Chapter II

Historical recollection and brief review on latest optimal and n-ary block designs
CHAPTER II

HISTORICAL RECOLLECTION AND BRIEF REVIEW ON LATEST OPTIMAL AND n-ary BLOCK DESIGNS.

2.1 INTRODUCTION

In the previous chapter, the definitions of various optimality, their formulations and n-ary blocks have been discussed. In the present chapter, contributions of various experts on various aspects of Optimality Theory - Particularly on A-, D- and E-Optimality theory have been presented.

2.2 HISTORICAL RECOLLECTION:

Smith (1918) has initially introduced specific optimality criteria in comparing designs in a given experimental set up. Followed by Smith, many experts have contributed a huge amount of literature in the field of optimality design. Developments occurred to a greater extent due to Kiefer (1958, 1959.) and Kiefer and Wolfowitz (1960) laid the foundations for a vigorous and systematic theory of optimum experimental designs.

In the meantime there was two parallel developments. One of the developments is associated partially with the name of G.E.P. Box who was concerned more with developing methods for tackling applied problems than with general mathematical theory. This work was reported in the papers of Box and Wilson (1951), Box and Lucas (1959), Box and Hunter (1965). The other developments took place in the USSR and are associated with the name of Fedorov. The earlier part of the work reported in the English literature in a book by Fedorov (1972).

During the 1980's the optimum theory has been explained and developed by many authors like Jacroux (1987). A detailed review was given in recent developments in the methods of optimum and related experimental designs by Atkinson (1982 a,b, 1988). After this review, many results have been published by Gupta and Singh (1989), Atkinson and doney (1989), Fedorov (1989), Shah and Sinha (1989) and Dey and Das (1989). Apart from the above-mentioned authors, many more experts have analyzed and revealed the matters about the optimum design of experiments.

The works of Ghosh, Patel, Dagri and Hemasingha (1997) on Construction of balanced ternary designs and proper balanced (m+1) - ary designs have been reviewed and research contributions in 1997 by Das (1998), Montepiedra and Yeh (1998), Rao and Oguryemi (1998), Srivastava and Morgan (1998) have been taken into consideration.

2.3 A BRIEF REVIEW ON LATEST IMPORTANT OPTIMAL DESIGNS:

2.3.1 Algorithms and exact designs

General Equivalence Theorem is used to provide algorithms for the construction of designs. Computer algorithms for optimal designs take two basic forms for continuous theory. We can improve a measure \( \xi \) by augmentation

\[
\xi_{n+1} = (1+\alpha_n) \xi_n + \alpha_n \eta_n
\]

where \( \xi_n \) is chosen with regard to the optimality criterion to put mass one at a single point. In the simplest algorithm for the construction of \( D \) - Optimum designs,

\[
\alpha_n = (n+1)^{-1} \quad \text{and} \quad \xi \text{ puts unit mass at the point where } d(\xi_n) \text{ is attained. Wu and Wynn (1978) prove convergence of algorithms for regular -optimal criteria.}
\]

Mitchell (1974) had developed the special DETMAX algorithms involving 'excursions' in which new points were added ad old points were thrown out. Galil & Kiefer (1980) suggest improvements to Mitchell's algorithm designed as save either computational space or time. A comparison of this algorithm was given by Cook and Nachtsheim (1980)
Because the General Equivalence Theorem does not hold for exact designs, exact G- and D-optimum designs can be appreciably different. Welch (1984) describes the extension of algorithms of the DETMX type to the calculation of V- and G-optimum designs. These criteria require calculations of the average or the maximum of the variance over a set of \( r \) points. If \( r \) is large, and it will often correspond to the points of a fine must is factor space, a straight forward adaptation of DETMAX is possible, but is computationally inefficient when compared to the D-Optimum algorithm, which require only on calculation of variance for each candidate point. For G-optimality, Welch (1984) has given bounds on the change in variance, which leads to an algorithm of improved efficiency. However, greater improvement comes from using the criterion of D-optimality for all but the last stage of an excursion. In addition to computational efficiency, this modification avoids the tendency of the G-optimum algorithm to become trapped at a poor local optimum.

The algorithms of this section find optimum designs through search over the grid of candidate points. If the factors are continuous, improved designs will be found by memorial optimization over the design region. The disadvantage is the appreciate increase in computational effort required, due in part to the presence of local optima Globally optimum designs can be found by the use of simulated annealing. This method makes it possible to escape from local optima by accepting steps away from the optima with a small probability, which decreases as a variable analogous to temperature decreases. Haines (1987) has analyzed D-, G- and linear-optimum designs for polynomial regression. Bohachevsky, Johnson and Stein (1986) have demonstrated the advantages of generalized annealing in which the optimum value of the criterion is known at least approximately. In their application of the method to a non-linear example, the optimization is performed at only one temperature.

Algorithm for the construction of a wide class of block designs including Balanced Incomplete Block (BIB) design has been described by Jergaw (1989). The algorithm, which allows the experimenter to give weights for a set f treatment contrasts, uses an initial starting design to generate an optimal block design sequentially with examples.

A new exchange algorithm for the construction of M.S-Optimal incomplete block design (IBD) has been developed by Nguyen and Dey (1990). This exchange algorithm is used to construct 973 M.S-Optima of IBD \((v,k,b)\) for \( v=4,5,\ldots,12 \)
(treatments) with arbitrary \( v, k \) (block size) and \( b \) (number of blocks). The efficiencies of the "best" MS optimal IBDs constructed by this algorithm have been compared with the efficiencies of the corresponding nearly balanced incomplete block (NBIBDs) designs of Cheng (1978), Cheng and Wu (1981) and of Mitchell and John (1976).

Pukelsheim and Torsney (1991) have derived an explicit formula for computing the A-Optimal design weights on linearly independent regression vectors, for the mean parameters in a linear model with homoscedastic variances. ABT (1992) has investigated linear covariance functions in one dimension, and show how exact optimal designs can be formed for several design criteria. Jansen, et. al., (1992) is concerned with a computer algorithm for searching optimal block designs.

For the past two decades, there has been increasing use of computers for the construction of experimental designs, which are 'good' in some well-defined sense. Nguyen (1992) have reviewed and compared some well-known exchange algorithms for the construction of discrete D-Optimal designs. An improve implementation of Fedorov's exchange algorithms has been suggested. Bhaumik's (1993) has discussed an algorithm for the construction of an A-optimal BIBD in the presence of a linear trend is provided.

Nguyen's (1994) has described an effective algorithm for constructing optimal or near-optimal incomplete block designs with up to 100 treatments. Eccleston and Street (1994) have analyzed an algorithm for the construction of optimal or near optimal change-over designs for arbitrary numbers of treatments, periods and units also he explained the performance of the algorithm with examples.

### 2.3.2 Designs with qualitative and quantitative factors

So far only qualitative factors have been discussed. In contrast, for the block designs discussed by Paterson (1988) all factors are qualitative. This section is concerned with designs when both kinds of factors are present, a subject which has been comparatively neglected. Pioneering works of Harville (1975), Kurotschka (1981), Lopes Troya (1982) and Wierich (1986) have given several examples. The qualitative factor could be 3 types of catalyst to be tested at different temperatures represented by the single quantitative factor.
2.3.3 Biased-Coin Designs

Our discussions deals with fixed sample size in previous sections. The physical problems considered in this section is the design of clinical trials in which patients arrive sequentially and are assigned to one of treatments. Usually the outcome of the treatment is not known before the next allocation is made. If the outcomes are known, an adaptive rule can be used, in which an attempt is made to balance the need for information against the desire to give the best treatment.

A biased coin design can be used (Efron, 1971) in which the under represented treatment is favoured in a probabilistic allocation. Pocock and Simon (1975) and Efron (1980) have described extensions to the comparison of any number of treatments with balance for prognostic factors. Wei (1977, 1978) has used urn designs to generate the probabilistic allocations Begg and Iglewicz (1980) have employed optimum design theory to yield a deterministic criterion to which they have derived an operationally useful approximation. Smith (1984 b) has discussed unified description of several biased-coin allocation rules.

The methods of Wei and Atkinson (1982) have provided algorithms for the construction of biased coin designs in a variety of settings. Smith (1984 a) has studied the distribution of the numbers of patients receiving each of several treatments when prognostic factors are present. Smith (1984 b) has applied the results to a variety of biases that could arise: selection bias, bias due to outliers, and accidental biases due to smooth trends and to correlated errors. Cox (1982) Smith (1984b) and Wei, Smythe and Smith (1986) have considered the problem of inference after random allocation. Heckmen (1985) has studied a local limit theorem for designs with two treatments.

Ensign (1994) has described an optimal 3 stage design for phase II clinical trials in cancer therapeutics. From sequential construction of \( \text{D}_\alpha \) – optimal designs Atkinson (1982) has built his design.

2.3.4 Response Surface Designs

In response surface designs it is assumed that the response is a smoothly varying function of continuous factors over a well defined experimental region. Box and Draper (1987) proposed the model as

\[
E(Y) = X_1 \beta_1 + X_2 \beta_2
\]
The method of least squares is used to fit the model with terms in $X_i$, which model is to be used for prediction over a specified region. The design should protect against bias due to the united term $X_2$. The resulting expression for the mean squared error of predictions breaks into two parts, one for variance, the other for bias. In general, the bias, and hence the design will depend upon the unknown parameters $\beta_1$ and $\beta_2$. Box and Draper (1987) have argued that unless the bias is very small, designs, which minimize bias lead to a near minimum, mean squared error of prediction.

Number of alternatives to Box and Draper's (1987) work differ both in the nature of the assumed departure from the fitted model and in the method used to fit the model. Pesotchinsky (1982) has assumed that the first-order model is distributed by an additive function, the bound on the magnitude, which is specified. Sacks and Ylvisaker (1984) have generalized the problem of consideration of linear estimates but are concerned only with contrast in the values of the estimated response function. In some cases least-squares estimation with a good design behaves well.

Welch (1983) has replaced the parameterized departure by a general departure, subject to an upper bound $Z_{\text{max}}$, on the departure at each point. Let there be $N$ candidate design points for the $n$-trial design. Exact designs are found by a variant of the exchange algorithm. Welch gives as an example the 9 trial designs for a two-factor first order model over the points of a $3^2$ factorial. When $Z_{\text{max}} = 0$, the all-variance design is obtained which is, as near as possible, the D-optimum $2^2$ factorial. As $Z_{\text{max}}$ increases, the design changes, by stages, to the uniform all bias design with one trial at each of the 9 experimental conditions. The design changes not continuously, but at a few specified points of $Z_{\text{max}}$. From these designs, that one is selected which behaves most reasonably for all-important values of $Z_{\text{max}}$.

To help to increase its utilization a simplified approach to one such statistical methodology, known as the determination of optimum conditions, has been developed by Baines (1992), which can be used by scientists and engineers with a minimum of statistical knowledge.

2.3.5 Off-line quality control

One of the aspects of off-line quality control is the reduction of variability in the performance of the product. The most important economic development has been the relative decline in the importance of manufacturing in USA and Western Europe, compared with the Pacific areas, especially in Japan. Part of the reason for
this change is the superior quality of Japanese goods achieved by the use of statistical techniques. The calculation of Deming Day underlines the strength of the belief the statistical methods have made an important contribution to Japanese achievement.

Suppose the required minimum value of the index can be obtained over a region of factor space, rather than at a single point. The experimental design yielding this information can be crossed with factors representing further stages in the manufacturing process or conditions of use. If some replication is present, the variance on each set of conditions can be considered as a second response. Conditions can then be formed for which \( E(Y) \) lies within specification and has acceptably small variance. Alternatively the factors representing conditions of use can be regarded as generating the variability in \( Y \). What is required is a product for which the expected response is high, but the variability is small. Box and Meyer (1986) have discussed the waste inherent in replication and suggest how to estimate dispersion effects are not replicated function factorials.

Yet, designs for the two sets of factors, conditions of manufacture and conditions of use, are often crossed to yield combined experiments which are unduly large simple assumptions about the absence of high-order interactions lead to smaller design. More efficient analysis than those associated with Taguchi is so possible. Box (1988) has discussed statistically sound alternative to the use of the signal-to-noise ratio, which avoid analysis of an arbitrary combination of mean and variance.

2.3.6 Non-Linear models:

For non-linear models optimum designs depend on the values of the parameters of the model. The purpose of the experiment is usually to estimate these parameters. One solution is to experiment sequentially, starting with some preliminary estimate of the parameter \( \theta_0 \), which is updated as the experimental results become available. Research on designs for non-linear models has been concerned mostly either with the normal theory of non-linear model or with models for binary data.

Ford (1992) have analyzed a certain class of generalized linear models the problem can be reduced to a canonical form. This simplifies the underlying problem
and designs are constructed for several contents with a single variable using Geometric and other arguments. Huang (1993) have considered the problem of constructing linear optimal designs for regression models, when some of the factors are not under control of the experiments. Such designs are referred to as marginally restricted linear optimal designs.

2.3.7. Computers and Optimum Designs:

Computers play a vital role not only in the analysis of experiments, but also in the selection of a design. One use of the machines is to store catalogues of designs. Another is for the generation of designs. Paterson and Paterson (1983) have described an algorithm for the construction of block designs.

A review has given by Nachtsheim (1987) about packages providing response surface designs. The advantage of these computer-based methods is that designs can readily be generated for non-standard problems. Snee (1985) gives examples in which seemingly standard problems are complicated by the non-availability of certain treatment combinations. For example too vigorous conditions in a chemical experiment may produce a sticky black tar instead of the desired product. Finally, recent developments and more sophisticated programs have been discussed by Nachtsheim(1987).

2.3.8 Bayesian Design:

Most of the work on the Bayesian design of experiments has led to analogues of exceed optimality criteria, particularly D-Optimality. If the prior dispersion matrix of the parameters is $H \det (X^TX + H)$ is to be maximized, with the appropriate generalization if the observations are not independent and identically distributed. In a sequential design $H$ will incorporate information from earlier trials.

For non-linear regression models, Box and Hunter (1965) have analyzed non-information priors lead to D-optimality for the linearized model. Designs for informative prices have been obtained by Smith and Verdinelli (1980), with $H$ modeled by hierarchical linear model of Linedley and Smith (1972). A full Bayesian approach to the design of experiments with a general decision function can lead to appreciable mathematical difficulties. Herrendorfer and Rasch (1980) have developed a general theory of cost optimal designs, which have been applied to the
construction of exact designs. After a series of approximations similar in spirit to those of Box and Hunter (1965), Brook (1977) has obtained D-optimality as the appropriate criterion for control. For Prediction A-Optimality is appropriate.

2.3.9 Control theory and Stochastic processes:

Related development has taken place in the literature of control theory where there has been appreciable interest in design for system identification and parameter estimation. Many of these results have been collected by Goodwin and Payne (1977). An introduction to control theory for statisticians and a survey of the design literature in the field has been given by Titterington (1980a). Dodge, Fedorov and Wynn (1988) have developed in great detail the stochastic processes model. A stochastic optimal way has been presented by Singh (1989) where probability of obtaining specified yields on components and monetary returns, have been compared for the shared system of sole crops and intercrops, using two practical examples.

2.3.10 Factorial Designs:

Patterson (1976) has described a 'design key', which generates the design. Introductions, intended for practicing statisticians, have been given by Patterson and Bailey (1978). This method requires the specification not only of the factorial structure of the treatment but also of the plots factors.

Given the design key, it is relatively easy to identify confounded effects. A more difficult problem is to find the design key from the list of effects to be confounded. This problem has been solved by Bailey (1977) using the framework and notation of Bailey, Gilchrist & Patterson (1977). An alternative approach to the generation of factorial design, based on generalized cyclic designs has been summarized by John (1980). The method has been developed for factorial structure in a block design and has to be modified for row and column designs or split plot designs. However, the specification of the generalized cyclic design usually appears much simpler than that of the design key for the same design. Cotter (1975) has described a method of obtaining partial confounding with these designs.

The block designs can also be used to provide blocking patterns for factorial experiments. Cotter (1978) has described the construction and analysis of designs with both complete and partial confounding. Lewis (1979) has studied the
construction of resolution III factorial designs from generalized cyclic designs. A very
different approach is that of Mitchell and Bayne (1978) who use DETMAX to find
optimum fractions of $3^k$ factorials.

The full $2^k$ factorials and their fractions are known to be A-, D- and E-
optimum. Their properties are closely related to weighing designs, from which
squaring the first column of the design matrix forms them. These designs require
that $n=2^p$. Designs with similar desirable properties can be obtained if $n$ is a multiple
of 4 by the use of Hadamard matrices. Cheng (1980b) has reviewed this theory and
has proved the general optimality theorems for designs based on balanced analyses.
Galil and Kiefer (1980b) have given a discussion of weighing designs for k-objects
and hence of two level designs for the $p=k=l$ factor first order model. A review of the
necessary theory of Hadamard matrices has been given by Hedayat and Wallis
(1978), in which the cyclic method of construction can be employed leading to
convenient algorithms for computer use.

Kuwada (1988a,b,c) has presented A-optimal partially balanced
fractional $2^{m_1+m_2}$ factorial designs of resolution V, with $4 \leq m_1 + m_2 \leq 6$. In addition, A-
optimal designs with $m_1=m_2 =3$ have been presented for $42 \leq N \leq 64$, where $N$ is the
number of observations (of resolution VII). He has also described those A-optimal
balanced fractional $2^m$ factorial designs of resolution V with two blocks, $4 \leq m \leq 6$.

Pesotan and Raktre (1988) have presented in invariance and randomization in
factorial designs with applications to D-optimal main effect designs of the
symmetrical factorial. Kolyva-Machera (1989) has been analyzed D-optimal
functions of three-level factorial designs for $k$-factors are constructed for factorial
effects models. In particular, the information matrix of the main model has been
studied in a result characterizing optimum designs, when $N-1 \mod 9$, has been
proved.

2.3.11 Miscellaneous Topics

a) Robust Designs

There has been enormous interest in robust inferences, which do not
depend crucially on departures from the assumed model. Box and Draper (1975)
have listed 14 properties of a good response surface design. Hedayat and Majumdar
(1988) have discussed the model robust optimal designs for comparing test
treatments with a control. They are simultaneously A- and M.V- Optimal for either
one-way or two-way elimination of heterogeneity when the model of response is homoscedastic and linear additive. Bhaumik and Whittinghill (1991) have investigated the optimally and robustness to the unavailability of blocks in block designs.

b) Auto correlated models

When the process $Y(.)$ is auto correlated, the optimal designs become more complex and and the experts have studied in several directions. Estimation of the parameter $\theta$ is more usual in the analysis of field experiments in which "uniformity" gives rises to spatial autocorrelation. This has generated a mathematically exciting area of "neighbor designs" that takes into account where treatments appear next to which and how often. For example in block design we may count neighbours inside a block if there is autocorrelation in a block.

Chaloner (1984) has presented a review of the whole field of spatial sampling with references to earlier joint work with Sacks and connections with Bayesian experimental design. Applications to computational experiments have been described by Sacks and Schiller (1987).

c) Low dose Carcinogenicity

One problem in low-dose carcinogenicity is that of extrapolation from experimental doses to the levels experienced in daily life. A review of this problem and its progress have been summarized by Crump (1979) and applied aspects of optimum design theory to extrapolation for a non-linear model. The resulting designs are shown to be over three times as efficient as the equal weighting designs reported in the literature.

d) Designs for discrimination between models

Designs for discrimination between models are concerned with the decision as to which, of two or more, models correctly describes the data. The literature have been described by Hill (1978). Even if the models are linear, the $T$-optimum designs of Atkinson Fedorov (1975 a,b ) depend upon the parameters of the unknown true model Jones and Mitchell (1978) have used this nesting approach and have developed non sequential alternatives to $T$-optimum designs.

e) Symmetric and spatial designs

The development of new designs continues to come from agricultural field trials. One theme which has recently re-emerged is that of systematic designs
in which allowance is made for the structure of the error. In one form of analysis, the yield of each plot is adjusted by the average residual of the adjacent plots. Atkinson (1969) has investigated the properties of the method for the one-dimensional designs of Williams (1952). Bartlett (1978) considered the method for the more challenging two-dimensional case of field trials. Several experts have presented systematic designs for the use with the method in two dimensions. If two or more species are grown together, by intention or by accident, the effect of spacing may depend also on competition from weeds, from plants of the same genotype or from a crop of a different species. A review of competition experiments has been given by Mead (1979).

2.4 RECENT DEVELOPMENTS ON OPTIMAL BLOCK DESIGNS:

The purpose of this section is to present various results so far known with respect to the E- optimum criteria. Indeed the E-optimality criterion particularly to block designs, has received comparatively more attention of the researcher than A-, D-, M.V-Optimality criterion.

Cheng (1981) has pointed out an interesting relation between the C-matrix of a design with blocks of size k and that of a design in which every block of the original design is replaced by kC2 blocks consisting of the pairs formed out of the treatments is that block.

For a given b, v, and k, an E-optimal design seeks to maximize the minimum positive eigenvalue $x_{(1)}$ of the C-matrix within the class $D(b,v,k)$ as well as that within the class of n-ary designs $D(b,v,k)$. There exist various seconds for various subclasses of binary designs and they have been compared with an E-Optimal design in the entire class $D(b,v,k)$ which have been extended to n-ary class of designs $D(b,v,k)$.

The basic strategy followed by Jacroux (1980 a,b) and Cheng (1980 a,b) in deriving E-Optimal designs, is first to develop bounds for $K x_{(1)}$ (a) for the equireplicate designs, (b) for equireplicate non-binary designs, and (c) for equireplicate binary designs. The conclusion is that if there is an equireplicate binary designs for which $K x_{(1)}$ exceeds each of the bounds under (a) and (b) above, the E-optimal design must necessarily be equireplicate and binary.

Following Takeuchi (1961,1963), Jacroux (1980 a,b) and Cheng (1980 a,b) have discussed some Lemmas and Theorems.
2.5 LATEST A-, D-, AND E-OPTIMAL BLOCK DESIGNS:

The usual additive model specifies the expectation of an observation on variety i in block j as $\alpha_i + \beta_j$ where $\alpha_i$ is the unknown effect of the i-th variety and $\beta_j$ is the effect of the j-th block. The $kb$ observations are assumed uncorrelated with variance ($\sigma^2$ usually unknown).

The information matrix for the variety effects, when the design 'd' is used, losses to be $Cd = \text{diag} (r_{d1}, r_{d2}, \ldots, r_{dv}) - k^{-1} N_d' N_d$ where $r_{di}$ is the number of replications of variety i in d and $N_d = (n_{ij})$, with $n_{ij}$ signifying the number of times variety i appears in block j. For convenience we denote $\sum_{i=1}^{b} n_{dij} n_{dj}$ by $\lambda_{dij}$. The $v \times v$ matrix $C_d$ contains all the relevant statistical information relating to the variety effects. It is well-known that $C_d$ is non-negative definite, has non-positive off-diagonal elements and has row sums zero for all $d \in D(b,v,k)$.

For a design $d \in D(b,v,k)$, let $\mu_d = \mu_{d1} \leq \mu_{d2} \leq \ldots \leq \mu_{dv}$ denote the design values of its information matrix $C_d$ with rank $v-1$. A design is called equireplicated if $r_{di} \leq \ldots \leq r$. A block of d is said to be binary if all block are binary.

A design $d^* \in D(b,v,k)$ is called E-Optimal if $\mu_{d^1} \leq \mu_{d^i}$ for all designs $d \in D(b,v,k)$. Krafft and Schaefer (1971), by their repeated comparison of matrix with its Schur complements have proved main effect plans which are D-, A- and also E-Optimal. Duthie (1991) has examined E-Optimality of designs, which have the cubic association scheme, and regular graph designs. I.

Several results are known concerning the E-Optimality of block designs as experimental situations where $v$ treatments are to be studied with respect to their effects on experimental units arranged in $b$ blocks of size $k$. However most of these optimality results have been derived under the assumptions of fixed block effects. Jacroux (1969) has investigated conditions under which designs determined to be E-Optimal under the assumption of fixed block effects remain E-Optimal under the assumption of random block effects. Das and Kageyama (1991) have analyzed the upper bounds for proper efficiency balanced designs. Recently, Dette and Haines (1994) have presented a method for constructing E-Optimal designs for a broad class of non-parameter models. Brzeskwiniewicz (1989) has derived an inequality for the smallest positive eigenvalues of the C-matrix of the block design, which is the generalization of Constantine (1982) for unequal block sizes. Giving
some E-optimal block designs, the coefficient $C_d$ has been introduced which assesses how close the block design is to an E-Optimal one. Gupta and Singh (1969) have presented present E-Optimality of block designs within various classes of connected designs having varying replications and unequal block sizes.

Dey and Das (1989) have described the E-Optimal of block designs that bounds for the smallest positive eigenvalue of the C-matrix of block designs, which have been obtained in some general classes of connected designs with equal block sizes. Lee (1990) has discussed some E-Optimal block designs having unequal block sized and gave a table of E-Optimal block designs for experimental situations in which v-treatments are to be tested on n-experimental units, arranged in b blocks and where the blocks sized and number of replications are assigned to the treatments are allowed to vary. Some sufficient conditions have been obtained for designs to be E-Optimal in these situations. Das Gupta and Das (1992) have studied the E-Optimality of block designs under a general heteroscedastic setting. The C-matrix of block design under a heteroscedastic setting has been obtained by using generalized least squares and then some bounds for the smallest positive eigenvalue of these c-matrix have been obtained. Optimal block designs with maximum block size and minimum replication constraints have given by Uddin and Morgan (1992).

Dette and Wong (1995) have applied robustness properties of D-optimal designs and $D_{nr}$ - optimal designs under varies polynomial model assumptions. Analytic formulae for the G-efficiencies of these designs have been derived with their D-efficiencies. Heis and Mters (1996) have modeled a bivariate quantal response using the Gumbel model for bivariate logistic regression. D-optimal and Q-optimal experimental designs have also been developed for this model. Chang and Heiligers (1996) have given all E-optimal designs for mean parameter vector in polynomial regression of degree d-without intercept in one real variable on a note on Bayesian, C - and D-optimal design in non-linear regression model DETTE (1996) has presented a version of Elfving’s theorem for the Bayesian D-optimal criterion in non-linear regression models, whereas Heiligers (1996) has studied on a computing method for E-optimal polynomial regression designs, Koukouvinos (1996) has dealt with the problem of constructing first order saturated designs not are optimal in some sense and Morgan, and Uddin (1996) have studied optimal designs for factorial
experiments in the nested row and column settings. This approach is analogous to that of orthogonal latin square.

E-optimal incomplete designs with two distinct block designs have studied in detail by Uddin (1996) whereas Brzeskwiniewcz (1996) has given lower bounds for A-, D-, and L-efficiency of block designs. These bounds have been obtained for wide class of block designs including also disconnected designs with unequal number of treatments replicates and unequal block designs.

Das, Dey and Dean (1998) have discussed optimal designs for diallel cross experiments whereas Chan, Meng, Jiang and Guan (1998) have discussed D-optimal axial designs for the additive, quadratic and cubic mixture models. The work on Jacroux (1998) conducted research a set of test treatments with a set of control in the presence of a linear trend.


2.6 BRIEF REVIEW OF n-ary BLOCK DESIGNS

Though Tocher (1952) has introduced the concept of n-ary designs, but no attempt was made to develop systematic method of construction until 1967. For the first time Das and Murthy (1967) have introduced the systematic method of construction of n-ary design by using a set of MOLS. They have also generalized the concept of BNB designs. Further Das and Rao (1968) have developed an alternative method of construction of BNB designs based on introduce matrix of BIB design. Dey (1970) has presented a method of construction of BIB designs using affine $\alpha$-resolvable balanced incomplete Block designs. He has constructed the BIB designs by collapsing the blocks of an affine $\alpha$ - resolvable 2x2 BIB design taking
one from one $\alpha$ - replicate and the second block from another $\alpha$ - replicate. By collapsing the blocks of BIB design evidently means the replacement of varieties of the GD-PBIB design by the block contents of the BIB design, the rule being that the i-th treatment of the GD design is replaced by the block contents of the i-th block of the BIB designs. As such when the blocks of the BIB design collapsed in a given number, the treatments of the GD design are in fact replaced by the block contents of the - BIB design.

Saha and Dey (1973) have made an attempt to construct BIB design through the method of generalized different set. BIB designs have been obtained by developing a single initial block and as such are symmetrical. A general class of BNB designs has also been obtained by developing an initial block. They also have proved that a generalized complement of a BNB design is also a balanced one.

Nigam (1971) has evolved some methods for constructing BNB designs by using the incidence matrix of a BIB. He has shown that BNB design can be obtained through any two p-ary and (n-p+1) - ary balanced block designs. The ternary designs of Dey (1970) turn out to be the particular cases of Nigam's ternary designs. Two methods of construction of BIB design (with K <v) have been described by Saha (1975). The first method of construction of BIB design is simple by considering a BIBD with even number of varieties relabelling every pair of subsequent varieties as a single new variety, to get the BIB design for v/2 treatments, while the second method uses, "initial block". The second method is based on a result of Saha and Dey (1973) on construction of BTB designs using differences.

Sharma and Agarwal (1976) have obtained a series of BNB designs by collapsing certain (n-1) tuplets of blocks of a BIB design. From general approach Dey's results can be obtained. Economy in number of blocks is arrived in Dey's complementary n-ary designs. Morgan (1977) has constructed some families of n-ary designs from (i) t-designs with $t \geq 3$; (ii) finite projective planes and (iii) symmetric BIBD's. It has shown that the dual of a SBTB design is a SBTB design with the same parameters. Morgan (1978) has also given two methods of construction of B(N+1)B design from a set of m BIBD's having same set of varieties.

Shafiq and Federer (1979) have extended the concept of N-ary balanced block designs (here N and n has no difference representing the same ary number) where the incidence matrix n contains the N values 0,1,2,......N-1, to generalized N-
ary balanced block designs, where the incidence matrix $n^*$ contains the $N$ values $m_a$ for $a = 0,1,2,\ldots, N-1$ for $m_a = am_1 + (a-1)m_0$ and $m_0$ and $m_1$ satisfying $0 \leq m_0 < m_1$. For ternary designs $m_2 = 2m_1 = m_0$ and $0 < m_0 < m_1 < m_2$. Given a fixed number $v$ of treatments and a fixed total number $N^*$ of experimental units, a class of $N$-ary balanced block designs with a different set of $m_a$ is possible. They have also developed criteria to select the designs in the class with the smallest variance of a contrast.

Tyagi and Rizwi (1979) have suggested some modifications to Nigam's (1974) method to effect reduction in the number of blocks of the ternary (n-ary) designs, where as Surendran and Sunny (1979) have suggested a new method of construction of proper BNB designs from associate designs. Kageyama (1980) has completely characterized BNB design with $RK = \Lambda V$ of Tocher (1952). It is noted that $B > V$ holds for a non-trivial BNB designs, where $B$ is the number of blocks and $V$ number of treatments. Rao (1955) has discussed the necessary and sufficient condition for any design to be balanced is that $C$-matrix of the adjusted intra-block normal equations, shall have equal diagonal elements and all the off-diagonal elements are also equal. Pearce (1964), Calinski (1971) and Calinski and Seranka (1974) have studied the use, of the designs in varying replications and unequal block sizes Kulshreshtha (1972) have presented the method of construction of balanced binary and ternary block designs with two block sized and varying replications considering the nearly BIB designs of Nigam (1976). Nigam, Mehta and Agarwal (1977) have generalized the work of Kulshreshtha et. al., to contract BNB designs with more than two block sizes and unequal replications, are seen to be almost as efficient as the totally balanced designs.

Recollecting basic concepts on PBIB designs and BNB designs, Paik and Federer (1973) and later Mehta, Agarwal and Nigam (1975) have defined partially balanced n-ary block designs as a generalization of PBIB (binary) designs do not change the type and order of the association scheme of the original PBIB designs. The note of Agarwal (1977) on the construction of PBTB designs has utilized the difference sets concept of Bose and Nair (1939).

Soundarapandian (1980a) has studied the methods of construction of partially balanced n-ary block designs using difference sets. Some more new series of PBNB designs have been developed from initial blocks. By that method a general class of partially balanced n-ary block designs has also been obtained." On a
property of balanced designs". Soundarapandian (1980b) has proved the necessary and sufficient condition for a design to be balanced that is the adjusted intra-block normal equations and its C-matrix should have all its off diagonal elements equal and this in turn gives that all the diagonal elements to be equal. Soundarapandian (1980 c) has studied the concept of N-ary partially balanced incomplete block designs from any binary partially balanced scheme. When the incidence matrix n contains the N values 0,1,2,....N-1, is extended to generalized N-ary partially balanced block designs where the new incidence matrix n* contains the n values m_a for a = 0,1,2,....N-1, for m_a = a m_1 - (a-1) m_0, and for any m_0 and m_1 satisfying 0 ≤ m_0 < m_1. Criteria have been developed to select the two associate scheme partially balanced n-ary block design(s) in the class with smallest variance of a contrast.

Soundarapandian (1981a) has established a new method of construction of higher associate cyclical partially balanced ternary designs by allowing the treatments to occur more than at least once in any of the initial blocks. Soundarapandian (1981 b) has stated the necessary and sufficient condition for a general class connected n-ary block designs to be balanced and its generalization to n-ary block design by using pseudo inverses. After giving some of the properties of symmetrical balanced n-ary block designs, he has discussed an equireplicated n-ary balanced block design with B=V is symmetrical balanced n-ary block design. A theorem from n-ary to binary designs has been stated and proved. Soundarapandian (1981 c) has further extended the expressions for binary linked block (LB) designs given by Rao (1956) to linked n-ary block (LNB) designs which admit easy estimation of parameters for these type of all n-ary block designs. Soundarapandian(1981) has broadly studied n-ary designs.