-degenerated basic feasible solution contains \(m + n + 1 - 2\) non-zero values of the variables. Haley [23] shows the necessary definitions to formulate the three axial sums problem (6.1), as a three planar sums problem. Haley [24] also described the solution procedure of the three planar sums, which basically consists of an extension of the modified distribution method [57] for the classical TP. Thus, the general methods of solving the ordinary TP can be adopted to solve the STP.
In many cases however the STP arises in an uncertain environment, and so that, different models of the problem have to be studied. In this paper we analyze fuzzy solid multi-objective transportation problem. Thus section two presents fuzzy solid transportation problem. In section three, we formulate multi-objective transportation problem. Section four, is devoted to solution approaches of the problems, and appropriate solution method is suggested to solve the problems finally.

6.2. FUZZY SOLID TRANSPORTATION PROBLEM

Other possible generalization of the STP is the fuzzy STP. This problem arises when decision maker has no crisp information about data problem. That is, if it has some lack of precision, and then the coefficients defining the problem can be modelled by means of fuzzy sets [18]. In this paper we consider the FSTP in which supplies, demands and conveyance capacities are trapezoidal fuzzy numbers, and a fuzzy solution to the problem is required. A parametric approach is used to obtain a fuzzy solution.
6.2.1. Definition

Let $F(R)$ be the set of all trapezoidal fuzzy number [18]. Let $\tilde{a} \in F(R)$ and denote as $\tilde{a} = (\underline{a^1}, \underline{a^2}, a^2, \overline{a^2})$ where its membership function $\mu_{\tilde{a}} : R \to [0,1]$ is defined as

$$
\mu_{\tilde{a}}(x) = \begin{cases} 
0 & \text{if } x \leq a^1 - a^1 \\
\frac{a^1 - a^1 + x}{\overline{a^1}} & \text{if } a^1 - a^1 < x < a^1 \\
1 & \text{if } a^1 \leq x \leq a^2 \\
\frac{a^2 - a^2 - x}{\overline{a^2}} & \text{if } a^2 < x < a^2 + a^2 \\
0 & \text{if } a^2 + a^2 \leq x
\end{cases}
$$

Let $g : F(R) \to R$ be a linear ranking function of fuzzy number [11]. We assume that $\forall \tilde{a} \in F(R)$, if $a^1 = a^2$, $\underline{a^1} = \overline{a^2} = 0$ then $g(\tilde{a}) = a^1 (= a^3)$. A fuzzy equality constraint (denoted as $\tilde{a}_i x \equiv \tilde{b}_i$) is defined as a fuzzy relation $\equiv$ with membership function $\mu_{\equiv} : F(R) \times F(R) \to [0,1]$ as follows

\[ \]
\[ H_{ai, bi} = \begin{cases} 
0 & \text{if } g(\bar{a}_i x) \leq g(\bar{b}_i - \bar{d}_i) \\
g(\bar{d}_i - \bar{b}_i + \bar{a}_i x) / g(\bar{d}_i) & \text{if } g(\bar{b}_i - \bar{d}_i) < g(\bar{a}_i x) < g(\bar{b}_i) \\
1 & \text{if } g(\bar{a}_i x) = g(\bar{b}_i) \\
g(\bar{d}_i + \bar{b}_i - \bar{a}_i x) / g(\bar{d}_i) & \text{if } g(\bar{b}_i) < g(\bar{a}_i x) < g(\bar{b}_i + \bar{d}_i) \\
0 & \text{if } g(\bar{b}_i + \bar{d}_i) \leq g(\bar{a}_i x) 
\end{cases} \]

where \( \bar{d}_i \in F(R) \) is the fuzzy violation allowed on the left and right in the fuzzy equality constraint.

It is easy to prove [28] that the fuzzy equality constraint \( \bar{a}_i x \equiv \bar{b}_i \) previously defined is a likeness relation, i.e., a reflexive, symmetrical and max \( ^\land \) fuzzy relation.

6.2.2. Formulation

We can formulate the FSTP model with fuzzy equality constraints and fuzzy supplies, demands and conveyance capacities denoted as

\[ \bar{a}_i = (a^1_i, a^1_i, a^2_i, a^2_i), i = 1, 2, \ldots m; \bar{b}_i = (b^1_i, b^1_i, b^2_i, b^2_i), j = 1, 2, \ldots n; \bar{c}_k = (c^1_k, c^1_k, c^2_k, c^2_k), k = 1, 2, \ldots l. \]
The formulation is the following

Minimize \( \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} C_{ij} x_{ijk} \)

Subject to \( \sum_{j=1}^{n} \sum_{k=1}^{l} \tilde{1} x_{ijk} \equiv \tilde{a}_k, \quad i = 1, 2, \ldots m \)

\( \sum_{i=1}^{m} \sum_{k=1}^{l} \tilde{1} x_{ijk} \equiv \tilde{b}_k, \quad j = 1, 2, \ldots n \) \hspace{1cm} (6.2)

\( \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{1} x_{ijk} \equiv \tilde{c}_k, \quad k = 1, 2, \ldots l \)

\( x_{ijk} \geq 0 \quad \text{for all } i, j, k. \)

where \( \tilde{1} = (1, 0, 1, 0) \), note that formulation (6.2) for the STP with classical equality constraints having \( a_i^1 = a_i^2, a_i^1 = \tilde{a}_i^2 = 0 \) for \( i = 1, 2, \ldots m, \)

\( b_j^1 = b_j^2, b_j^1 = \tilde{b}_j^2 = 0 \) for \( j = 1, 2, \ldots n, \)

\( e_k^1 = e_k^2, e_k^1 = \tilde{e}_k^2 = 0 \) for \( k = 1, 2, \ldots l \) with violations \( \tilde{d} = (0, 0, 0, 0) \) in all constraints.

6.3. SOLID MULTI-OBJECTIVE TRANSPORTATION PROBLEM

The classical transportation problem refers to a special class of linear programming problems. In a typical problem, product is to be transported from \( m \) sources to \( n \) destinations and their capacities are \( a_1, a_2, \ldots a_m \) and \( b_1, b_2, \ldots b_n \) respectively. In addition there is a penalty
$c_{ij}$ associated with transporting a unit of product from source $i$ to destination $j$. This penalty may be cost or delivery time or safety of delivery. A variable $x_{ij}$ represents the unknown quantity to be shipped from source $i$ to destination $j$. In general, the real life problems are modelled with multi-objective, which are measured in different scales and at the same time in conflict. Furthermore, it is frequently difficult for the decision maker to combine the objective functions in one overall utility function. The mathematical model of the solid multi-objective transportation problem (SMOTP) as follows

Minimize $F^R(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} C_{ijk}^R x_{ijk}$, $R = 1, 2, \ldots q$

Subject to

\[ \sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = a_i \quad , \quad i = 1, 2, \ldots m \]

\[ \sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = b_j \quad , \quad j = 1, 2, \ldots n \quad \ldots \ldots \ldots (6.3) \]

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = e_k \quad , \quad k = 1, 2, \ldots l \]

, $x_{ijk} \geq 0$ for all $i, j, k$. 

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6.3.1. Formulation

We can formulate the fuzzy solid multi-objective TP model with fuzzy equality constraints and fuzzy supplies, demands and conveyance capacities denoted as \(\bar{a}_i = (a^1_i, a^2_i, a^3_i)\), \(i = 1, 2, \ldots m\), \(\bar{b}_j = (b^1_j, b^2_j, b^3_j)\), \(j = 1, 2, \ldots n\), \(\bar{e}_k = (e^1_k, e^2_k, e^3_k)\), for \(k = 1, 2, \ldots l\).

The formulation is the following

Minimize \(F^R(x) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l C_{ijk} x_{ijk}\), \(R = 1, 2, \ldots q\)

Subject to

\[
\sum_{j=1}^n \sum_{k=1}^l \bar{1} x_{ijk} \equiv \bar{a}_i, \quad i = 1, 2, \ldots m
\]

\[
\sum_{i=1}^m \sum_{k=1}^l \bar{1} x_{ijk} \equiv \bar{b}_j, \quad j = 1, 2, \ldots n
\]

\[
\sum_{i=1}^m \sum_{j=1}^n \bar{1} x_{ijk} \equiv \bar{e}_k, \quad k = 1, 2, \ldots l
\]

\(x_{ijk} \geq 0\) for all \(i, j, k\).

where \(\bar{1} = (1, 0, 0, 0)\) note that formulation (6.4) generalizes formulation (6.1) for the STP with classical equality constraints having \(a^1_i = a^2_i, a^3_i = \bar{a}^2_i = 0\) for \(i = 1, 2, \ldots m\), \(b^1_j = b^2_j, b^3_j = \bar{b}^2_j = 0\) for \(j = 1, 2, \ldots n\).
... n, \( e_k^1 = e_k^2, \overline{e_k}^1 = \overline{e_k}^2 = 0 \) for \( k = 1, 2, \ldots, l \) with violations \( \overline{d} = (0, 0, 0, 0) \) in all constraints.

### 6.4. SOLUTION PROCEDURE

According to the decomposition theorem [43] and using the parametric approach to solve fuzzy mathematical programming problem [59], particularly its application to the fuzzy transportation problem [58, 16] the following parametric programming problem can be associated with the problem (6.4).

Minimize \[ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ijk} x_{ijk}, \quad R = 1, 2, \ldots, q \]

Subject to \[ \mu \equiv \left( \sum_{j=1}^{n} \sum_{k=1}^{l} \tilde{a}_{ij} x_{ijk} \right) \geq r, \quad i = 1, 2, \ldots, m \]
\[ \mu \equiv \left( \sum_{i=1}^{m} \sum_{k=1}^{l} \tilde{b}_{j} x_{ijk} \right) \geq r, \quad j = 1, 2, \ldots, n \]
\[ \mu \equiv \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{k} x_{ijk} \right) \geq r, \quad k = 1, 2, \ldots, l \]
\[ x_{ijk} \geq 0 \quad \text{for all } i, j, k \text{ and } r \in [0, \bar{r}] \]
In order to solve this problem for a chosen value of the parameter, let \( r = 0.50 \).

According to the definition of the membership functions \( \mu \equiv \) with violations \( \tilde{d}^a_i, i = 1, 2, \ldots, m, \tilde{d}^b_j, j = 1, 2, \ldots, n \) and \( \tilde{d}^c_k, k = 1, 2, \ldots, 1 \) in the fuzzy equality constraints of supply demand and conveyance respectively, the problem (6.5) becomes

Minimize \( F^R(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} C_{ijk}^R \ x_{ijk} \), \( R = 1, 2, \ldots q \)

Subject to

\[
\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} \in [a_i^1(r), a_i^2(r)] \quad , i = 1, 2, \ldots m
\]

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \in [b_j^1(r), b_j^2(r)] \quad , j = 1, 2, \ldots n \quad \ldots \ldots \ldots \ldots \ldots (6.6)
\]

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \in [c_k^1(r), c_k^2(r)] \quad , k = 1, 2, \ldots l
\]

\( x_{ijk} \geq 0 \) \quad for all \( i, j, k \)

\( r = 0.50 \)
6.4.1. A fuzzy programming approach for solving MOTP problem

This problem is characterized by the membership function

\[ \mu_D(C) = \min \{ \mu_G(x), \mu_C(x) \} \]

To define the membership function of MOTP problem. Let \( L_R \) and \( U_R \) be the lower and upper bounds of the objective function \( F^R(x) \). These values are determined as follows. Calculate the individual minimum of each objective function as a single objective transportation problem subject to the given set of constraints. Let \( x^1, x^2, \ldots, x^R \) be the respective optimal solutions for the \( R \) different transportation problems and evaluate each objective function at all these \( q \) optimal solutions. It is assumed here that at least two of these solutions are different for which the \( R^{th} \) objective function has different bounded values. For each objective function \( F^R(x) \), find the lower bound (minimum value) \( L_R \) and the upper bound (maximum value) \( U_R \) on the basis of definition. \( L_R \) and \( U_R \), Biswal [6] gives a membership function of a multi-objective geometric programming problem which can be implemented for the MOTP problem as follows
\[
\mu_k(F^R(x)) = \begin{cases} 
1 & \text{if } F^R(x) \leq L_R \\
\frac{U_R - F^R(x)}{U_R - L_R} & \text{if } L_R < F^R(x) < U_R \\
0 & \text{if } F^R(x) \geq U_R 
\end{cases} \quad \text{............ (6.7)}
\]

where \( L_R \neq U_R, \ k = 1, 2, \ldots q \), if \( L_R = U_R \) then \( \mu(F^R(x)) \) for any value of \( R \).

By introducing an auxiliary variable \( \beta \), the problem can be transformed into the following equivalent conventional linear programming (LP) problem.

Maximize \( \beta \)

Subject to

\[
\begin{align*}
\beta & \leq \mu(F^R(x)) \quad , \ R = 1, 2, \ldots q \\
\sum_{j=1}^n \sum_{k=1}^1 x_{ijk} &= a_i \quad , \ i = 1, 2, \ldots m \\
\sum_{i=1}^m \sum_{k=1}^1 x_{ijk} &= b_j \quad , \ j = 1, 2, \ldots n \\
\sum_{i=1}^m \sum_{j=1}^1 x_{ijk} &= e_k \quad , \ k = 1, 2, \ldots l \\
0 \leq \beta \leq 1, \ x_{ijk} \geq 0 & \quad \forall \ i, j, k
\end{align*}
\]
In problem (6.8), the first constraint can be reduced to the following form.

\[ \beta (U_R - L_R) \leq (U_R - F^R(x)) \]

\[ \beta (U_R - L_R) + F^R(x) \leq U_R \]

\[ \beta (U_R - L_R)/U_R + (1/U_R)F^R(x) \leq 1 \]

Then, the solution procedure of the interval solid MOTP problem is summarized in the following steps.

Step 1: Pick out the first objective function and solve it as a single objective transportation problem subject to formulation (6.4) continue this process R times for different objective functions. If all the solutions (ie., \( x^1 = x^2 = \ldots = x^R = \{x_{ijk}\} \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \), \( k = 1, 2, \ldots, l \)) are the same, then one of them is the optimal compromise solution and go to step 6 otherwise so to step 2.

Step 2: Evaluate the \( R^{th} \) objective function, at the optimal solutions \( (R = 1, 2, \ldots, q) \). For each objective function, determine its lower and upper bounds \( (L_R \& U_R) \) according to the set of optimal solutions.

Step 3: Define the membership function as mentioned in Eqn. (6.7).
Step 4: Construct the linear programming problem Eqn. (6.8).

Step 5: Solve the problem using linear programming technique to get the solution and evaluate R objective functions at this optimal compromise solution.

Step 6: Stop

6.5. NUMERICAL EXAMPLE

We consider now a simple fuzzy TP with two sources and three destinations fuzzy supplies, fuzzy demands, fuzzy violations in the constraints and costs are the following:

Supplies: \( \tilde{a}_1 = (10, 5, 10, 5), \tilde{a}_2 = (16, 5, 16, 5) \)

Demand: \( \tilde{b}_1 = (12, 5, 12, 5), \tilde{b}_2 = (9, 4, 9, 4), \tilde{b}_3 = (5, 3, 5, 3) \)

Violations: \( \tilde{d}_1^a = (5, 1, 5, 1), \tilde{d}_2^a = (5, 1, 5, 1) \)
\( \tilde{d}_1^b = (5, 1, 5, 1), \tilde{d}_2^b = (4, 1, 4, 1), \tilde{d}_3^b = (3, 1, 3, 1) \)

Costs:

\[
\begin{array}{ccc}
2 & 7 & 7 \\
9 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{ccc}
8 & 9 & 10 \\
2 & 5 & 1 \\
\end{array}
\]
Then we obtain the following interval – parametric TP associated to the fuzzy problem

\[
\begin{align*}
    a_1^1(r) &= 10 - 5(1 - r) = 5 + 5r \\
    a_1^2(r) &= 10 + 5(1 - r) = 15 - 5r \\
    a_2^1(r) &= 16 - 5(1 - r) = 11 + 5r \\
    a_2^2(r) &= 16 + 5(1 - r) = 21 - 5r \\
    b_1^1(r) &= 12 - 5(1 - r) = 7 + 5r \\
    b_1^2(r) &= 12 + 5(1 - r) = 17 - 5r \\
    b_2^1(r) &= 9 - 4(1 - r) = 9 - 4 + 4r = 5 + 4r \\
    b_2^2(r) &= 9 + 4(1 - r) = 13 - 4r \\
    b_3^1(r) &= 5 - 3(1 - r) = 2 + 3r \\
    b_3^2(r) &= 5 + 3(1 - r) = 8 - 3r
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
2 & 7 & 7 & [5 + 5r, 15 - 5r] \\
9 & 3 & 4 & [11 + 5r, 21 - 5r] \\
[7 + 5r, 17 - 5r] & [5 + 4r, 13 - 4r] & [2 + 3r, 8 - 3r] \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
8 & 9 & 10 & [5 + 5r, 15 - 5r] \\
2 & 5 & 1 & [11 + 5r, 21 - 5r] \\
[7 + 5r, 17 - 5r] & [5 + 4r, 13 - 4r] & [2 + 3r, 8 - 3r] \\
\hline
\end{array}
\]
For a chosen value of the plasmatic $r = 0.5$. In this case, since the problem is bidimensional, the above interval parametric TP.

We obtain the following TP

\[
\begin{array}{cccccccccccccccccccccccc}
2 & 7 & 7 & 2 & 7 & 7 & 8 & 8 & 9 & 10 & 8 & 9 & 10 & 8 \\
9 & 3 & 4 & 9 & 3 & 4 & 14 & 2 & 5 & 1 & 2 & 5 & 1 & 14 \\
2 & 7 & 7 & 2 & 7 & 7 & 5 & 8 & 9 & 10 & 8 & 9 & 10 & 5 \\
9 & 3 & 4 & 9 & 3 & 4 & 5 & 2 & 5 & 1 & 2 & 5 & 1 & 5 \\
10 & 8 & 4 & 5 & 3 & 2 & 10 & 8 & 4 & 5 & 3 & 2 & \end{array}
\]

As the first step, the solution of each single objective transportation problem is,

\[
x^1 = (8, 0, 0, 0, 0, 2, 8, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 2) \\
x^2 = (0, 8, 0, 0, 0, 0, 10, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 2)
\]

\[
F^1(x) = 2x_{11} + 7x_{12} + 7x_{13} + 2x_{14} + 7x_{15} + 7x_{16} + 9x_{21} + 3x_{22} + 4x_{23} + 9x_{24} + 3x_{25} + 4x_{26} + 2x_{31} + 7x_{32} + 7x_{33} + 2x_{34} + 7x_{35} + 7x_{36} + 9x_{41} + 3x_{42} + 4x_{43} + 9x_{44} + 3x_{45} + 4x_{46} \\
F^2(x) = 8x_{11} + 9x_{12} + 10x_{13} + 8x_{14} + 9x_{15} + 10x_{16} + 2x_{21} + 5x_{22} + x_{23} + 2x_{24} + 5x_{25} + x_{26} + 8x_{31} + 9x_{32} + 10x_{33} + 8x_{34} + 9x_{35} + 10x_{36} + 2x_{41} + 5x_{42} + x_{43} + 2x_{44} + 5x_{45} + x_{46}
\]
\[ F^1(x^1) = 101, \quad F^1(x^2) = 189, \quad F^2(x^2) = 153, \quad F^2(x^1) = 169 \]

ie., \( 101 \leq F^1 \leq 189 \) and \( 153 \leq F^2 \leq 169 \)

The membership function of both \( F^1(x) \) and \( F^2(x) \) are

\[
\mu_1(F^1(x)) = \frac{(189 - F^1(x))}{(189 - 101)} \\
\mu_2(F^2(x)) = \frac{(169 - F^2(x))}{(169 - 153)}
\]

Now, the problem is written as follows

**Maximize** \( \beta \)

Subject to

\[
\begin{align*}
x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} &= 8 \\
x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} &= 14 \\
x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} &= 5 \\
x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} &= 5 \\
x_{11} + x_{21} + x_{31} + x_{41} &= 10 \\
x_{12} + x_{22} + x_{32} + x_{42} &= 8 \\
x_{13} + x_{23} + x_{33} + x_{43} &= 4 \\
x_{14} + x_{24} + x_{34} + x_{44} &= 5 \\
x_{15} + x_{25} + x_{35} + x_{45} &= 3 \\
x_{16} + x_{26} + x_{36} + x_{46} &= 2
\end{align*}
\]
\[ 0.0106x_{11} + 0.370x_{12} + 0.0370x_{13} + 0.0106x_{14} + 0.0370x_{15} + 0.0370x_{16} + \\
0.0476x_{21} + 0.0159x_{22} + 0.0212x_{23} + 0.0476x_{24} + 0.0159x_{25} + 0.0212x_{26} + \\
0.0106x_{31} + 0.0370x_{32} + 0.0370x_{33} + 0.01606x_{34} + 0.0370x_{35} + 0.0370x_{36} + \\
0.0476x_{41} + 0.0159x_{42} + 0.0212x_{43} + 0.0476x_{44} + 0.0159x_{45} + 0.212x_{46} + \\
0.4656\beta \leq 1. \\
0.0473x_{11} + 0.0533x_{12} + 0.0592x_{13} + 0.0473x_{14} + 0.0533x_{15} + 0.0592x_{16} + \\
0.118x_{21} + 0.0296x_{22} + 0.0059x_{23} + 0.0118x_{24} + 0.0296x_{25} + 0.0059x_{26} + \\
0.0473x_{31} + 0.0533x_{32} + 0.0592x_{33} + 0.0473x_{34} + 0.0533x_{35} + 0.0592x_{36} + \\
0.0118x_{41} + 0.0296x_{42} + 0.059x_{43} + 0.0118x_{44} + 0.0296x_{45} + 0.059x_{46} + \\
0.0947\beta \leq 1. \\
\]

6.5.1. Computer output

The above example is solved by using TORA Computer software package and we get the following optimal compromise solutions are
\[ x_{11} = 0.98, x_{14} = 4.02, x_{15} = 3.00, x_{22} = 7.02, x_{23} = 4, x_{24} = 0.98, x_{26} = 2, \\
x_{31} = 4.02, x_{32} = 0.98, x_{41} = 5.00 \] the overall satisfaction \( \beta = 0.50 \) and the objective function value are \( F^1(x^*) = 144.78 \) and \( F^2(x^*) = 161.04. \)
6.5.2. Optimum solution summary

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Objective Coefficient</th>
<th>Objective Value Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>0.9826</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>x2</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>x3</td>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>x4</td>
<td>4.0174</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>x5</td>
<td>3.0000</td>
<td>0.0000</td>
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</tr>
<tr>
<td>x6</td>
<td>0.0000</td>
<td>0.0000</td>
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<tr>
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<td>4.0000</td>
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<tr>
<td>x10</td>
<td>0.9826</td>
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<tr>
<td>x11</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>x12</td>
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<tr>
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</tr>
<tr>
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<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
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<tr>
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<td>0.0000</td>
</tr>
<tr>
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<tr>
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<tr>
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</table>
## OPTIMUM SOLUTION SUMMARY

Objective value = 0.5015

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<thead>
<tr>
<th>Constraint</th>
<th>RHS</th>
<th>Slack(-)/Surplus(+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (=)</td>
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<tr>
<td>5 (=)</td>
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</tr>
<tr>
<td>6 (=)</td>
<td>8.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>7 (=)</td>
<td>4.0000</td>
<td>0.0000</td>
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<tr>
<td>8 (=)</td>
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</tr>
<tr>
<td>9 (=)</td>
<td>3.0000</td>
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<td>10 (=)</td>
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</tr>
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<tr>
<td>12 (&lt;)</td>
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SENSITIVITY ANALYSIS

Objective value = 0.5015
Objective Coefficients -- Single Changes ==> (Degenerate/alternative optimal; ranges not unique)

<table>
<thead>
<tr>
<th>Variable Coefficients</th>
<th>Current Coefficients</th>
<th>Min Coefficients</th>
<th>Max Coefficients</th>
<th>Reduced Cost</th>
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SENSITIVITY ANALYSIS

Objective value = 0.5015
Righthand Side -- Single Changes ==> (Degenerate/alternative optimal; ranges not unique)

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<tr>
<th>Constraint</th>
<th>Current RHS</th>
<th>Min RHS</th>
<th>Max RHS</th>
<th>Dual Price</th>
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<td>-0.2018</td>
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<tr>
<td>4 (=)</td>
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<td>3.0000</td>
<td>5.0000</td>
<td>-0.0539</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>7 (=)</td>
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</tr>
<tr>
<td>8 (=)</td>
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<td>-0.0595</td>
</tr>
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<td>9 (=)</td>
<td>3.0000</td>
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<td>-0.1195</td>
</tr>
<tr>
<td>10 (=)</td>
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</tbody>
</table>
SENSITIVITY ANALYSIS

Objective value = 0.5015

Objective Coefficients -- Simultaneous Changes d:

<table>
<thead>
<tr>
<th>Nonbasic Variable</th>
<th>Optimality Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>x2</td>
<td>$0.3573 + 3.8678 , d_1 + 3.8678 , d_{14} + 0.3573 , d_{25} + 2.8678 , d_{10} + -2.8678 , d_{4} + -3.8678 , d_{13} + -2.8678 , d_{8} - d_2 \geq 0$</td>
</tr>
<tr>
<td>x3</td>
<td>$0.1507 + -0.2990 , d_1 + 1.0000 , d_{9} + -0.2990 , d_{14} + 0.1507 , d_{25} + -1.2990 , d_{10} + 1.2990 , d_{4} + 0.2990 , d_{13} + 0.2990 , d_{8} - d_3 \geq 0$</td>
</tr>
<tr>
<td>x6</td>
<td>$0.1507 + -0.2990 , d_1 + -0.2990 , d_{14} + 0.1507 , d_{25} + -1.2990 , d_{10} + 1.2990 , d_{4} + 0.2990 , d_{13} + 1.0000 , d_{12} + 0.2990 , d_{8} - d_6 \geq 0$</td>
</tr>
<tr>
<td>x7</td>
<td>$0.5611 + -3.4967 , d_1 + -4.4967 , d_{14} + 0.5611 , d_{25} + -3.4967 , d_{10} + 3.4967 , d_{4} + 4.4967 , d_{13} + 4.4967 , d_{8} - d_7 \geq 0$</td>
</tr>
<tr>
<td>x11</td>
<td>$0.0000 + -1.0000 , d_{1} + -1.0000 , d_{14} + 1.0000 , d_{13} + 1.0000 , d_{5} + 1.0000 , d_{8} - d_{11} \geq 0$</td>
</tr>
<tr>
<td>x15</td>
<td>$0.1507 + -1.2990 , d_1 + 1.0000 , d_{9} + -0.2990 , d_{14} + 0.1507 , d_{25} + -1.2990 , d_{10} + 1.2990 , d_{4} + 1.2990 , d_{13} + 0.2990 , d_{8} - d_{15} \geq 0$</td>
</tr>
<tr>
<td>x16</td>
<td>$0.0059 + -0.9530 , d_1 + 0.0470 , d_{14} + 0.0059 , d_{25} + 0.0470 , d_{10} + 0.9530 , d_{4} + 0.9530 , d_{13} + -0.0470 , d_{8} - d_{16} \geq 0$</td>
</tr>
<tr>
<td>x17</td>
<td>$0.0000 + -1.0000 , d_{1} + 1.0000 , d_{13} + 1.0000 , d_{5} - d_{17} \geq 0$</td>
</tr>
<tr>
<td>x18</td>
<td>$0.1507 + -1.2990 , d_1 + -0.2990 , d_{14} + 0.1507 , d_{25} + -1.2990 , d_{10} + 1.2990 , d_{4} + 1.2990 , d_{13} + 1.0000 , d_{12} + 0.2990 , d_{8} - d_{18} \geq 0$</td>
</tr>
<tr>
<td>x20</td>
<td>$0.0000 + -1.0000 , d_{1} + -1.0000 , d_{10} + 1.0000 , d_{4} + 1.0000 , d_{19} + 1.0000 , d_{8} - d_{20} \geq 0$</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
x_{21} & \quad 0.2806 - 3.2484 d_1 + 1.0000 d_9 + -2.2484 d_{14} + 0.2806 d_{25} + \\
      & \quad -3.2484 d_{10} + 3.2484 d_4 + 1.0000 d_{19} + 2.2484 d_{13} + \\
      & \quad 2.2484d_8 - d_{21} \geq 0 \\
x_{22} & \quad 0.0000 - 1.0000 d_1 + 1.0000 d_4 + 1.0000 d_{19} - d_{22} \geq 0 \\
x_{23} & \quad 0.0000 - 2.0000 d_1 - 1.0000 d_{14} + -1.0000 d_{10} + 1.0000 d_4 \\
      & \quad + 1.0000 d_{19} + 1.0000 d_{13} + 1.0000 d_5 + 1.0000 d_8 - d_{23} \geq 0 \\
x_{24} & \quad 0.4853 - 1.6052 d_1 - 0.6052 d_{14} + 0.4853 d_{25} + -1.6052 d_{10} \\
      & \quad + 1.6052 d_4 + 1.0000 d_{19} + 0.6052 d_{13} + 1.0000 d_5 + \\
      & \quad 0.6052 d_8 - d_{24} \geq 0 \\
s_{36} & \quad 1.0731 + 8.6121 d_1 + 8.6121 d_{14} + 1.0731 d_{25} + 8.6121 d_{10} + \\
      & \quad -8.6121 d_4 + -8.6121 d_{13} + -8.6121 d_8 \geq 0 \\
s_{37} & \quad 5.2837 + -42.3420 d_1 + -42.3421 d_{14} + 5.2837 d_{25} + \\
      & \quad -42.3421 d_{10} + 42.3421 d_4 + 42.3421 d_{13} + 42.3421 d_8 \geq 0 \\
\end{align*} \]
### SENSITIVITY ANALYSIS

**Objective value =** 0.5015

**Right-hand Side Ranging -- Simultaneous Changes D:**

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Value/Feasibility Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>(0.9826 + 1.8890 D1 + 0.0672 D2 + 0.8890 D3 + -0.9328 D4 + 1.0225 D5 + 1.0492 D6 + 0.0225 D8 + 0.0491 D9 + 8.6121 D11 + -42.3420 D12 \geq 0)</td>
</tr>
<tr>
<td>x9</td>
<td>(4.0000 + 1.0000 D7 \geq 0)</td>
</tr>
<tr>
<td>x14</td>
<td>(0.9826 + 1.8890 D1 + 0.0672 D2 + 1.8890 D3 + 0.0672 D4 + 0.0225 D5 + 1.0492 D6 + 0.0225 D8 + 0.0491 D9 + 8.6121 D11 + -42.3421 D12 \geq 0)</td>
</tr>
<tr>
<td>x25</td>
<td>(0.5015 + -0.2018 D1 + -0.0539 D2 + -0.2018 D3 + -0.0539 D4 + -0.0595 D5 + -0.1195 D6 + -0.0595 D8 + -0.1195 D9 + 1.0731 D11 + 5.2837 D12 \geq 0)</td>
</tr>
<tr>
<td>x10</td>
<td>(0.9826 + 0.8890 D1 + 0.0672 D2 + 0.8890 D3 + -0.9328 D4 + 1.0225 D5 + 1.0492 D6 + 1.0225 D8 + 1.0491 D9 + 8.6121 D11 + -42.3421 D12 \geq 0)</td>
</tr>
<tr>
<td>x4</td>
<td>(4.0174 + -0.8890 D1 + -0.0672 D2 + -0.8890 D3 + 0.9328 D4 + -1.0225 D5 + -1.0492 D6 + -0.0225 D8 + -1.0491 D9 + -8.6121 D11 + 42.3421 D12 \geq 0)</td>
</tr>
<tr>
<td>x19</td>
<td>(5.0000 + 1.0000 D4 \geq 0)</td>
</tr>
<tr>
<td>x13</td>
<td>(4.0174 + -1.8890 D1 + -0.0672 D2 + -0.8890 D3 + -0.0672 D4 + -0.0225 D5 + -1.0492 D6 + -0.0225 D8 + -0.0491 D9 + -8.6121 D11 + 42.3421 D12 \geq 0)</td>
</tr>
</tbody>
</table>
x12  \[ 2.0000 + 1.0000 D1 + 1.0000 D2 + 1.0000 D3 + 1.0000 D4 + \\
    -1.0000 D5 + -1.0000 D6 + -1.0000 D7 + -1.0000 D8 + \\
    -1.0000 D9 \geq 0 \]

Rx35  \[ 0.0000 + -1.0000 D1 + -1.0000 D2 + -1.0000 D3 + -1.0000 D4 \\
    + 1.0000 D5 + 1.0000 D6 + 1.0000 D7 + 1.0000 D8 + \\
    1.0000 D9 + 1.0000 D10 \geq 0 \]

x5  \[ 3.0000 + 1.0000 D9 \geq 0 \]

x8  \[ 7.0174 + -1.8890 D1 + -0.0672 D2 + -1.8890 D3 + -0.0672 D4 \\
    + -0.0225 D5 + -0.0492 D6 + -0.0225 D8 + -0.0491 D9 + \\
    -8.6121 D11 + 42.3421 D12 \geq 0 \]