Chapter I

Introduction
CHAPTER I
INTRODUCTION

1.1 CONCEPT OF OPTIMAL DESIGN

The twin essential features of general scientific methodology are experimentation and making inferences. Statistics as a scientific discipline is mainly designed to achieve these objectives. It is generally concerned with problems of inductive inferences in relation to stochastic models describing random phenomena. When the scientist faced with the problem of studying a random phenomenon, he may not have complete knowledge of the true variant of the phenomenon under study. A Statistical problem arises when he is interested in the specific behaviour of the unknown variant of the phenomenon. After a statistical problem has been set-up, the next step is to perform experiments for collecting information on the basis of which inferences can be made in the best possible manner.

During the First and the Second World Wars (1918-1939) Sir Ronald A. Fisher dominated the history of the experimental designs. He was incharge of Statistics at the Rothamsted Agricultural Experiment station near London and in 1933 he succeeded Karl Pearson, Alton Professor at the University of London and later he moved to Cambridge University. Fisher was the man to introduce the analysis of variance and design of experiments. Dr.F.Yates joined Fisher at Rothamsted in 1931. While these two eminent men, being in touch with experimental scientists, were restrained in their efforts to obtain new designs and new analytical techniques by the requirements of the experiments. Bose concentrated more on the methods of construction of the Balanced and other Incomplete Block Designs and was not necessarily constrained by the consideration of practical utility.

Design of experiments forms a fascinating branch of statistics and though primarily it originated from agricultural experiments. It is finding more and more applications in various other fields. In varietal trials if the number of varieties is large, the ordinary Randomized Block Design (RBD) or Latin Squares Design (LSD) are not at all suitable as the efficiency of varietal comparisons become very much reduced, because of lack of effective control on experimental error. In order to overcome this drawback, a series of designs were introduced. Among these designs, Yates (1936) introduced a design known as Balanced Incomplete Block Designs (BIBD), which could accommodate more types of varietal trials. But these designs have the effect
that larger number of replications for each of the varieties is required for its application. This eminently demands more resources. Bose and Nair (1939) have introduced another series of designs known as Partially Balanced Incomplete Block Designs (PBIBD) removed this difficulty to some extent. Such designs can be obtained with smaller number of replications of each the varieties. A brief review is done on all n-ary block designs available of till date.

First of all review the optimal designs with important definitions of various optimality criteria have been presented in this chapter. In the remaining chapters new solid works on optimum designs, particularly A-, D- and E- optimalities and latest universal optimality have been analyzed. Now days an experimental with modern statistical experiments, at his disposal, can use a variety of experimental designs as available in different literature of designs of experiments. Hence, whenever the condition of an experiment allows the possibility of simultaneous existence of a number of experimental designs, the question of selection of an appropriate design, which is easy for analysis and has optimum proportions naturally, arises. The idea of efficiency usually relative to some standard designs has also been discussed.

1.2 OPTIMALITY THEOREM.

Kiefer (1955, 1959) has introduced the central theory of optimum experimental design and later it has been reviewed by Wynn (1984). The optimum experimental design is concerned with linear model $E(Y) = X \beta$, where $Y$ is a vector of responses, $\beta$ is a vector of unknown parameters and $X$ is a design matrix of full rank. Unobservable random errors are assumed to be independent and to have constant variance $\sigma^2$. The parameters of the vector $\beta$ are estimated by least squares giving estimates with a covariance matrix given by.

$$\text{Cov}(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$$

The optimum design of experiments is concerned with the choice of $X$, ie., the design points, so as to optimize various characteristics of $X^T X$, for example to maximize $\det(X^T X)$. We are concerned with low-order polynomial and we rewrite the model as $E(Y) = F\beta$ where the i-th row of the n x p matrix $F$ is $f_i^T(x_i)$, representing functions of $m (\leq p)$ explanatory variables. Now the least square
estimates have variance matrix $\sigma^2 (F^T F)^{-1}$, where $F^T F = \Sigma f(x_i) f^T(x_i)$ is called the information matrix.

A good design is a choice of $n$ points in $\chi$ which makes the matrix function $F^T F$ large, in some sense to be determined. Mathematically, it is convenient to consider instead of the continuous or approximate, theory in which a design is described by a measure $\xi$ over $\chi$. Then the information matrix is written as

$$M(\xi) = \int f(x) f^T(x) \xi(dx) = \int m(x) \xi(dx) \quad (1.2.1)$$

A design for which exactly $n$ points in $\chi$ are chosen is called an exact design, and is represented by a discrete measure $\xi_n$, that puts weight $n^{-1}$ at each of the $n$ points $x_1, x_2, \ldots, x_n$. Then $M(\xi_n) = n^{-1} (F^T F)$. In the theory for continuous design it is customary to consider minimization of some measure of imprecision $\psi(M(\xi))$.

### 1.2.1 Directional Derivatives

There are two types of directional derivatives. They play a basic role in optimality designs.

a) Let the measure $\xi$ put unit mass at the point $x$. The Gateaux derivative of $\psi$ at $\xi$ in the direction of $\xi$ is

$$\Phi(x, \xi) = \lim_{\alpha \to 0} \frac{1}{\alpha} \{ \psi[M(\xi) + \alpha M(\xi)] - \psi[M(\xi)] \} \quad (1.2.1)$$

b) The Frecher derivative of $\psi$ at $\xi$ in the direction of $\xi$ is

$$\Phi(x, \xi) = \lim_{\alpha \to 0} \frac{1}{\alpha} \{ \psi[(1-\alpha) M(\xi) + \alpha M(\xi)] - \psi[M(\xi)] \} \quad (1.2.2)$$

and this derivative will serve our purposes better than the previous one.

### 1.2.2 Equivalence Theorem.

The general equivalence theorem (Kiefer and Wolfowitz, 1960), then states the equivalence of the three following conditions.

i) $\xi$ minimizes $\psi \{M(\xi)\}$

ii) $\min \Phi(x, \xi^*) \geq 0 \quad (1.2.3)$

iii) $\Phi(x, \xi^*)$ achieves its minimum at the points of the design.
1.3 OPTIMALITY DEFINITIONS:

1.3.1 A-Optimality.

The criterion of A-Optimality is defined by the criterion function

\[ \psi(M(\xi)) = \text{tr} \left( M^{-1}(\xi) \right) \quad \text{if det} \{M(\xi)\} \neq 0 \]
\[ = \infty \quad \text{if} \quad \text{det} \{M(\xi)\} = 0 \]

This means that the trace of \( M^{-1}(\xi) \) is minimized so that the average variance of the parameter estimates is minimized.

1.3.2 D-Optimality:

The most widely used design criterion is that of D-optimality, which has been introduced by Wald (1943) and it is defined by the criterion function,

\[ \psi(M(\xi)) = -\log |M(\xi)| \]

so that the determinant of the information matrix is to be maximized or, equivalently the determinant of \( M^{-1}(\xi) \) is to be minimized.

1.3.3 D_A-Optimality.

Suppose that our interest is not in all the parameters in the model, but in the linear combinations of \( \beta \) which are the elements of \( A^T \beta \), \( s<k \). To minimize the generalized variance of this subsystem, the analogue of D-Optimality is called D_A-Optimality, to stress the dependence of the optimum design on the particular linear combinations of interest. It has been introduced by Sibson (1974) and this criterion,

\[ -\log |A^T M^{-1}(\xi) A| \]

is maximized.

1.3.4 D_s-Optimality.

A special case of importance arises when there is only one model, a subset of the parameters of which is of interest. Let the model be defined as

\[ E(Y_1) = f_1^T(x) \beta = f_1^T(x) \beta_1 + f_2^T(x) \beta_2 \]

where \( \beta_1 \) are the S-parameters of interest, \( \beta_2 \) being treated as nuisance parameters.

The coefficients matrix \( A \) becomes \( A = (I_s; 0) \) where \( I_s \) is the SxS identity matrix. If the matrix is correspondingly partitioned as
the determinant to be maximized is

\[ \left| M_{11}(\xi) - M_{12}(\xi) \right| \]

This criterion, which depends on the particular subset of parameters, is customarily called \( D_5 \) - Optimality. The variance to be minimized is

\[ d_4(x, \xi) = f^T(x) M^{-1}(\xi) f(x) - f_2^T(x) M_{22}^{-1}(\xi) f_2(x) \]

1.3.5 E-Optimality

It is defined by the criterion function,

\[ \psi(M(\xi)) = \lambda_{\xi}^{-1} \quad \text{if} \quad \det(M(\xi)) \neq 0 \]

\[ = \infty \quad \text{if} \quad \det(M(\xi)) = 0 \]

Here the minimum eigenvalue of \( M(\xi) \) by \( \lambda_{\xi} \). Otherwise, it is explained that E-Optimum designs minimize the maximum eigenvalue of \( M^T(\xi) \), which is equivalent to minimizing the variance of the contrast \( a^T \beta \) with largest variance, subject to \( a^T a = 1 \).

The criteria of \( D_5 \)-, \( A \)- and E-optimum designs are special cases of a power function of the eigenvalues of \( M(\xi) \) used by Kiefer (1975) to study the variation in structure of optimum designs as the criterion changes in a smooth way.

1.3.6. G-Optimality.

The G-Optimality criterion is defined by the criterion function,

\[ \psi(M(\xi)) = \max_{x \in X} f^T(x) M^{-1}(\xi) f(x), \quad \text{if} \quad \det(M(\xi)) \neq 0 \]

\[ = \infty, \quad \text{if} \quad \det(M(\xi)) = 0 \]

The experimenter optimizing the design according to the G-Optimality criterion intends to get a good estimate of the whole state function \( \theta \in \Theta \).
1.3.7 **Linear optimality.**

These criteria are stated by criterion function of the form

\[ \psi(M(\xi)) = tr WM^{-1}(\xi) \text{ if } \det \{M(\xi)\} \neq 0 \]

\[ = \infty \text{ if } \det \{M(\xi)\} = 0 \]

Where \( W \) is a positive definite \( m \times m \) matrix. The A-optimality criterion corresponds to the particular case of \( W = 1 \).

1.3.8 **The \( L_p \)-class (or) Trace-class of optimality.**

Many optimality criteria are particular cases of a \( L_p \)-class of optimality criteria. A criterion belonging to the class is defined by a criterion function of the form

\[ \psi(M(\xi)) = M^{-1} \text{tr} [H M^{1/2}(\xi) H^{1/2}]^{1/p} \text{ if } \det \{M(\xi)\} \neq 0 \]

\[ = \infty \text{ if } \det \{M(\xi)\} = 0 \]

where \( p > 0 \) and \( H \) is a non-singular \( m \times m \) matrix.

The linear optimality criteria are obtained if \( p = 1 \). Also, the E-optimality criteria belongs to the \( L_p \)-class if \( H = 1 \) and \( p \to \infty \).

1.3.9 **MV-Optimality.**

In a block design Takeuchi(1961) has argued that an optimal design may seek to minimize the maximum variance of the corresponding estimates. This criterion has been called MV - optimality criterion by Jacroux (1983).

A design in the class \( \mathcal{D} (b.v.k) \) is said to be MV-Optimal if it minimized the maximum variance for a paired treatment contrast among all designs \( \mathcal{D} (b.v.k) \).

Thus, MV-optimality is somewhat different from E-optimality in which the comparison refers to all treatment contrasts. It is also different from A-optimality which seeks to relate to the average variance for all paired treatment contrasts. However, unlike the A-D- and E-optimality criteria, this criterion is not exclusively a function of the eigen values and, as such, it needs a separate treatment.

1.3.10. **Schur-Optimality.**

Magda (1979) has introduced the notion of Schur-Optimality by utilizing Schur-convex function as follows:

For any vector \( X (n \times 1) \), a real valued function \( \Phi (x) \) satisfying \( \Phi(S_x) \leq \Phi (x) \).
for every doubly stochastic matrix $S$, is said to be Schur-Convex. Such a function is permutation-invariant in the sense that $\Phi(x) = \Phi(Px)$ for every permutation matrix.

Let $X(C)$ denote the vector of non-zero eigenvalues of $C$. Then a Schur-optimality criterion seeks to minimize,

$$\Phi(C) = \Phi\{X(C)\}$$

among all relevant $C$-matrices for a given Schur-Convex function $\Phi$.

### 1.3.11 S-Optimality.

Further extension is, suppose that there are $m$-models and that, for the $i$-th, the subsystem of interest is given by $A_i^T \beta_i$ with $s_i < k_i$ where $k_i$ is the rank of the $i$-th linear model. Interest in the different models is represented by the non-negative weights $w_i$.

The following three requirements on the optimum measure $\xi^*$ are equivalent:

i) $\xi^*$ maximizes

$$\sum_{i=1}^{m} w_i \log |A_i^T M_i^{-1}(\xi) A_i|$$

ii) $\xi^*$ minimizes the maximum over $\chi$ of

$$\sum_{i=1}^{m} \sum_{i=1}^{m} w_i x^T M_i^{-1}(\xi) A_i \{ A_i^T M_i^{-1}(\xi) A_i \}^{-1} A_i^T M_i^{-1}(\xi) x$$

iii) the maximum value of

$$\sum_{i=1}^{m} w_i d_{A_i}(x, \xi) \;	ext{is} \; \sum_{i=1}^{m} w_i s_i$$

The equivalence theorem is a generalization of those given by Atkinson and Cox (1974) and by Lauter (1976), who calls the criterion S-Optimality.

### 1.3.12 M.S-Optimality.

Eccleston and Hedayat (1974) have proposed an optimality criterion, called M.S-Optimality, extending the notion of S-Optimality introduced by Shah (1960). These criteria are known for minimizing the dispersion of the latent roots of the information matrix of a design.
1.3.13 T-Optimum Design.

Atkinson and Fedorov (1975) have described T-Optimum experimental design as in the form of the following theorem:

i) A necessary and sufficient conditions for a design $\xi^*$ to be Bayesian T-Optimum are fulfillment of the inequality $\psi(x,\xi^*) \leq \Gamma(\xi^*)$ for all $x \in \chi$.

where 

$$\psi(x,\xi^*) = \sum \prod_{ij} e_{ij} j (n_j (x, \theta _j) - n_j (x, \theta _* j))^2$$

ii) At the points of Bayesian T-Optimum design $\psi(x,\xi^*)$ achieves its upper bound.

iii) For any non-optimum design $\xi$, that is a design for which $\Gamma(\xi) < \Gamma(\xi^*)$

$$\sup_{x \in \chi} \psi(x,\xi) > \Gamma(\xi^*)$$

iv) The set of Bayesian T-Optimum designs is convex.

1.3.14 Universal Optimality.

Kiefer (1975) has introduced the notion of universal optimality in the following manner.

Consider the optimality functional $\Phi$ defined on the set of all $C$-matrices which satisfy namely,

i) $\Phi(C)$ is non-increasing in $t$, $t \geq 0$

ii) $\Phi (\alpha C_1 + (1-\alpha) C_2) \leq \alpha \Phi (C_1) + (1-\alpha) \Phi (C_2)$ for $0 < \alpha < 1$ and for any pair of C matrices $C_1$ and $C_2$.

If a design is optimal with respect to all such optimality functional $\Phi$, it is said to be Universally optimal.

1.3.15 V-optimality.

The criteria of A- and E-optimality have been much used in the construction of Block designs (Paterson, 1988). The criterion of G-optimality is concerned with the maximum of the variance of the estimated response. The second variance-based criterion is $V$-optimality in which the design is found to minimize

$$d_{ave} (\xi) = \frac{1}{r} \sum_{i=1}^{r} d (x_i, \xi)$$
Interest is now in the average of the variance at r-points which need not belong to the design region \( \chi \).

### 1.4 ANOTHER FORM OF OPTIMALITY DEFINITIONS

Let \( \text{E}(Y) = X \theta, \text{cov}(Y) = \sigma^2 I_n \)

Where \( Y_{nx1} \) is observation vector follows a standard linear model, \( X_{nx1} \) is the design matrix \( \theta_{nx1} \) is the unknown parameter \( \sigma^2 \) is the constant error variance.

Normal Equations for obtaining BLUE of \( \theta \) is \( X'X \theta = X' Y \).

Here, \( X'X \) is called the "Information matrix" of \( \theta \) and \( \text{var}(\theta) = \sigma^2 (X'X)^{-1} \) (provided \( \text{Rank}(X) = t \)).

Let \( t' \theta \) be an estimable linear parametric function.

Then \( \text{var}(t' \theta) = \sigma^2 t' (X'X)^{-1} t \)

Thus, we want to choose a design \( d \), with design matrix \( X_d \), whose information matrix \( X_d'X_d \) is "large" (equivalently, \( X_d'X_d \) is "Small") in some sense.

Now, suppose we are interested in a component \( \theta_i \) of \( \theta \).

We write \( \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \)

and accordingly partition \( X \) as

\[ X = (X_1, X_2) \]

So that the model becomes

\[ \text{E}(Y) = X_1 \theta_1 + X_2 \theta_2, \quad \text{cov}(Y) = \sigma^2 I_n \]

The information matrix of \( \theta_1 \) is

\[ I(\theta_1) = X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1, \quad I(\theta_1) \text{ is n.n.d.} \]

#### 1.4.1 Completely randomized designs

\[ \theta_1 = \tau_{vx1} \]

\[ \theta_2 = \mu_{1x1} \]

Given a design \( d \)

\[ X_{1d} = \{ (X_{ij}) \}_{nxv} \quad \text{obs. Vs. treat. matrix.} \]

\[ X_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ treat. in } i^{th} \text{ obs.,} \\ 0 & \text{otherwise} \end{cases} \]
\[ X_{2d} = l_n \]

\[ l_d(\theta_1) = R_d - n^{-1} r_d r_d' (= C_d) \]

\[ R_d = \text{diag} (r_{d1}, r_{d2}, \ldots, r_{dv}) \]

\[ r_d = \begin{bmatrix} r_{d1} \\ r_{d2} \\ \vdots \\ r_{dv} \end{bmatrix} \]

\[ r_d = \# \text{ times treat } i \text{ is rep. in } d. \]

1.4.2 Block Design.

A block design is an arrangement of \( v \) treatments in \( b \) blocks each of size \( k_{d1}, k_{d2}, \ldots, k_{dv} \) respectively. The replication of treat \( i \) is \( r_{di}, i = 1, \ldots, v. \)

\[ \theta_1 = \tau_{v \times 1} \]

\[ \theta_2 = \begin{pmatrix} \mu \\ \beta \end{pmatrix}_{(b+1) \times 1} \]

\[ X_{1d} = ((X_{1i}))_{n \times v \text{ obs. vs. treat. matrix}} \]

\[ X_{2d} = ((l_n X_{\beta d}))_{(n \times (b+1))} \]

\[ X_{\beta d} = (X_{\beta ij})_{n \times (b+1)} \text{ obs. vs. block matrix}. \]

\[ X_{\beta ij} = \begin{cases} 1 & \text{if } l^\text{th} \text{ obs. out of } j^\text{th} \text{ block} \\ 0 & \text{otherwise} \end{cases} \]

\[ l_d(\theta_1) = R_d - N_d K_d^{-1} N_d' (= C_d). \]

where

\[ R_d = \text{diag} (r_{d1}, \ldots, r_{dv}) \]

\[ K_d = \text{diag} (k_{d1}, \ldots, k_{dv}) \]

\[ N_d = ((n_{dij}))_{(v \times b)} \text{ treat. block incidence.} \]

\( n_{dij} \) is the number of times treat \( i \) appears in block \( j. \)
1.4.3 Row - Column Design

A row - column design \( d \) is an arrangement of \( v \) treatments in a \( k \times b \) array of \( k \) rows and \( b \) columns.

\[
\begin{align*}
\theta_1 &= \begin{pmatrix} \mu \\ r \end{pmatrix} \\
\theta_2 &= \begin{pmatrix} c \end{pmatrix} \\
\end{align*}
\]

\((k + b + 1) \times 1\)

\[l_d(\theta) = R_d^{-1/k} N_d N_d' - 1/b M_d M_d' + 1/bk r_d r_d' \quad (= C_d)\]

where \( R_d = \text{diag}\left(r_{d1}, \ldots, r_{dv}\right) \)

\(N_d = ((n_{dij})_{(v \times b)}) \text{ treat - row incidence}\)

\(m_d = ((m_{dij})_{(v \times k)}) \text{ treat - row incidence}\)

As we are interested in the treatment effects, the problem of inference may be specified as

\[\Pi : \eta = L \tau\]

where \( L \) is \( p \times u \) matrix with \( LI = 0 \).

Thus, \( \eta \) contains \( p \) treat. contrasts (In fact, only treat. contrasts can be estimated).

With reference to \( \Pi \), we call a design \( d \) as acceptable if all components of \( \eta \) are estimable using \( d \).

Let \( D_\pi \) be the class of all acceptable designs with reference to the problem \( \pi \).

Problem \( \pi \) is referred to as

1. non-singularly estimable iff \( \text{Rank} \left(L\right) = p \).

2. non-singularly estimable full rank iff \( \text{Rank} \left(L\right) = p = v-1 \).

For a full rank problem \( \pi, D_\pi \) consists only of such designs \( \{d\} \) for which \( \text{Rank} \left(C_d\right) = v - 1 \). Such designs are called connected designs.

In the above types of designs, \( C_d1 = 0 \) and \( \tau \cdot \tau \) is estimable iff \( \tau \) belongs to the column - space of \( C \). Thus, \( \tau \cdot \tau \) need to be a treat contrast in order to be estimable.

\( \text{Rank} \left(C_d\right) = v - 1 \) iff all treat. Contrasts are estimable and in that case, the underlying design is said to be connected.
1.5 OPTIMALITY CRITERION TO SELECT GOOD DESIGN

Suppose \( \eta_d \) is the BLUE of \( \eta \) using a design \( d \)

\[ \text{Var} (\eta_d) = V_d. \]

It is reasonable to define an optimality criterion as a meaningful function of \( V_d \).

1.5.1 A - Optimality

A design \( d^* \in D \) is said to be A - optimal in \( D \) iff \( \text{tr} (V_d^*) \leq \text{tr} (V_d) \) for any other design \( d \in D \).

The trace of \( V_d \) is minimized in the A-optimality criterion, which implies the minimization of the average variance of the BLUE of the components of \( \eta \).

1.5.2 D-Optimality.

A design \( d^* \in D \) is said to be D-optimal in \( D \) iff \( \text{det} (V_d^*) \leq \text{det} (V_d) \) for any design \( d \in D \).

The D - optimality criterion has the following statistical significance.

Let the observation vector \( Y \) follow a multivariate normal distribution. Then, \( \eta_d \) also follows a multivariate normal distribution with mean \( \eta \) and dispersion matrix \( V_d \).

\( A(1-\alpha) \% \) joint confidence region for \( \eta \) is the ellipsoid.

\[
(\eta - \eta_d)^T V_d^{-1} (\eta - \eta_d) \leq \sigma^2 \chi^2_{a(u-1)} \quad \text{(A)}
\]

Where \( \sigma^2 \) = per obs. Var. (known)

\[
\chi^2_{a(u-1)} = (1-\alpha) \text{ percentile of a central } \chi^2 \text{ with } (u-1) \text{ d.f.}
\]

or is the ellipsoid

\[
(\eta - \eta_d)^T V_d^{-1} (\eta - \eta_d) \leq s^2 F_{a} (u-1,\eta_{e}) \quad \text{(B)}
\]

where \( s^2 \) = unbiased estimator of \( \sigma^2 \) (unknown)

\[
F_{a} (u-1,\eta_{e}) = (1-\alpha) \text{ percentile of } F \text{ with } (u-1) \text{ and } \eta_{e} \text{ d.f.}
\]

\( \eta_{e} \) = error d.f.
The volume of (A) (expected volume in (B)) is proportional to the square root of det. \( \text{det}(V_d) \).

Thus the D-optimality criterion chooses that design as the "best" for which the volume (expected volume) of the joint confidence ellipsoid is least.

### 1.5.3 E-Optimality

A design \( d^* \in \mathcal{D} \) is said to be E-optimal in \( \mathcal{D} \) iff for all normalized treat contrasts \( i' \tau \) with BLUE \( i' \tau \),

\[
\max_{i,j=1} \left( \text{var}_{d}(i' \tau) \right) \leq \max_{i,j=1} \left( \text{var}_{d}(i' \tau) \right)
\]

for any other design \( d \in \mathcal{D} \).

Let \( \mu_{d_0} < \mu_{d_1} \leq \ldots \leq \mu_{d_{d-1}} \) be the eigenvalues of \( C_d \). Then,

\[
\text{var}_{d}(i' \tau) = \sigma^2 \cdot C_{d,1}^\tau
\]

Where \( C_{d,1}^\tau \) is the Moore-Penrose inverse of \( C_d \). Also

\[
\mu_{d_{d-1}}^{-1} \leq \frac{\|C_{d,1}^\tau\|}{\|i'\|} \leq \mu_{d_1}^{-1}
\]

Thus, if \( i'1 = 1 \), implies

\[
\max_{i,j} \text{var}_{d}(\tau_i \tau_j) = \mu_{d_1}^{-1}
\]

Hence,

A design \( d^* \in \mathcal{D} \) is E-optimal in \( \mathcal{D} \) iff \( \mu_{d^*} \geq \mu_{d_1} \) where \( d \) is any other competing design in \( \mathcal{D} \) (\( d^* \) minimizes (over \( d \)) the maximum variance of all normalized treat. Contrasts).

### 1.5.4 MV-Optimality

A design \( d^* \in \mathcal{D} \) is said to be MV-optimal iff

\[
\max_{i \neq j} \left( \text{var}_{d}(\tau_i \tau_j) \right) \leq \max_{i \neq j} \left( \text{var}_{d}(\tau_i \tau_j) \right)
\]

Where \( d \) is any other competing design in \( \mathcal{D} \).

Here the interest is only on elementary treat-contrasts and accordingly the MV-optimality criteria is based on only specific contrasts.
1.5.5 Optimality Criterion as a Functional

Let $P\tau$ represent a compete set of orthonormal treatment contrasts with BLUE $P\tau P1$

Thus $P$ is of order $(v-1) \times v$ and Rank $(p) = v-1$ with $P1 = 0$, $PP' = I_{v-1}$

Also $\sigma^{-2} \text{Var}_d(P\tau) = P C^* d P' = (P C^* d P')^{-1}$

Consider $A = \begin{pmatrix} 0 & -\frac{1}{2} & 1 \end{pmatrix}$

With $AA' = I_v = A'A$

Then, since $C^* d = 0$

$AC^* d A' = \begin{pmatrix} 0 & 0^t \\ 0 & PC^* d P' \end{pmatrix}$

And $\det(C^* d - \lambda I_v) = \det(AA' - \lambda I_v) = -\lambda \det(PC^* d P' - \lambda I_{v-1})$

Thus, the non-zero eigenvalues of $C^* d$ and the eigenvalues of $PC^* d P'$ are the same.

It follows that instead of minimizing a function of the eigenvalues of $PC^* d P'$ to arrive at an optimal design, one may as well minimize the same function of the non-zero eigenvalues of $C^* d$

One may thus think of an optimality criterion as a function on the set of n.n.d. Symmetric matrices of order $v$ with zero row sums.

Let, $B_{u,o} \rightarrow$ set of all n.n.d. Symmetric matrices of order $v$ with zero row sums.

An optimality criterion $\phi$ is a function.

$\phi : B_{u,o} \rightarrow (-\infty, \infty)$

A design $d$ is $\phi$ - optimal if it minimizes $\phi(C_d)$.

Note $C_d \in B_{u,o}$

$A$ - optimality: $\phi_A(C_d) = \sum_{i=1}^{v-1} \mu^{-1}_d$

$D$ - optimality: $\phi_D(C_d) = \prod_{i=1}^{v-1} \mu^{-1}_d$

$E$ - optimality: $\phi_E(C_d) = \max_{i} \mu^{-1}_d$
1.5.6 Universal Optimality

Kiefer (1975) has defined if \( d^* \) minimizes \( \phi(C_d), d \in D \) for any \( \phi: B_{u,0} \rightarrow (-\infty, \infty) \) satisfying

1. \( \phi \) is matrix convex, i.e.,
   \[ \phi \left\{ aC_1 + (1-a)C_2 \right\} \leq a \phi(C_1) + (1-a) \phi(C_2) \]
   for \( C_1 \in B_{u,0} \) (\( i = 1,2 \)) and \( 0 \leq a \leq 1 \),
2. \( \phi(bc) \) is non increasing in the scalar \( b \geq 0 \) for each \( C \in B_{u,0} \),
3. \( \phi \) is invariant under each simultaneous permutation of rows and columns of \( C \in B_{u,0} \).

Then \( d^* \in D \) is universally optimal over \( D \).

1.5.7 \( \phi_p \) Optimality

Kiefer (1974) established this optimality as given below.

Let \( \phi_p(C_d) = \left[ (v-1)^{-1} \sum_{i=1}^{\mu} \mu_{di}^p \right]^{1/p}, 0 < p < \infty \)

A design is called \( \phi_p \) - optimal if \( \phi_p(C_d) \) is minimum over \( D \) for all \( P \).

The \( \phi_p \) family of optimality criteria has \( A, D, E \) - criterion as particular cases.

\[
\begin{align*}
\phi_p & \to 1 \quad (C_d) = \phi_A(C_d) \\
\phi_p & \to 0 \quad (C_d) = \phi_D(C_d) \\
\phi_p & \to \infty \quad (C_d) = \phi_E(C_d)
\end{align*}
\]

If \( d^* \) is universally optimal in \( D \) then \( \text{tr} \left( C_{d^*} \right) \geq \text{tr} \left( C_d \right) \) for any other \( d \in D \) i.e., maximization of \( \text{tr} \left( C_d \right) \) is a necessary condition for universal optimality.

1.5.8 S-and (M-S) Optimality

Shah (1960), Eccleston and Hedayat (1974) have defined the S and (M-S) optimal in the following manner.

A design \( d^* \in D \) is called S-optimal if \( d^* \) minimizes

\[
\text{tr} \left( C_d^2 \right) = \sum_{i=1}^{\mu_d} \mu_{di}^2 \text{ for all } d \in D .
\]

Let \( t_\mu(C_d) = A, \) a constant, for all \( d \in D \). Then a balanced design has all its eigenvalues equal to \( A/(\mu-1) \). It is possible that such a balanced design do not exist. In that situation an S- optimal design would be one, which is "Closest" to a balanced
design. For this the Euclidian distance between \((\mu_{di}, \mu_{di}, ..., \mu_{dv-1})\) and \((A/v-1)\) is minimized and an s-optimal design \(d^*\) is obtained.

\[
\text{Distance} = \left\{ \sum_{i=1}^{v-1} \frac{\mu_{di}^2 - A^2 / (v-1)}{i} \right\}^{1/2}
\]

A design \(d^* \in D\) is said to be (M-S) - optimal if

\[
\max_{d \in D} \text{tr}(C_d) = \text{tr}(C_d^*) \quad \text{where} \quad d \in D
\]

and

\[
\min_{d} \text{tr}(C_d^2) = \text{tr}(C_d^*) \quad \text{where} \quad d \in D'
\]

Where \(D'\) is the sub-class of all designs \(d \in D\) for which \(\text{tr}(C_d)\) is maximum.

\[
\text{Distance} = \left\{ \sum \mu_{di}^2 - \left( \sum \mu_{di} \right)^2 / (v-1) \right\}^{1/2}
\]

1.5.9 Special Cases:

- A - Criteria \(f(x) = 1/x\)
- D - Criteria \(f(x) = -\log(x)\)
- S - Criteria / (M-S) - Criteria \(f(x) = x^2\)

Distance Criterion

Sinha (1970) has studied a specific optimality criterion based on concept of distance choose a design \(d^* \in D\) such that

\[
P\left[ | \eta_{d^*} - \eta | < \epsilon \right] \geq P\left[ | \eta_d - \eta | < \epsilon \right] \text{ for all } \epsilon > 0 \text{ and } d \text{ is any other design in } D.
\]

1.6 OPTIMAL DESIGNS

Kiefer (1975) has defined optimal designs as follows. Suppose \(d^* \in D\) and \(C_{d^*}\) satisfies

a) \(C_{d^*}\) is completely symmetric, i.e., \(c_{d^*} = a v + b J_0\)

b) \(\text{tr}(C_{d^*}) = \max_{d \in D} \text{tr}(C_d)\)

then \(d^*\) is universally optimal in \(D\)
1.6.1 Completely Randomized Design

For a design \( d \in \mathcal{D}(u,n) \) with \( n/v = r \), an integer

\[
\text{tr}(C_d) = \text{tr}(R_d^{-1}r_d' r_d') = n^{-1}\sum_{i=1}^{v-1} r_i^2 d_i
\]

\[
\min \sum r_i^2 d_i \text{ such that } \sum r_i d_i = n \text{ is attained when } r_i d_i = n/u = r. \text{ So, a design } d^* \text{ with } r_i^* d_i = r \forall i \text{ is universally optimal in } \mathcal{D}(u,n).
\]

1.6.2 Block Design

Let \( d \) be a block design with \( u \geq 3 \) treatments and \( b \) blocks, each of size \( k \geq 2 \).

Then \( d \) is called a balanced block design (BBD) if

\begin{align*}
\text{i) } & \sum_{j=1}^{b} n_{dij} n_{dmi} = \lambda, \quad \text{for } i \neq m \\
\text{ii) } & | n_{dij} - k/v | < 1, \quad \forall i,j.
\end{align*}

Condition (i) \( \Rightarrow n_{dij} = [k/v] \) or \([k/v] + 1 \) and (ii) \( \Rightarrow \) variance balance.

Also, conditions (i) and (ii) \( \Rightarrow \) BBD is equireplicate design.

If \( k < v \), then \( n_{dij} = 0 \) or \( 1 \) and BBD is a BIBD

A BBD \( d^* \in \mathcal{D}(u,b,k) \) is universally optimal in \( \mathcal{D}(u,b,k) \)

\[
\text{tr}(C_d) = \text{tr}(R_d - K^{-1}N_d N'_d) = bk - k^{-1}\sum_{i} n_{dij}^2
\]

Again \( \min \sum n_{dij}^2 \text{ s.t. } \sum n_{dij} = bk \) is attained when \( n_{dij}'s \) are as nearly equal as possible, i.e., when \( n_{dij} = [k/v] \text{ or } [k/v] + 1 \). When the class of designs does not contain any c.s. C-matrix with max, trace, Kiefer's sufficient condition of optimality cannot be used. Yeh (1986) has generalized Kiefer's(1975) result in the following manner. Suppose a class \( C = \{C_d: d \in \mathcal{D}\} \) of matrices in \( B_{u,0} \) contain a \( C^* \) such that

\begin{align*}
\text{i) } & \forall d \in \mathcal{D}, \text{ and } C_d \neq 0, \text{ there exist scalars } a_{di} \geq 0 \ (i=1,...,m) \text{ satisfying} \\
& C^* = \sum_{i=1}^{m} a_{di} P_i C_d P_i^t
\end{align*}

\begin{align*}
\text{ii) } & \text{tr}(C_d) = \max_{d \in \mathcal{D}} \text{tr}(C_d)
\end{align*}

where \( m = v! \) and \( P_i \) is the ith permutation matrix. Then \( d^* \) is universally optimal in \( \mathcal{D}. \)
Class of Binary Designs to for d∈ \mathcal{D}_1, n_{di} = \lceil k/v \rceil or \lceil k/v \rceil + 1

Let \( u \geq 3 \), \( b = um + n \geq 2 \), \( k = u-1 \) where \( m, n \) are integers with \( m \geq 0 \) and \( 1 \leq n < u \), suppose \( d^* \) is a BIBD \((u,b = um, r = (u-1)m, k = (u-2) m)\) plus the last \( n \) distinct binary blocks of size \( u-1 \). Then \( d^* \) is Universally optimal in \( \mathcal{D}_1 (u,b,k) \)

**Example**

\( u=5, m=1 \) \( n=2 \).

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 3 & 3 & 2 & 2 \\
3 & 3 & 4 & 4 & 3 & 3 & 3 \\
4 & 5 & 5 & 5 & 5 & 4 & 5 \\
\end{array}
\]

As per Das, Gupta and Notz (1996), if there exists a universally optimal design in \( \mathcal{D} (u,b,k) \) then \( d^* \) is universally optimal over \( \mathcal{D} (u,b,k) \).

Let \( u \geq 3 \), \( b = um + n \geq 2 \), \( k = ux + 1 \) with \( x \geq 0 \), \( m \geq 0 \), and \( 1 \leq n < u \).

Suppose \( d^* \) is a BBD \((u,b = um, k = uk+1)\) plus the last \( n \) distinct binary blocks of size \( u x + 1 \). Then \( d^* \) is universally optimal over \( \mathcal{D}_1 (v,b,k) \).

**Example:** \( u = 5, x = 1, m=1, n=2, b=7, k=6 \)

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 2 & 3 & 4 & 5 & 1 & 2 \\
\end{array}
\]

1.6.3 MB-GD Design

i). \( n_{di} = m \lceil k/v \rceil or \lceil k/v \rceil + 1 \)

ii). \( r_{ai}'s \) are all equal.

iii). The treatments can be divided into \( m \) groups of \( n \) each such that \( \lambda_{ai} = \lambda_1 \) if \( i \) and \( i' \) are in the same group, and \( \lambda_{ai} = \lambda_2 \) otherwise. Here \( NN' = ((\lambda_{ai})), i,i' = 1, \ldots, u \)
iv). $\lambda_2 = \lambda_1 \pm 1$, then the design $d$ is said to be Most Balanced group divisible design (MB-G DD)

Takeulchi (1961, 1963) has analyzed that any GDD, with $\lambda_2 = \lambda_1 + 1$ is $E$-optimal in $D\ (\nu, b, k)$ and later Cheng (1978) has proved that MB-GDD with $m=2$ and 

$\lambda_2 = \lambda_1 + 1$ is $\Psi$ - optimal in $D\ (\nu, b, k)$, when $m=2$, a MB - GDD has 2 distinct eigenvalues with multiplicities $1$ and $\nu - 2$. Also cheng (1980) studied that MB-GDD with $m=2$ and $\lambda_2 - \lambda_1 - 1$ is $E$-optimal in $D\ (\nu, b, k)$.

### 1.6.4 Optimality of Dual Designs

Let $d \in D\ (\nu, b, k)$, with $bk/\nu = r$ be an equireplicate design with $N_d$, then $d^*$ is said to be dual of $d$ if $\bar{N}_d = N^\prime a$ and then $\bar{d} \in D\ (\bar{\nu} = b, \bar{b} = \nu, \bar{k} = r)$ and is an equireplicate design with $r = k$.

$$C_{\bar{d}} = r I - k^{-1} N_d N^\prime_d$$

$$C_{\bar{d}}^r = r^{-1} N^\prime_d N_d$$

$$\mu_{\bar{d}} b = k_\mu / r, \quad 1 \leq i \leq \nu - 1$$

(and if $\nu < b$) $= k, \quad \nu \leq i \leq b - 1$

Thus, if $d$ is $\Psi$ - optimal in $D^*\ (\nu, b, k, r)$ the $d$ is $\Psi$ - optimal in $D^*\ (\nu = b, b = \nu, k = r, \nu = r)$. Hence equals of BBD’s one $\Psi$ - optimal in the equireplicate class of designs. Cheng (1980) has established the duals of BBD’s are $E$ - and $D$ - optimal in the unrestricted class $D\ (\nu, b, k)$.Constantine (1981) has used "averaging technique" to show optimality of certain designs in situations when $bk/\nu$ is not an integer.

### Averaging technique:

For any $C_d$, let

$$C_{\bar{d}} = (1/n) \sum_{i=1}^{n} C_{d'} P_i$$

$$= (1/n) \sum_{i=1}^{n} P_i C_d P_i^\prime$$
Where \( \{ \sigma_i \} \) denotes a collection of \( n \) permutations on the symbols \( 1, \ldots, v \); and representing the \( v \times v \) matrix representation of \( \sigma_i \)

Then

\[
\sum_{i=1}^{v-1} f(\mu_i) \geq \sum_{i=1}^{v-1} f(\mu_{-i})
\]

If we are able to express the r.h.s. of the above inequality in terms of the parameters \( u, b, k \) we would then get a lower bound for \( \sum f(\mu_d) \) and identify a design \( d^* \) which attains this bound.

For \( d \in D(u, b, k) \),

\[
\mu_d \leq r(k-1)u/u-1)k
\]

Constantine (1981) has established that

let \( d^* \in D(u, b, b+x, k) \) be a design obtained by adding \( x \) disjoint blocks to a BIB \( (u, b, k) \) or a GDD with \( \lambda_2 = \lambda_1 + 1 \). Then \( d^* \) is E-optimal in \( D(u, b = b+x, k) \) provided \( x < v/k \) (in case of BIBD) and provided \( x < (v-m)/k \) (in case of GDD).

Similarly, \( d^* \in D(u, b = b-x, k) \) obtained by deleting \( x \) disjoint blocks from a BIB \( (u, b, k) \) E-optimal in \( D(u, b, k) \) provided \( x < v/k \).

Sathe and Bapat (1985) have established by deleting any \( x \) blocks \( x \leq (v - \sqrt{v}/v-k) \) from a BIB \( (u, b, k) \) yield an E-optimal design in \( D(u, b = b - x, k) \).

JACROUX (1983) has showed that

for \( d \in D(u, b, k) \) with \( bk = v - r + s \), \( 0 \leq s \leq u \), \( r(k-1) = (v-1)\lambda + t \), \( 0 \leq t < v-1 \). Then for \( d \in D(u, b, k) \),

\[
\mu_d \leq \left( r(k-1 + \lambda) \right)/k, \quad \text{provided} \quad u \leq (v-s)(v-t).
\]

According to DAS (1993)

let \( d^* \in D(u = v - p, b = b + x, k) \) be a design obtained from BIB \( (u, b, r, k, \lambda) \) by adding \( x \) arbitrary blocks and collapsing \( p+1 \) treatments into one. Then \( d^* \) is E-optimal in \( D(u, b, k) \) provided

\[
\begin{align*}
v - pr - xk & \geq 1 \\
v - p\lambda & \geq 2 \\
v & < (v - pr - xk)(v - Prlambda)
\end{align*}
\]
E-optimal in \( D(12, 14, 4) \). Similar results hold when starting from a GDD with \( \lambda_2 = \lambda_1 + 1 \). To obtain A-optimal and D-optimal of designs in \( D(u, b, k) \) with \( K = \nu - 1 \) averaging technique is used by Das and Notz (1995).

Let \( d^* \in D(\nu, b = b-s, k) \) be a design obtained by deleting \( s \) distinct blocks from a BIB \( (\nu, b = \mu, r, k, \lambda), m \geq 1 \).

Then, \( d^* \) is D-optimal in \( D(\nu, b, k) \) if \( s > \sqrt{\nu} \) or \( s = 1 \).

\( d^* \) is A-optimal in \( D(\nu, b, k) \) if \( s = 1 \).

or \( s > \sqrt{\nu} + 1 \) and \( \nu < 99 \)

or \( s > \sqrt{\nu} + 2 \) and \( 99 \geq \nu < 575 \)

1.6.5 Minimally connected Designs (MCD)

If designs belong to \( D_b(\nu, b, k) \) with \( b \) \( k = \nu + b - 1 \), more than one MCD designs belong to \( D_1(\nu, b, k) \) with \( bk = \nu + b \).

According to Mandal, Shah, Sinha (1991) if \( d^* \in D_0(\nu, b, k) \) such that any one treat appears in each of the \( b \) binary blocks, then \( d^* \) is A-optimal in \( D_0(\nu, b, k) \). Such a design has also been shown to be D- and E-optimal by Bapat and Dey (1991).
Bala Subramaniam and Dey (1996) used graph-theoretic methods to establish $D$-optimality of certain types of designs in $\mathcal{D}(u, b, k)$.

### Example $u = 19, b = 6, k = 4.$

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### Example $u = 15, b = 5, k = 4$

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$D$-optimal in $\mathcal{D}(15, 5, 4)$

### 1.7 HISTORICAL RECOLLECTION OF n-ary BLOCK DESIGNS.

When all the existing incomplete block designs are binary, that is when either a treatment does occur only once or does not occur at all in a block, Tocher (1952) has considered a generalization of the structure of these designs by allowing a treatment to occur more than once in a block. In this design giving equal accuracy for all different treatment comparisons, he allowed the elements of the incidence matrix to take the values either 0, 1 or 2 and for the first time introduced the concept of balanced ternary designs. He did not stipulate that each treatment should occur in the design a constant number of times, and in fact that the replication number for each treatment was allowed to vary between two limits determined by parameters of the design. By trial and error method, he has constructed some Balanced Ternary Block designs (BTBD) with varying replications with a constant block size. Further, he has also contemplated for allowing the elements of the incidence matrix of the
design to take the values 0,1,2,3.........(n-1), and he named such designs as n-ary designs.

Tocher (1952) has defined a balanced n-ary block (BNBD) design as an arrangement of V treatments in B blocks each of size K, such that the i-th treatment occurs in the j-th block n_{ij} times, and altogether R times where n_{ij} can take values 0,1,2,......(n-1). Such a design is named as variance balanced if the inner product of any two row vectors of the incidence matrix N_{vxB} of the n-ary design, \( \sum_{j=1}^{B} n_{ij}n_{kj} \) is a constant and equal to \( \Lambda \) (say) for all \( i \neq k = 1,2,...v \).

This implies also that \( \sum_{j=1}^{B} n_{ij}^2 = \Delta \) (another constant) for all \( i = 1,2,...,V \). According to Hedayat and Federer (1974), n-ary block design is said to be pairwise balanced if \( NN' = D(\Lambda) + \Lambda J \), where \( N' \) is the transpose of the incidence matrix \( N \) and \( D \) is a diagonal matrix with elements \( \Delta \), \( \Lambda \) a scalar and \( J \) a matrix with unit entries everywhere. A BNBD design is said to be incomplete if any of the entries in the incidence matrix \( N \) is zero; otherwise it is said to be complete.

For a basic N-ary balanced incomplete block (BNBIB) design with parameters \((v, b, r, k; n_{ij} = 0, 1,2,...(N-1))\), and v x b incidence matrix \( n \), Shafiq and Federer (1979) have defined a Generalized N-ary Balanced Block (GNBB) design with parameters \((v, b, r^*, k^*, \lambda^* n^*_{ij})\), with \( n^*_{ij} \) in the set \((m_0, m_1,...,m_{N-1})\) where \( m_a = am_1 - (a-1)m_0 \) for \( a = 0, 1,2,...N-1 \), to be an arrangement of V treatments in B blocks each of size \( K^* \) (\( K^* \) not necessarily less than \( V \)) such that its incidence matrix is defined by \( n^* = n(m_1 - m_0) + Jm_0 \) where \( J \) is a \((v \times b)\) matrix with unit elements everywhere and \( 0 \leq m_0 \leq m_1 \).

Now, one can define a Generalized N-ary partially balanced block (GNPBB) design as follows without having any confusion between incidence matrices of balanced and partially balanced designs.

For a basic N-ary partially balanced incomplete block design with parameters \((v, b, r, k, \lambda_0; \lambda_1, n_{ij} P_{ij} = (P^a)_{jk})\) \( \alpha, j, k = 1, 2,...,m; n_{ij} = 0, 1,2,...,N-1 \) and v x b incidence matrix \( n \), a generalized N-ary partially balanced block design with
parameters \([v, b, r^*, k^*, \lambda_0^*, \lambda_0^*, n, P, \alpha = (P^\alpha)\), \(\alpha, j, k = 1, 2, \ldots, m; n^*_{ij} = m_0, m_1, m_2, \ldots, m_n\] is defined to be an arrangement of \(v\) treatments in \(b\) blocks each of size \(k^*\) (\(k^*\) not necessarily less than \(v\)) such that its incidence matrix is defined by

\[n^* = n (m_1 - m_2) + Jm_0\]

there \(J\) is a \((v \times b)\) matrix with unit elements everywhere and \(0 \leq m_0 \leq m_1\)

For the first time Paik and Federer (1973) have introduced the concept of Partially balanced \(N\)-ary Block (PBNB) designs which reduced the number of replicates required of each treatment in our BNB designs of Tocher (1952). Attempt has been made Soundarapandian (1980) taking the above BNPBB designs to Generalized \(n\)-ary partially Balanced Block (GNPBB) designs which have the advantage of reducing the number of replicates required by each treatment in GNB designs of Shafiq and Federer (1979). The number of frequencies of treatments in a block of GNPBB design will be non-negative integers.

\[m_a = am_1 - (a-1)m_0, a=0, 1, 2, \ldots, (N-1),\]

such that \(0 \leq m_0 < m_1 < \ldots < m_{N-1}\)

The GNPBB designs presented herein will be useful for the situation of within block variance is a constant for block size \(k\) over a range of block sizes \(0 \leq k < b\) in experimentation as presented by Shafiq and Federer (1979) for balanced designs.

1.8 ANALYSIS OF BALANCED AND PARTIALLY BALANCED \(n\)-ary BLOCK (BNB AND PBNB) DESIGNS

Yates (1940) has suggested the method of combined intra and inter block analysis for binary Incomplete block designs and later derived expressions in the case of E-optimal \(n\)-ary block designs. Considering the BNB designs of Tocher (1952) one can suggest and extend Rao's (1956) binary method to \(n\)-ary designs which suitably admit easy estimation of block parameters in the case of E-Optimal \(n\)-ary Block designs and other similar E-Optimality of partially balanced \(n\)-ary block designs.

The usual linear additive model is

\[Y_{ijk} = \mu + i + b_j + e_{ijk}\] (1.8.1)

where

\[Y_{ijk} = \text{the yield to the } k\text{-th plot in the } j\text{-th block of the } i\text{-th treatment}\]

\[i = 1, 2, \ldots, v;\]

\[j = 1, 2, \ldots, B;\]
\[ k = 1.2 \ldots n_i \]
\[ t_i = \text{the effect of the } i\text{-th treatment}, \]
\[ b_j = \text{the effect of the } j\text{-th block and} \]
\[ e_{ijk}'s \text{ are uncorrelated random variables having } E(e_{ijk}) = 0 \text{ and} \]
\[ V(e_{ijk}) = \sigma^2 \text{ for all } i, j \& k. \]

In the intrablock analysis of the above BNB designs the reduced normal equation for the estimation of treatment differences can be written, assuming equal replication for all treatments as

\[ Q_i = \frac{(RK-\Delta)}{k} \frac{1}{t_i} + \frac{1}{t_i} + \frac{1}{t_i} + \frac{1}{t_i} \frac{t_i}{k} \frac{t_i}{k} \quad (1.8.2) \]

with consistent equation \[ \sum_{j=1}^{k} t_i = 0 \] where

\[ Q_i \] is the total yield for the \( i \)-th treatment minus the sum of block means, in which it occurs,
\[ \Lambda \] is the number of block pairs (balanced blocks) in which the \( i \)-th and \( j \)-th treatments occur together, and
\[ R \] is \((RK-\Delta)\) where \( R \) = the number of replications, \( K \) is the block size, and
\[ \Delta \] is \[ \sum_{j=1}^{B} n_i^2 \] \text{ for every } \( i = 1, 2, \ldots, V. \)

The extended inter block analysis from binary (Rao, 1947) to n-ary is easily seen to be

\[ Q_i' = \frac{\Delta}{k} t_i + \frac{\Lambda_{i}}{k} t_i + \frac{\Lambda_{iv}}{k} t_v \quad (1.8.3) \]

Adding corresponding equations in (1.8.2) and (1.8.3) with respective weights \( w \) and \( w' \), we get the equations giving the combined estimates as

\[ P_i = R^* \frac{t_i}{k} + \frac{\Lambda_{i}}{k} t_i + \frac{\Lambda_{iv}}{k} t_v \quad (1.8.4) \]

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together with the consistent condition \( \sum_{i=1}^{V} t_i = 0 \) where

\[
R^* = \begin{bmatrix} (RK - \Delta) w + \Delta w' \\
\Lambda_i \\
P_i = w Q_i + w' Q_i'
\end{bmatrix} = \begin{bmatrix} R w + \Delta w' \\
\Lambda_i (w-w') \\
w Q_i + w' Q_i'
\end{bmatrix}
\]

\( Q_i = \) the sum of means of blocks in which the i-th variety occurs minus R times the grand mean,
\( w = \) the reciprocal of the estimated intrablock variance and \( w' \) that of the intrablock variance (Rao, 1947)

For balanced designs \( \Lambda_i = \Lambda_k \Rightarrow \Lambda_i = \Lambda_k \)

Solution of equations (1.8.2) as functions of Q, R and distinct \( \Lambda_i \) provides solution for the later combined equation (1.8.4) by writing Q (c), \( R^* \), \( \Lambda_i \) for Q, R, \( \Lambda_i \). The same is true for the expressions for variances. If \( (RK-\Delta) = \sum \Lambda_i \) explicitly used then \( R^* \) and \( \Lambda_i \) should be defined as

\[
R^* = \begin{bmatrix} (RK-\Delta) w + w' \{\Delta(V-K)/V}\end{bmatrix} \tag{1.8.5}
\]

and \( \Lambda_i = \Lambda_i (w-w') + (w' RK/V) \tag{1.8.6} \)

A certain amount of care may be necessary involving the actual examination of solution and methods instead of fully depending on the available published formulae.

Let us consider some examples.

1.8.1 The Balanced n -ary Block Designs

The intrablock normal equations are

\[
Q_i = \frac{(RK-\Delta)}{K} t_i - \frac{\Lambda}{K} \sum_{i=1, 2, \ldots, V} t_j, \quad i=1, 2, \ldots, V \tag{1.8.7}
\]

with \( \sum t_i = 0 \)

On simplification, we get

\[
t_i = \frac{K}{\Lambda_v} Q_i \quad \text{and} \quad \frac{V}{\Lambda_v} (t_i - t_j) = \frac{2K}{\sigma^2}
\]

The normal equations for corresponding combined intra/inter-block analysis is
\[ Q(c) = R^* - \sum_{j \neq i} t_j \quad i = 1, 2, \ldots, v, \sum t_i = 0 \]

when \( \Lambda = \Lambda(w - w') \) and \( R^* = \left[ (RK - \Delta)w + \frac{w'}{K} \right] \)

Here variance \( \sigma^2 \) has to be dropped to have difference in notation. The Partially balanced \( n \)-ary Block Designs (\( m=2 \)). Considering PBNB design with two associate (\( m=2 \)) classes Soundarapandian (1980 a), the intrablock equations are

\[ Q_i = \frac{(RK-\Lambda)}{K} t_i - \sum_{1}^{\Lambda_1} t_j - \sum_{2}^{\Lambda_2} t_j \]

and \( \sum t_i = 0 \) for \( i = 1, 2, \ldots, v \)

where \( \sum_{1}^{\Lambda_1} \) and \( \sum_{2}^{\Lambda_2} \) indicate the summation over first and second associates, respectively, of the \( i^{th} \) treatment.

By using the method of solving two associate PBNB designs [Soundarpandian (1980b)], we have

\[ \sum_{11} Q_i = A_{22} t_1 + B_{22} \sum_{11} t_j \]

where \( KA_{12} = (PK - \Delta) + \Lambda_2 \)

\( KB_{12} = \Lambda_2 - \Lambda_1 \)

\( KA_{22} = (\Lambda_2 - \Lambda_1) P_{12}^2 \)

\( KB_{22} = (RK - \Delta) + \Lambda_2 + (\Lambda_2 - \Lambda_1) (P_{11}^2 - P_{12}^2) \)

Solving, we get

\[ \frac{Q_i B_{22} + B_{12} \sum_{11} Q_i}{(B_{22} + B_{12}) Q_i + B_{12} \sum_{22} Q_i} = \frac{\Delta^*}{\Delta^*} \quad (1.8.8) \]

Where \( \Delta^* = A_{12} B_{22} - A_{22} B_{12} \)

Taking (1.8.8) for \( (n_2 < n_1) \), We get the various of difference
\( V(t_i - t_k) = \frac{2(B_{22} + B_{12}) \sigma^2}{\Lambda} \) if \( t_i \) and \( t_k \) are first associate

and

\( V(t_i - t_k) = \frac{2B_{22} \sigma^2}{\Lambda} \) if \( t_i \) and \( t_k \) are second associate.

Combined analysis (intra / inter block) can be obtained by changing \( Q, R, \Lambda \) by \( Q(c), R', \Lambda \) and dropping \( \sigma^2 \) as the expression for variance. More associate cases and other special cases of n - ary designs range also are considered as above.

The review of the remaining part of this thesis is summarized in the following lines.

In chapter II a brief review on latest optimal and n-ary block designs based on the literature available has been discussed. A strong reference is made regarding n-ary block designs. The developments of A-, D- and E-optimal designs have also been discussed.

In chapter III a deep attention has been paid to A-, D- and E-optimal n-ary block designs with unequal replications and equal block sizes. Further some more theorems and examples on various methods of constructions of A-, D- and E-optimal n-ary block designs are established.

In chapter IV more attention has been paid to A-, D- and E-optimal n-ary block designs with unequal replications and unequal block sizes. Further some definitions and lemmas on n-ary block designs have also been discussed. Also results on E-optimal, A-optimal and D-optimal and their constructions have been presented.

In chapter V several families of PBNB designs have been investigated. Theorems and lemmas for various bounds for BNB and PBNB designs are also presented. The new findings of A-, D- and E-optimal PBNB designs have been discussed.

In chapter VI the generalization work of Shafig and Eedecen(1979) for BNIB designs is presented. Definitions, theorems and constructions of A-, D- and E-optimal have also been discussed.