CHAPTER IV
STORAGE MODEL

Every business concern or industrial establishment maintains an inventory of various equipments and articles of daily use with the express purpose of meeting the demand of customers and in some cases to keep buffer stocks also for emergency. The purpose of elementary statistical treatment is to spot out the items which are in heavy demand and also those which are primarily idle or more slow. The inventory models are characterized by an ordering policy, the supply of material being under control or subject to randomness. We now describe a class of models in which both the supply of material (the input) and the demand (output) are random variables, and the objective is to regulate the demand so as to achieve a storage of desirable level. We describe these input--output models or storage models in section 4.1, we also give simple $(S, s)$ inventory policy with the specific probability distribution of storage. We elaborate our discussion with the work on deep dam by Phatarfod [49] and an embedded level crossing technique adopted by Brill [7]. Smith and Yeo [60] have taken up the study of the storage models under the purview of GI/G/r(x) queue models. Their work with detail derivation is presented in section 4.2. A study on $(S, s)$ type policy for controlling the cash level is presented in the last section.

SECTION 4.1. SIMPLE STORAGE MODEL

Determination of storage level

Let $Z_t$ is the storage level at time $t$ just before an input $X_t$, $Y_t$ is the release at the end of the time period $(t, t + 1)$, and $k$ is the maximum capacity which can be held. Then the model for storage problem is as follows
\[
Z_{t+1} = \begin{cases} 
Z_t + X_t - Y_t & \text{if } Z_t + X_t - Y_t < k, \\
k & \text{if } Z_t + X_t - Y_t \geq k, 
\end{cases} \tag{1}
\]

Let \( F(x) = \Pr(Z_t \leq x) \), \( G(x) = \Pr(Y_t \leq x) \), and let \( X_t = m \), a constant, then on the assumption that an equilibrium distribution of \( Z_t \) exists and that there is no back-logging of orders, we have the following integral equation (Ghosal [19]):

\[
F(x) = 1 - G(m+k-x) + \begin{cases} 
\int_{x-m}^{k} F(t) \, d G(m + t-x) & \text{if } m < x < k, \\
\int_{0}^{k} F(t) \, d G(m + t - x) & \text{if } x < m. 
\end{cases} \tag{2}
\]

If \( Y_t \) follows gamma distribution,

\[
dG(x) = \frac{\lambda^p}{(p-1)!} x^{p-1} e^{-\lambda x} \quad (0 \leq x < \infty ; \ p > 0), \tag{3}
\]

we can determine \( F(x) \) by applying a transformation

\[
\phi(x) = F(x) \ e^{\lambda x} \tag{4}
\]

whence we get (where \( mn \leq x < (n+1) m ; n = 0, 1, \ldots \)),

\[
\phi(x) = \sum_{r=0}^{p-1} d_r \sum_{s=0}^{n} e^{\lambda sm} \frac{\{-\lambda \ (x - sm)\}^{rsp}}{(r + sp)!} \tag{5}
\]

where

\[
d_r = e^{-\lambda (m+k)} (-\lambda)^r \sum_{s=0}^{p-1} \frac{(\lambda (m+k))^{s}}{s!} + \frac{\lambda^p e^{-\lambda m}}{(p-1)!} (-1)^r \int_{0}^{k} \phi(t) \ (m+t)^{s-1} \ dt \quad (r = 0, 1, \ldots, p-1)
\]
Probability of Emptiness

An important practical problem is to determine, for a storage system governed by a specific policy of replenishment or release, the probability that the store becomes empty before filling completely when the level is \( x (0 < x < k) \). We call this probability \( V(x) \). It has been proved by Ghosal [20] that the probability \( V(x) \) bears the following relationship with

\[
F(x) = \Pr(Z_t < x), \\
V(x) = F(k-x) (0 < x < k) \quad .... (6)
\]

This implies that the probability \( V(x) \) that at level \( x \) the system goes empty before filling completely is the same as the probability that the level is less than \( k - x \), i.e., the amount by which it falls short of the capacity at present. The calculation of \( V(x) \) for various values of \( x \) enables us to determine the risk level. We define the risk level \( x_o \) as that level for which \( V(x_o) \) is high, say 0.80 or 0.90. The risk level so defined can also be interpreted as the buffer stock in inventory problems.

An Optimization Problem

Next we deal with the problem of determining the optimum storage capacity of a system if we know the distribution functions of input and release and the cost factors, viz. the cost of storing an item for unit time, the capital cost of building a storage space for \( k \) items (for various values of \( k \)), etc. If the capital cost of constructing unit storage space be \( c \), the cost of maintenance and repair for a storage system of unit capacity be \( h \), and \( \beta \) be the revenue as rental for storing an item for unit time, then the present value of cost for a system of capacity \( k \) over time horizon \((0, T)\)
\[ ck + hk \int_0^T e^{\delta t} \, dt, \quad \ldots (7) \]

where \( \delta \) is the force of interest; hence present value of earnings.

\[ \beta \int_0^T e^{\delta t} Z_t \, dt, \quad \ldots (8) \]

so that present value of net earnings over \((0, T)\) is

\[ f(k) = \beta \int_0^T e^{\delta t} Z_t \, dt - [ck + hk \int_0^T e^{\delta t} \, dt]. \quad \ldots (9) \]

A more general version of the cost function \( C(k) \), which is not necessarily linear in \( k \), can also be assumed in (9). An optimization problem which is of interest here is to maximize the expected value of \( f(k) \) with respect to \( k \). Writing \( E(.) \) for the expected value, we get

\[ E[f(k)] = \beta E(Z_t ; k) \int_0^T \exp (-\delta t) \, dt \cdot \]

\[ -[ck + hk \int_0^T \exp (-\delta t) \, dt] \]

\[ = \beta E(Z_t ; k) \bar{a}_T - ck - hk \bar{a}_T \quad \ldots (10) \]

where \( \bar{a}_T \) stands for continuous annuity of unit amount for a time horizon of \( T \) years. For approximation we may assume \( a_T \) for \( \bar{a}_T \); in other words, we assume that the costs and benefits are incurred once a year.

We assume that \( Z_t \), for a given capacity \( k \), follows a stationary probability distribution. Theoretically the optimal \( k \) should be determined by equating \( E'[f(k)] \) to zero, i.e.,

\[ 0 = dE[f]/dk = \beta a_T [dE (Z_t ; k)/k] - c - h a_T. \]
However, determination of \( \frac{dE(Z;k)}{dt} \), when \( k \) is continuous, is a difficult problem in most of the practical problems. We therefore solve the problem by assuming \( k \) as discrete (integral valued) and demand distribution also discrete.

Then we get

\[
E(Z_t; k) = \sum_{i=0}^{k-1} V_i^{(k)}
\]

where \( V_i^{(k)} = 1 - F_i^{(k)} = \text{Prob} (Z > i/capacity = k), i = 0, 1, \ldots \)

We try to estimate a value of \( k \) such that

\[
E[f(k)] \sim E[f(k+1)]
\]

We get

\[
\Delta E(Z; k) = V_k^{(k+1)} + \sum_{i=0}^{k-1} \{ V_i^{(k+1)} - V_i^{(k)} \},
\]

whence we derive optimal \( k(k^*) \) such that

\[
0 = E[f(k)] = \beta a_k \Delta E(Z; k) - c - ha_T.
\]

Thus we have evolved an algorithm for solving the optimization problem associated with the storage problem.

**Illustration 4.1.1 Grain storage problem**

Now we shall discuss a grain storage problem with the definition of \( Z_t, X_t \) and \( Y_t \) as given above, Rosenblatt's [53] model assumes \( k = \infty \) and \( Y_t = (1-d) (Z_t + X_t) \), i.e. a fixed proportion of the storage, existing just after an input \( X_t \) is released for consumption. Hence
\[ Z_{t+1} = d \left( Z_t + X_t \right). \]  \hspace{1cm} \ldots (13)

The problem is to choose \( d \) such that the cost of storage is minimum.

From (12) we get:

\[
Z_{t+1} = d(Z_t + X_t) = dX_t + d^2 (Z_{t-1} + X_{t-1}) = dX_t + d^2 X_{t-1} + d^3 X_{t-2} + \ldots + d^i X_i.
\]

Since \( X_i \)'s have all identical distribution, we assume that in the long run the distribution function of \( Z_t \) is the same as that of \( dX_1 + d^2 X_2 + \ldots \); in other words

\[ \Pr(Z_t \leq y) = \Pr(dX_1 + d^2 X_2 + \ldots \text{ ad inf} \leq y). \]

If \( E(X_t) = m, \ \text{var} (X_t) = \sigma^2 \), then on the assumption that \( X_i \)'s are mutually independent, we have

\[
E(Z_t) = \frac{md}{1 - d}, \ \text{var} (Z_t) = \frac{d^2 \sigma^2}{1 - d^2}, \ \text{and}
\]

\[
E(Y_t) = (1 - d) \{E(Z_t) + E(X_t)\} = m, \ \text{var} (Y_t) = \frac{1 - d}{1 + d} \sigma^2.
\]

Let the cost of storage be \( h \) per unit, \( K_1 \) a constant cost not depending on the amount to be stored, and let there be a cost component which depends only on the deviation of the actual amount of release from the desired amount of release \( m \). Thus, we can write the cost function as

\[ L(Z_t) = hZ_t + K_1 + K_2 (Y_t - m)^2, \]  \hspace{1cm} \ldots (14)

\( (K_2 \text{ is being a constant}). \)

The expectation of this quantity is given by

\[ E \{ L(Z_t) \} = hE(Z_t) + K_1 + K2 \text{ var} (Y_t) \]
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\[
= K_1 + \frac{hmd}{1-d} + K_2 \sigma^2 \frac{1-d}{1+d} 
\]

\[
= K_1 + \frac{hm}{1-d} + \frac{2K_2 \sigma^2}{1+d} - (hm + K_2 \sigma^2).
\]

Differentiating (15) with respect to \( d \) and equating to zero, we get optimum \( d \) as

\[
d_o = \begin{cases} 
1 - \frac{2}{1 + \sqrt{2K_2 \sigma^2/hm}} & \text{if } 2K_2 \sigma^2 > hm, \\
0 & \text{if } 2K_2 \sigma^2 \leq hm
\end{cases}
\]

**Determination of the optimum capacity of a storage system**

Bose [6] has developed an algorithm for determining the optimum capacity of a storage system when the demand is geometric. Here we illustrate his algorithm on \((s, S)\) policy.

**The \((s, S)\) Policy**

It is assumed that there is no time lag between the placing of orders and compliance with them. We presume that within each time period the ordering is performed at the end of the period while the delivery is made before the beginning of the next. For \((s, S)\) policy we have the transition law

\[
Z_{t+1} = \begin{cases} 
Z_t - Y_t & \text{if } s < Z_t \leq S \\
S - Y_t & \text{if } Z_t \leq s
\end{cases}
\]

Assume \(\{Z_t\}\) to be the steady state. Let

\[
P(Z_t = i) = f_i
\]

\[
P(Z_t \geq i) = \sum_{j=i}^{S} f_j = V_i
\]
and

\[ P(Y_t = j) = g_j, \]

We will get for (16) the relation

\[ f_i = \begin{cases} 
(1 - V_{s+1}) g_{s-i} + \sum_{j=i}^{s} f_j g_{j-i} & (i > s + 1) \\
(1 - V_{s+1}) g_{s-i} + \sum_{j=s+1}^{s} f_j g_{j-i} & (i \leq s + 1) 
\end{cases} \]

We discuss the particular case of geometrically distributed demand. The case of negative binomial demand will be discussed elsewhere.

**Geometric demand**

Let \[ g_j = a b^j \quad (j = 0, 1, \ldots; a = 1 - b) \] .... (18)

Then, from (17) and (18), we will get

\[ f_i = \begin{cases} 
(1 - V_{s+1}) a b^{s-i} + a \sum_{j=i}^{s} f_j b^{j-i} & (i > s + 1) \\
(1 - V_{s+1}) a b^{s-i} + a \sum_{j=s+1}^{s} f_j b^{j-i} & (i \leq s + 1) 
\end{cases} \]

Using the transformation \( f_i = b^i \phi_i \) in (19), we get

\[ \phi_i = \begin{cases} 
ab^s (1 - V_{s+1}) + a \sum_{j=i}^{s} \phi_j & (i > s + 1) \\
ab^s (1 - V_{s+1}) + a \sum_{j=s+1}^{s} \phi_j & (i \leq s + 1) 
\end{cases} \]
By proceeding on the same line as Ghosal [19], we can derive the following:

\[
f_i = \begin{cases} 
\alpha b^i & (i \leq s + 1) \\
\alpha b^{(s+1)} & (i > s + 1)
\end{cases}
\]

where

\[\alpha = ab^s (1 - V_s) + \sum_{j=s+1}^{s} \phi_j\]

We can evaluate the value of \(\alpha_i\) (i) when \(Z_t\) can be negative and (ii) when \(Z_t\) cannot be negative.

When \(Z_t\) cannot be negative, we have following Ghosal [19]:

\[\alpha = ab^{s+1} / \{ b - b^{s+2} + a(S - s) \}\]

When \(Z_t\) can be negative, i.e., when backlogging is allowed, we get

\[\alpha = ab^{s+1} / \{ b + a(S - s) \}\]

**Algorithm**

We consider here the discrete time \(t = 0, 1, 2, ...\) Our problem is to build up the economics of the system, given the capital cost of creating the storage capacity, the revenue from the system, etc. Let \(\beta\) be the revenue as rental for storing for a unit time; \(h(S)\), the cost of maintenance and repair of a storage capacity \(S\); \(c(S)\), the capital cost of building it at present rates.

Our basis of optimizing the capacity \(S\) is to calculate the present value of earning from the storage system and the outlay during the time horizon \((0, T)\). If \(\epsilon\) be the force of interest, we can write the present value of net profit over the
period \((0, T)\) from the system of capacity \(S\). Following Ghosal [21], after simplifications for discrete time, we get the present value of net profit \(N(S)\) as

\[
N(S) = \beta \sum_{t=1}^{T} e^{-et} Z_t - h(S) \left\{ \frac{1-e^{-eT}}{e} \right\} - c(s)
\]

Taking expectation on both sides, we obtain

\[
E\{N(S)\} = \beta \ E(Z_t) \left\{ \frac{1-e^{-eT}}{e} \right\} - c(S) - h(S) \left\{ \frac{1-e^{-eT}}{e} \right\}
\]

\[
\ldots (20)
\]

We will also use the relation for optimum \(S'\),

\[
E \{ N(S) \} \sim E \{ N(S+1) \}
\]

\[
\ldots (21)
\]

We must calculate the value of \(E(Z_t)\) to determine the optimum value of \(S\) given \(s\). This can be done in two different ways:

**Case 1.** Let \(Z_t\) can be negative, Then

\[
E(Z_t) = \sum_{i=-\infty}^{s} i \alpha b^{i} + \sum_{i=s+1}^{S} i (\alpha b_{i-1}^{s+i})
\]

\[
= \left\{ S(S+1) - s(s+1) \right\} a^2 + 2abs - 2b^2 \right\} / 2a \ {b + a(S-s)}
\]

\[
\ldots (22)
\]

If \(c(S) = cS\),

\(h(S) = hS\),

Using (20), (21) and (22), we obtain

\[
\beta \left\{ \frac{1-e^{-eT}}{e} \right\} \left\{ \frac{2a^2 \ (S+1) \ {b+a(S-s)} - a \ [ \ {S(S+1)-s(s+1)} \ a^2 + 2abs - 2b^2 ]} {2a \ {b + a(S-s)} \ {b + a(S+1-s)}} \right\}
\]

\[
= c + h \left\{ (1-e^{-eT}) / e \right\}
\]

\[
\ldots (23)
\]
Case 2. When $Z_t$ cannot be negative. Then

$$E(Z_t) = \frac{2abs + 2b^{s+2} - 2b^2 + a^2 \{S(S+1) - s(s+1)\}}{2a \{b-b^{s+2} + a(S-s)\}} \ldots (24)$$

If

$$c(S) = c.S,$$
$$h(S) = h.S,$$

Using (20), (21) and (24), we obtain

$$\beta \left\{ \frac{1-e^{-\epsilon T}}{\epsilon} \right\} = c + h \left\{ \frac{1-e^{-\epsilon T}}{\epsilon} \right\} \ldots (25)$$

Equations (23) and (25) can be used as a mechanism to determine the optimum value of store of capacity $S$ to any industry when $Z_t$ can and cannot be negative.

Continuous time dam model

Definition 4.1.1.

A process $\{X(t), t \geq 0\}$ is called a process with stationary independent increments if it satisfies the following properties:

(i) For $0 \leq t_1 < t_2 < \ldots < t_n (n \geq 2)$ the random variables

$$X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$$

are independent.

(ii) The distribution of the increment $X(t_p) - X(t_{p-1})$ depends only on the difference $t_p - t_{p-1}$. 
Definition 4.1.2

A Levy process is a process with stationary independent increments which satisfies the following additional conditions:

(i) \( X(t) \) is continuous in probability. That is, for each \( \varepsilon > 0 \)

\[ P \{ |X(t)| > \varepsilon \} \to 0 \text{ as } t \to 0^+ . \]

(ii) There exist left and right limits \( X(t-) \) and \( X(t+) \) and we assume that \( X(t) \) is right continuous: that is, \( X(t+) = X(t) \).

Basic storage model

Consider a dam of large enough capacity, and let \( X(t) \) denote the input of water into it during a time interval \((0, t]\). Our description of a process with stationary independent increments described earlier is in agreement with our intuitive concept of inputs into the dam. It is also natural to impose the regularity conditions on the input process. Thus we assume that \( \{X(t), t \geq 0\} \) is a Levy process. The c.f. of \( X(t) \) is given by \( e^{it\phi(\omega)} \), where

\[ \phi(\omega) = \int_{0^+}^{\infty} (e^{ix\omega} - 1) x^{-2} M(dx), \]

with \( M^+(0) = \int_{0^+}^{\infty} x^2 M(dx) = \infty \). Let the release from the dam be continuous and at a unit rate except when the dam is empty. If \( Z(t) \) denotes the content of the dam at time \( t \), then \( Z(t) = Z(0) + X(t) - t \) if the dam remains wet throughout the interval \((0, t]\).

For the storage model we define the random variable \( T \) as follows:

\[ T = \inf \{ t : Z(t) = 0 \}, \quad Z(0) > 0. \]
We shall call $T$ the busy period initiated by a workload $W(0)$ if the model describes the $M/G/1$ queue and we shall call $T$ the wet period initiated by a water level $Z(0)$ if our model describes the continuous time model. It is to be noted that $T$ has the same distribution as the random variable

$$T(x) = \inf \{ t : Y(t) \leq -x \} \ (x > 0).$$

Here $T(x)$ is the first passage time of the Levy process $Y(t)$ into the set $(-\infty, -x]$.

**Theorem 4.1.1 (Prabhu [50])**

For $\theta > 0$, $s > 0$ we have

$$\int_0^\infty e^{\theta t} E[e^{\theta t} | Z(0) = x] dt = \frac{1}{s} \left\{ 1 - e^{-\theta \eta} \frac{\theta}{\theta + \eta} \right\},$$

$$\int_0^\infty e^{\alpha t} E[e^{\beta t} | Z(0) = x] dt = \frac{e^{\alpha x} - \theta F_x^\ast (0, s)}{s - \theta + \phi(\theta)},$$

where $F_x^\ast (0, s) = e^{\alpha x} \eta^{-1}$.

**Theorem 4.1.2**

If $X(t)$ has an absolutely continuous distribution with density $k(x, t)$, then the random variable $T(x)$ has also an absolutely continuous distribution with density $g(t, x)$, where

$$g(t, x) = \frac{x}{t} k(t-x, t) \quad \text{for } t > x > 0,$$

$$= 0 \quad \text{otherwise.}$$
Proof

Let $M(t)$ and $m(t)$ be the supremum and infimum functionals of the net input process $Y(t)$. It is to be noted that $Z(t) \sim M(t)$, $z(0) = 0$. .... (26)

Using equation (26) we have from the above theorem

\[ r_+ (s, \omega) = s \int_0^\infty e^{st} E[e^{imM(t)}] \, dt = \frac{s}{\eta + i\omega} \cdot \frac{\eta + i\omega}{s + \Phi(\omega)} \]

\[ r_- (s, \omega) = s \int_0^\infty e^{st} E[e^{imM(t)}] \, dt = \frac{\eta}{\eta + i\omega} \]

where $M(t)$ and $m(t)$ are the supremum and infimum functionals of the net input process $Y(t)$ and $Ee^{i\omega Y(t)} = e^{i\Phi(\omega)}$. We therefore have

\[ \frac{s}{s + \Phi(\omega)} = r_+ (s, \omega) \cdot r_- (s, \omega) \quad (s > 0, \ \omega \ \text{real}) \quad .... (27) \]

This identity is a Wiener-Hopf factorization for the Levy process $Y(t)$ in the following sense. We have

\[ \frac{s}{s + \Phi(\omega)} = \exp \left\{ \log \frac{s}{s + \Phi(\omega)} \right\} = \exp \left\{ \int_0^\infty e^{st} t^{-1} E[e^{i\omega Y(t)} - 1] \, dt \right\} = \exp \left\{ \int_\infty^\infty (e^{i\omega x} - 1) v_s (dx) \right\}, \]

where

\[ v_s \{0\} = 0, \ v_s (dx) = \int_0^\infty e^{st} t^{-1} k(t+x, t) \, dt \, dx \quad (x \neq 0). \quad .... (28) \]
It is to be noted that for fixed $s > 0$, $\nu_s$ is a Levy measure, so that the expression on the left side of (27) is an infinitely divisible c.f. It can also be proved that for fixed $s > 0$, $r_+ (s, \omega)$ and $r_- (s, \omega)$ are also infinitely divisible c.f.'s of distributions concentrated on $(0, \infty)$ and $(-\infty, 0)$ respectively. Thus (27) is a Wiener-Hopf factorization of $s[s + \Phi (\omega)]^{-1}$, and this factorization is unique if restricted to infinitely divisible c.f.'s on the right side of (27).

Now fixed $s > 0$, $\eta (\eta + i \omega)^{-1}$ is the c.f. of the exponential density on $(-\infty, 0)$, which is infinitely divisible and has Levy measure with density $e^{\eta x} (-x)^{-1} (x < 0)$. On account of the uniqueness of the factorization (27) we therefore have from (28)

$$\frac{e^{\eta x}}{(-x)} \cdot \int_0^\infty e^{-st} t^{-1} k(t + x, t) \, dt \quad (x < 0)$$

or

$$e^{\eta x} = \int_0^\infty e^{-st} g(t, x) \, dt \quad (x > 0).$$

Since $e^{\eta x}$ is the L.T. of the random variable $T(x)$, it follows that $g(t, x)$ is the required density.

**Storage model with compound Poisson input process**

Consider the storage model in which the net input process $Y(t) = X(t) - t$, is a compound Poisson process. Let $T = \inf \{t : Y(t) > 0\}$ and $\bar{T} = \inf \{t : Y(t) < 0\}$.

**Theorem 4.1.3**

For a compound Poisson process $Y$ with zero drift and c.f. given by $E e^{i\omega Y(t)} = e^{i\phi (\omega)}$, we have the factorization
where

\[
s r_0(s) = \exp \left\{ -\int_0^{\infty} t^{-1} e^{-st} P\{Y(t) \neq 0\} \, dt \right\}.
\]

**Proof**

We have

\[
[1 - E(e^{sT+i\omega Y(T)})][1 - E(e^{st+i\omega Y(T)})],
\]

\[
= \exp \left\{ -\int_0^{\infty} t^{-1} e^{-st} E\left[ e^{i\omega Y(t)} \right] \, dt \right\}
\]

\[
= \exp \left\{ -\int_0^{\infty} t^{-1} e^{-st} E[e^{i\omega Y(t)-1}] \, dt \right\} \cdot \exp \left\{ -\int_0^{\infty} t^{-1} e^{-st} P\{Y(t) \neq 0\} \, dt \right\}.
\]

Since

\[
\int_0^{\infty} t^{-1} e^{-st} P\{Y(t) \neq 0\} \, dt \leq \int_0^{\infty} t^{-1} e^{-st} (1-e^{-\nu t}) \, dt = \log \left( \frac{\nu + s}{\nu} \right) < \infty,
\]

where \(\nu (0 < \nu < \infty)\) is the jump rate of the process, and

\[
\int_0^{\infty} t^{-1} e^{-st} E[e^{i\omega Y(t)-1}] \, dt = \int_0^{\infty} t^{-1} e^{-st} [e^{i\omega (s + \phi(\omega)) -1}] \, dt = \log \frac{s}{s + \phi(\omega)},
\]

\[
\text{R.H.S.} = \exp \left\{ -\log \left( \frac{s}{s + \phi(\omega)} \right) \right\} \cdot \exp \left\{ -\log \frac{\nu + s}{\nu} \right\}
\]

\[
= e^{\log s + \log \nu - \log (\nu + s)}
\]

\[
= r_0(s) (s + \phi(\omega)).
\]
The infinitely deep dam with a Markovian input

In the infinitely deep dam model we consider the depletion from the maximum content $K-M$ of the dam, where $K$ is the size of the dam and $M$ is the release at the end of the time interval $(n, n+1)$. Denoting the input during $(n, n+1)$ by $X_n$ and the depletion by $Y_n$, we have

$$Y_{n+1} = [Y_n + M - X_n]^+,$$

Let the inputs $\{X_n\}$ during the time interval $(n, n+1), (n = 0, 1, 2, \ldots)$ form an ergodic Markov chain over the space $(0, 1, 2, \ldots, r)$ with transition probability matrix $P = (p_{ij})$. Let us denote the stationary distribution of $X$ by $\nu = (\nu_0, \nu_1, \ldots, \nu_r)$ and let us assume that $E(X) > M$. We consider the operator $P(\theta) - \theta^M I$ where $P(\theta) = (p_{ij} \theta^j)$.

Singularities of $P(\theta) - \theta^M I$

Let us first consider $P(\theta) = (p_{ij} \theta^j)$ for non-negative values of $\theta$. From the Perron-Frobenius theory of non-negative matrices it is known that for the matrix $P(\theta)$ which is irreducible and aperiodic, the maximum eigenvalue $\lambda_0(\theta)$ is positive, simple and strictly monotonic increasing in $\theta$, the other eigenvalues $\lambda_i(\theta)$ ($i = 1, 2, \ldots, r$) being such that $|\lambda_i(\theta)| < \lambda_0(\theta)$. It is also known that $\lambda_0(\theta)$ is superconvex and hence continuous. Moreover $\lambda_0(0) = \rho_{oo}$, while $\lambda_0(1) = 1$. Further $\lambda_0'(1) = E(X)$.

If $E(X) > M$, the equation $\lambda_0(\theta) = \theta^M$ has a unique solution $\xi_1 (0 < \xi_1 < 1)$. $\xi_1$ is therefore one of the singularities of $P(\theta) - \theta^M I$. We may note in passing that $\theta^M > \lambda_0(\theta), (\xi_1 < \theta < 1)$. 


Let us now consider \( P(\theta) - \theta^M I \) for all values of \( \theta \).

If \( P \) is irreducible, the operator \( P(\theta) - \theta^M I \) has \((r+1)M\) singularities in the region \(|\theta| < 1\).

**Result 4.1.1**

Let us denote a singularity other than \( \xi_1 \) by \( \xi \). Then we have \(|\xi| < \xi_1\).

To see this we first state the following result on non-negative matrices.

**Result 4.1.2**

Let \( B = (b_{ij}) \) be a square matrix with complex elements and let \( B^* = (|b_{ij}|) \).

If \( \beta \) is any eigenvalue of \( B \), if \( A \geq 0 \) is irreducible and if \( B^* \leq A \), then \(|\beta| \leq \lambda_o\), the maximum eigenvalue of \( A \). Moreover, \(|\beta| = \lambda_o\) and \( B^* \leq A \) together imply that \( B^* = A \). Using this result and noting the nature of the matrix \( P(\theta) \), it is clear that \(|\lambda_i(\theta)| < \lambda_o (c), \) (\(i = 0, 1, 2, \ldots, r\)) where \( c \) is positive and real and \( \theta \) either negative or complex such that \(|\theta| \leq c\).

Consider now the contour \(|\theta| = \xi_1 + \delta\). On this contour we have \(|\theta|^{M > |\lambda_o(\theta)| > |\lambda_i(\theta)|}, \) (\(i = 1, 2, \ldots, r\)). Hence we have \(|\xi| < \xi_1\).

**Result 4.1.3**

To show that at most \( N = M(M+1)/2 \) of these singularities are non-zero

Consider the matrix \( P(\theta) - \theta^M I \). We note that for the \( i^{th} \) row, \( \theta^i \) is a common factor for \( 0 \leq i \leq M \), and \( \theta^M \) is a common factor for \( M < i \leq r \), so that we have

\[
\det [P(\theta) - \theta^M I] = \det [L(\theta)] \theta^{M(M+1)/2+(r-M)M}.
\]

Thus at least \((M(M+1)/2) + M(r - M)\) of the singularities are at the origin. This leaves at most \( M(M + 1)/2 \) non-zero singularities.
Let us denote these by $\xi_1, \xi_2, \ldots, \xi_N$, so that we have $\xi_k^M$ as an eigenvalue of $P(\xi_k)$ ($k = 1, 2, \ldots, N$). Let us assume, for simplicity, that they are distinct. Let $a_k = (a_{k0}, a_{k1}, \ldots, a_{kr})$ be the normalized eigenvector of $P(\xi_k)$ corresponding to the eigenvalue $\xi_k^M$, so that we have, for $k = 1, 2, \ldots, N$,

$$a_k P(\xi_k) = \xi_k^M a_k; \quad \sum_{i=0}^r a_{ki} = 1. \quad \text{(29)}$$

**Stationary distribution**

Let

$$\lim_{n \to \infty} \Pr \{ Y_n = u, X_n = i \} = \pi_{ui}, \quad \pi_u = (\pi_{u0}, \pi_{u1}, \ldots, \pi_{ur}).$$

Then, $\pi_u$ ($u = 0, 1, 2, \ldots$) satisfy the equations

$$\pi_o = \sum_{i=0}^\infty \pi_i \quad \text{Q}_{i+M-1} \quad \text{(30)}$$

$$\pi_j = \sum_{i=M-j}^\infty \pi_{i+j-M} P_i \quad (1 \leq j \leq M) \quad \text{(31)}$$

$$\pi_j = \sum_{i=0}^\infty \pi_{i+j-M} P_i \quad (j \geq M + 1) \quad \text{(32)}$$

where $P_i$ ($i = 0, 1, 2, \ldots, r$) is the matrix obtained from $P$ by retaining the $i^{th}$ row and replacing the others by zeros, and $Q_i = \sum_{j \geq i} P_j$ ($i \leq 0 < r$), $Q_i = 0$ ($i \geq r$).

We first show that

$$\pi_i = \sum_{k=1}^N c_k a_k \xi_k^i \quad (i \geq 1) \quad \text{(33)}$$
satisfies (32). For substituting (33) on the right-hand side of (32) noting that
\[ \sum_{i=0}^{\infty} P_i \xi^i = P(\xi) \] and using (29) we obtain
\[ \sum_{i=0}^{\infty} \sum_{k=1}^{N} c_k a_k \xi^i P_i = \sum_{k=1}^{N} j \]
\[ \sum_{i=0}^{\infty} \sum_{k=1}^{N} c_k a_k \xi^i P_i = \sum_{k=1}^{N} c_k a_k \xi^k = \pi_j \]

Now, since \[ \sum_{u=0}^{\infty} \pi_{ui} = \Pr \{ X = i \} = v_i \], we have
\[ \pi_o = v - \sum_{k=1}^{N} c_k a_k \frac{\xi_k}{1 - \xi_k}, \quad \pi_{oo} = v_o - \sum_{k=1}^{N} c_k a_k \frac{\xi_k}{1 - \xi_k} \]
\[ \pi_{oo} = v_o - \sum_{k=1}^{N} c_k a_k \frac{\xi_k}{1 - \xi_k} \] \[ \cdots \] (34)

We now use the M equations in (31) to give N relations to determine the unknown c’s. Consider first (31) with \( j = M \). Substituting \( \pi_j \) from (33) in (31) we obtain
\[ \pi_o P_o = \sum_{k=1}^{N} c_k a_k P_0, \] so that \( \pi_{oo} = \sum_{k=1}^{N} c_k a_k \).
This together with (34) gives us
\[ \sum_{k=1}^{N} c_k a_k \xi_k / (1 - \xi_k) = v_0. \]

Consider now (31) with \( j = M - 1 \). Substituting \( \pi_j \) from (33) and \( \pi_o \) from (34) in (31) we obtain
\[ (v - \sum c_k a_k / (1 - \xi_k)) P_1 - (\sum c_k a_k / \xi_k) P_0 = 0, \]
so that we have
\[ \sum_{k=1}^{N} c_k a_k \xi_k = 0, \quad \sum_{k=1}^{N} c_k a_k (1 - \xi_k) = v_1. \]

And, in general using (31) with \( j = M - i, (i = 1, 2, \ldots, M-1) \), we obtain
\[ \sum_{k=1}^{N} c_k a_k \xi_k = \sum_{k=1}^{N} c_k a_k (1 - \xi_k) = v_1. \]
Thus we have the following $N$ equations:

\[ \sum_{k=0}^{N} c_k a_{ki} \cdot \frac{Z}{1 - \xi_k} = v_i \quad (i = 0, 1, 2, \ldots, M - 1); \quad \ldots \quad (35) \]

\[ \sum_{k=0}^{N} c_k a_{ki} \cdot \frac{Z}{\xi_k^j} = 0 \quad (1 \leq j \leq M - 1; i = 0, 1, 2, \ldots, M - 2). \]

It now remains to show that (30) is satisfied as well. Consider first

\[ \sum_{i=0}^{M-1} \pi_i Q_{M+1-i}. \]

Using $Q_i = \sum_{j>i} P_{j}$ and (29), this can be simplified to

\[ \sum_{i=0}^{M-1} c_k a_k \cdot \frac{\xi_k^{i-M+1} - \xi_k}{(1 - \xi_k)} \cdot P_i + \sum_{i=0}^{M-1} c_k a_k \cdot \frac{\xi_k (P - I)}{(1 - \xi_k)} \quad \ldots \quad (36) \]

Substituting (36) on the right of (30) and using (34) we obtain,

\[ \nu - \sum_{k} \frac{c_k a_k \cdot \xi_k}{(1 - \xi_k)} + \sum_{i=0}^{M-1} \left\{ \sum_{k} \frac{c_k a_k \cdot \xi_k^{i-M+1}}{1 - \xi_k} - \nu \right\} P_i. \quad \ldots \quad (37) \]

It can be easily verified using (35) that the last term of (37) vanishes, and hence (37) reduces to $\pi_\nu$.

Now since the Markov chain $\{Y_n, X_n\}$ is irreducible and ergodic, the stationary distribution of $\{Y, X\}$ is unique, and hence the distribution given by (33) and (34) with $c'$s satisfying (35) is the required stationary distribution. From the bivariate stationary distribution we find the marginal distribution of $Y$ to be

\[ g_0 = \Pr \{ Y = 0 \} = 1 - \sum_{k=0}^{N} c_k \xi_k / (1 - \xi_k); \quad g_j = \Pr \{ Y = j \} = \sum_{k=0}^{N} c_k \xi_k \]

\[ (j \geq 1) \quad \ldots \quad (38) \]
For the case where \( E(X) \gg M, |\xi_2| \ll \xi_1 \), and we obtain the crude approximation
\[
g_0 \sim 1 - \frac{\nu_0 \xi_1}{a_1 \xi_1}, \quad g_j \sim \frac{\nu_0 (1 - \xi_1)^j}{a_1 \xi_1}, \quad j \geq 1.
\]

Illustration 4.1.2

Let the transition probability matrix of the inputs be

\[
P = \begin{bmatrix}
0 & 0.05 & 0.25 & 0.30 & 0.40 \\
1 & 0.05 & 0.20 & 0.30 & 0.45 \\
2 & 0.05 & 0.20 & 0.25 & 0.50 \\
3 & 0.05 & 0.15 & 0.30 & 0.50
\end{bmatrix}
\]

and let \( M = 2 \).

We have here, \( \nu = (0.05, 0.1782, 0.2857, 0.4861) \), so that \( E(X) = 2.2079 \). The draft-ratio, i.e. release/mean input = 90.58\%. Using the methods outlined above, we obtain \( \xi_1 = 0.664372, \xi_2, \xi_3 = -0.084686 \pm 0.01882i \), and \( c_1 = 0.30228, \ c_2, c_3 = 0.056756 \pm 0.164418i \).

Substituting these values in (38) gives the stationary distribution of the depletion. In Table (4.1.1) column 2 gives the stationary distribution function while column 3 gives the corresponding distribution obtained for a finite dam of size \( K = 10 \).

It is seen that even for this case of a very high draft-ratio the approximation is quite close.
Table 4.1.1. Stationary distribution function of the depletion

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate Infinitely deep dam</th>
<th>Exact Dam of size 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4185</td>
<td>0.4252</td>
</tr>
<tr>
<td>1</td>
<td>0.6008</td>
<td>0.6144</td>
</tr>
<tr>
<td>2</td>
<td>0.7361</td>
<td>0.7527</td>
</tr>
<tr>
<td>3</td>
<td>0.8245</td>
<td>0.8432</td>
</tr>
<tr>
<td>4</td>
<td>0.8834</td>
<td>0.9034</td>
</tr>
<tr>
<td>5</td>
<td>0.9225</td>
<td>0.9435</td>
</tr>
<tr>
<td>6</td>
<td>0.9485</td>
<td>0.9700</td>
</tr>
<tr>
<td>7</td>
<td>0.9658</td>
<td>0.9877</td>
</tr>
<tr>
<td>8</td>
<td>0.9773</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Thus we have derived the limiting distribution of the depletion for the case where the input is Markovian with a finite state space and the release is non-unit.

SECTION 4.2. GENERAL STORAGE MODEL

Transformation of content process to the time domain in general storage problem

We consider a stochastic process \( X(t), 0 \leq t < \infty \), defined on \([0, \infty)\), which represents the content of a dam or store or the virtual waiting time or workload in a queue etc. Inputs, at times \( t_1 < t_2 < \ldots (t_i > t_0 = 0) \), occur in a renewal process with the times \( T_i = t_{i+1} - t_i \) being independently and identically distributed random variables with common distribution function \( P(T_i \leq x) = A(x) \), \( 0 \leq x < \infty \), \( A(0^+) = 0 \). The sizes \( S_n \) of the inputs, \( n = 1, 2, \ldots \) are independently and identically distributed
random variables, independent of the $T_i$, with $P(S_n \leq x) = B(x)$, $0 \leq x < \infty$, $B(0+) = 0$.

The outflow from the store is determined by a general release rule $r(.)$ such that, for $t \neq$ some $t_i$,

$$\frac{dX(t)}{dt} = - r(X(t)),$$

with $r(0-) = 0$, $0 < r(x) \leq \infty$ for $0 < x < \infty$.

In obvious analogy with the GI/G/1 queue (where $r(x) = 1$, $0 < x < \infty$, and $B(x)$ is the service-time distribution function), we denote this as the GI/G/r(x) store.

It will be convenient, although not strictly necessary, to assume that $A(x)$ and $B(x)$ are absolutely continuous on $[0, \infty)$ with $a(x) = A'(x)$ and $b(x) = B'(x)$ being proper density functions. To apply our approximation procedure below we require that the moments $\mu_i = E(T^i)$ and $\nu_i = E(S^i)$ exist and are finite at least up to a finite order $M$.

For some $s \geq 0$ we define

$$\chi_\varepsilon (x) = \int_{y=s}^{x} \frac{dy}{r(y)} \quad 0 < x < \infty.$$

If at time $\tau$ the content is $X(\tau) = x > \varepsilon$, then $\chi_\varepsilon (x)$ is the time to run down to content $\varepsilon$ if no inputs occur in the interval $[\tau, \tau + \chi_\varepsilon (x)]$. If $\lim_{\varepsilon \to 0} \chi_\varepsilon (x) < \infty$, then the store can empty in a finite time, and the zero state may be recurrent and will be so if the system is recurrent, while if $\lim_{\varepsilon \to 0} \chi_\varepsilon (x) = \infty$, then the store can not empty in a finite time from any positive content and the zero state is not recurrent.
We consider the transformed process

\[ Y(t) = \chi_\varepsilon(X(t)), \quad \ldots \quad (39) \]

with \( \varepsilon = 0 \) if \( \chi_\varepsilon(x) < \infty \), and call this the 'extinction time' of the store from level \( X(t) \), while if \( \chi_\varepsilon(x) \) is divergent we choose a suitable \( \varepsilon > 0 \), and call \( Y(t) \) the 'pseudoextinction time' of the store from level \( X(t) \). If \( X(t) < \varepsilon \) this can be negative, so \( -\chi_\varepsilon(X(t)) \) is the pseudoextinction time from \( \varepsilon \) to \( X(t) \) in this case. For convenience in the sequel we omit the subscript \( \varepsilon \) in \( \chi_\varepsilon(x) \), except where necessary for understanding.

The relationship (39) is a mapping from \( x \in [0, \infty) \) to \( y \in [0, \infty) \) if the store can empty, whereas if it cannot then (39) is a mapping of \( x \in [0, \infty) \) to \( y \in (-\infty, \infty) \). It is necessary to complete the first case; to the line \( y = \chi(x) \) defined for non-negative \( x \) and \( y \) adjoin the half-line \( \{x = 0, \ y \in (-\infty, 0]\} \). We write \( x = \chi^{-1}(y) \) as the unique inverse mapping of \( y = \chi(x) \). If the store can empty we define \( \chi^{-1}(y) = 0 \) for \( y < 0 \). In this case a negative extinction time indicates an empty store, and its magnitude \( |y| \) is the time elapsed since it last became empty.

In the case when \( r(x) = 1 \), this is the single-server queue GI/G/1. The store can clearly empty in finite time. We have \( \chi(x) = x = \chi^{-1}(x) \) for \( x \in [0, \infty) \). Adjoin to this the half-line \( \{x = 0, \ y \in (-\infty, 0]\} \). The inverse mapping is \( \chi^{-1}(y) = y^+ \), where \( y^+ = \max(0, y) \), for \( y \in (-\infty, \infty) \).

If we consider the imbedded Markov process \( (Z_n, Y_n) = (\text{content at time } t_n, \text{extinction time at time } t_n) \) in the GI/G/1 queue for \( n = 1, 2, \ldots \) we have
\[ Z_{n+1} = \chi^{-1}(\chi(Z_n + S_n) - T_n), \quad \ldots (40) \]

and this is the horizontal resolvent of a discrete-time, continuous-state-space random walk along the line \( y = \chi(x) \) in \((x, y)\) space, wherein the increments \( S \) are applied to the right, i.e. in content, and \( T \) are applied downwards, i.e. in elapsed time.

Similarly, the second Lindley or (pseudo-) extinction-time recurrence is
\[ Y_{n+1} = Y_n + S_n - T_n \]
\[ = \chi(\chi^{-1}(Y_n) + S_n) - T_n, \quad \ldots (41) \]

which is the vertical resolvent of the same random walk along the line \( y = \chi(x) \). In this instance the negative values are times since the end of the last busy period.

The one-step integral equation for the general form of (41) is
\[ G_{n+1}(y) = \int_{w=-\infty}^{\infty} L(y, w) \, dG_n(w), \quad \ldots (42) \]

where
\[ L(y, w) = \int_{w=\chi^{-1}(w)}^{\infty} \{1-A(\chi(u)-y)\} \, du \cdot B(u-\chi^{-1}(w)). \quad \ldots (43) \]

The distribution function \( G_n(y) \), and the density function \( g_n(y) \) and its limit \( g(y) \) if these exist, may be approximated for \( y \in (-\infty, \infty) \) with a generalized Fourier series. The appropriate functions are the Hermite polynomials defined by the weight function
\[ \alpha(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \quad -\infty < x < \infty \quad \ldots (44) \]

via the relation
The Hermite polynomials $H_n(x)$ are characterized by

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) \alpha(x) \, dx = m! \delta_{mn}, \quad n, m = 0, 1, 2, \ldots$$

so they are orthogonal but not orthonormal.

More modern version of the density function $g(y)$ is the generalized Gram-Charlier (GGC) representation

$$g(y) = \sum_{j=0}^{\infty} g_j^{(p)} H_j(y) \{\alpha(y)\}^p,$$

where $p \in (0, 1]$, and typically $p = 0.5$.

We now formally describe the principle of the GC representation, leaving conditions for validity and proofs to an appendix. Assume for the present that the following representations are permissible:

$$g_{n+1}(y) = \int_{w=-\infty}^{\infty} \xi(y, w) g_n(w) \, dw; \quad -\infty < y < \infty$$
\[ \begin{align*}
&= \int_{\omega=0}^{\infty} \left( \sum_{j=0}^{\infty} g_{nj} H_j(w) \alpha(w) \right) \left( \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \epsilon_{mr} H_r(w) \right) H_m(y) \alpha(y) \, dw \\
&= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} g_{nj} \epsilon_{mr} H_m(y) \alpha(y) \int_{-\infty}^{\infty} H_r(w) H_j(w) \alpha(w) \, dw \\
&= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} g_{nj} \epsilon_{mr} H_m(y) \alpha(y) \, \delta_{ij} \\
&= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} g_{mr} \epsilon_{mr} H_m(y) \alpha(y) \, \delta_{ij} \\
&= \sum_{m=0}^{\infty} H_m(y) \alpha(y) \sum_{r=0}^{\infty} g_{mr} \epsilon_{mr} \delta_{ij} \\
\end{align*} \]

Also \( g_{n+1}(y) = \sum_{m=0}^{\infty} g_{(n+1)m} H_m(y) \alpha(y) \)

Hence \( g_{(n+1),m} = \sum_{r=0}^{\infty} \epsilon_{mr} g_{mr} \) .... (51)

Also \( Lg_n \)

\[ \begin{pmatrix}
\cdots \cdots \\
L_{m1} L_{m2} \ldots L_{mr} \\
\cdots \cdots \\
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{21} \\
\cdots \\
\epsilon_{r1} \\
\end{pmatrix}
\begin{pmatrix}
\cdots \\
g_{n1} \\
\cdots \\
\end{pmatrix}
\]

\[ = L_{m1} g_{n1} + \ldots + L_{mr} g_{nr} + \ldots \]

\[ = \sum_{r=0}^{\infty} L_{mr} g_{nr} \]
That is, \( g_{n+1} = Lg_n \) \( \ldots (52) \)

If an initial vector \( g_0 \) is suitably defined, then

\[
\begin{align*}
g_1 &= Lg_0 \\
g_2 &= L(g_1) = L(Lg_0) = L^2g_0
\end{align*}
\]

Similarly,

\[
\begin{align*}
g_3 &= L^3g_0 \\
\vdots \\
g_n &= L^ng_0 \ldots (53)
\end{align*}
\]

Further,

\[
G_n(y) = \int_{-\infty}^{y} g_n(w) \, dw
\]

\[
= \int_{-\infty}^{y} \sum_{j=0}^{\infty} g_{nj} \{(-D)^j \alpha(y)\}
\]

\[
= g_{no} (-D)^0 \alpha(y) + \int_{-\infty}^{y} \sum_{j=1}^{\infty} g_{nj} (-D)^j \alpha(y)
\]

\[
= g_{no} \alpha(y) - \sum_{j=1}^{\infty} g_{nj} ((-D)^{j-1} \alpha(y))
\]

\[
= g_{no} \phi(y) - \sum_{j=1}^{\infty} g_{nj} ((-D)^{j-1} \alpha(y))
\]

where \( \phi(y) \) is the standard normal distribution function.

In order to apply (53) we must determine the matrix elements 
\( (L_qv, q, v = 0, 1, \ldots, M) \). Consider first for fixed finite \( w \).
\[ I_q(w) = \int_{-\infty}^{\infty} H_q(y) \, \delta(y, w) \, dy, \]

which exists since \( H_q(y) \) is a polynomial of order \( q \) in \( y \) and \( \delta(y, w) \) is a density function in \( y \) for fixed finite \( w \), with the moments \( \mu_i \) and \( \nu_i \) of \( a(.) \) and \( b(.) \) existing at least up to order \( M \). Using the truncated form of (49) to order \( M \) and reversing the order of sum and integral, we find,

\[
I_q(w) = \int_{-\infty}^{\infty} H_q(y) \, \delta(y, w) \, dy \\
= \int_{-\infty}^{\infty} H_q(y) \sum_{m=0}^{M} \epsilon_m(w) H_m(y) \, \alpha(y) \, dy \\
= \sum_{m=0}^{M} \epsilon_m(w) \int_{-\infty}^{\infty} H_q(y) \, H_m(y) \, \alpha(y) \, dy \\
= \sum_{m=0}^{M} \epsilon_m(w) \, q! \, \delta_{qm} \\
= q! \, \epsilon_q(w).
\]

Repeating this kind of argument with

\[
I_{qv} = \int_{-\infty}^{\infty} H_v(w) \, \epsilon_q(w) \, \alpha(w) \, dw,
\]

with (50) we find \( I_{qv} = v! \), \( \epsilon_{qv} = L_{qv} \), so that

\[
L_{qv} = I_{qv} \\
= \int_{-\infty}^{\infty} H_v(w) \, \epsilon_q(w) \, \alpha(w) \, dw
\]
for all $q, v = 0, 1, \ldots, M$. 

It remains to evaluate $(L_{qv})$. As $\ell(y, w)$ is a conditional density function in $y$ and $H_0(y) = 1$ for all real finite $y$, it follows that $L_{oo} = 1$, $L_{ov} = 0$, $v = 1, 2, \ldots, M$. 

Using the standard form

$$H_q(x) = \sum_{j=0}^{[q/2]} h_{q,j} x^{q-2j} = q! \sum_{j=0}^{[q/2]} \frac{(-1)^j x^{q-2j}}{j! 2^j (q-2j)!}$$

of the Hermite polynomials it follows that

$$L_{qv} = 1/q! \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{-\infty}^{\infty} q! \sum_{j=0}^{[q/2]} \frac{(-1)^j y^{q-2j}}{j! 2^j (1-2j)!} \ell(y, w) dy dw$$

$$= \int_{-\infty}^{\infty} H_v(w) \alpha(w) \left( \int_{-\infty}^{\infty} \sum_{j=0}^{[q/2]} \frac{(-1)^j y^{q-2j}}{j! 2^j (1-2j)!} \right) dy dw$$

$$\left( \int_{u=\max(\psi^{-1}(y), \psi^{-1}(w))}^{\infty} a(\psi(u)-y) b(u-\psi^{-1}(w)) du dy \right) dw \quad \quad \ldots \quad (55)$$
\[
\left( \int_{ \left[ \psi^1(y), \psi^1(w) \right] } a(\psi(u) - y) b(u-\psi^1(w)) \, du \right) dy \, dw
\]
Consider \( \int_0^\infty y^{q-2j} a (\psi(u) - y) \, dy \)

put \( \psi(u) - y = t \)

\[
= \int_0^\infty (\psi(u) - t)^{q-2j} a(t) \, dt
\]

\[
= \int_0^\infty \left[ \sum_{i=0}^{q-2j} \binom{q-2j}{i} (\psi(u))^{q-2j-i} (-t)^i \right] a(t) \, dt
\]

\[
= \sum_{i=0}^{q-2j} \frac{(q-2j)!}{i! (q-2j-i)!} (\psi(u))^{q-2j-i} (-1)^i \int_0^\infty t^i a(t) \, dt.
\]

\[
= \sum_{i=0}^{q-2j} \frac{(q-2j)!}{i! (q-2j-i)!} (\psi(u))^{q-2j-i} (-1)^i \mu_i
\]

.... (56)

Using (56) in (55) we have

\[
L_{qv} = \int_0^\infty H_v(w) \alpha(w) \sum_{j=0}^{[q/2]} \frac{(-1)^j}{j! 2^j(q-2j)!} \sum_{i=0}^{q-2j} \frac{(q-2j)!}{i! (q-2j-i)!} (\psi(u))^{q-2j-i} (-1)^i \mu_i
\]

\[
= \sum_{j=0}^{[q/2]} \sum_{i=0}^{q-2j} \frac{(-1)^{i+j}}{j! 2^i (q-2j)!} \frac{\mu_i}{i! (q-2j-i)!} \int_0^\infty H_v(w) \alpha(w)
\]

\[
= \int_0^\infty \int_{\mu = \psi(w)} b(u - \psi^{-1}(w)) \, du \, dw
\]

\[
= \sum_{j=0}^{[q/2]} \sum_{i=0}^{q-2j} \frac{(-1)^{i+j}}{j! 2^i (q-2j)!} \frac{\mu_i}{i! (q-2j-i)!} \int_0^\infty H_v(w) \alpha(w)
\]

\[
= \int_{u = \psi(w)}^{\infty} (\psi(u))^{q-2j-i} b(u - \psi^{-1}(w)) \, du \, dw
\]
\[
L_{qv} = \sum_{j=0}^{[q/2]} \sum_{i=0}^{q-2j} \frac{(-1)^{i+j}}{j! 2^j} \mu_i^* J_{q-2j-i,v},
\]
where \( \mu_i^* = \frac{\mu_i}{i!} \), \( i = 0, 1, 2, \ldots M \) and

\[
J_{sv} = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{u=\psi^{-1}(w)}^{\infty} (\psi(u))^s b(u-\psi^{-1}(w)) \, du \, dw
\]
put \( y = u - \psi^{-1}(w) \)

\[
= \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} (\psi(y + \psi^{-1}(w))^s b(y) \, dy \, dw \ldots (57)
\]

We can make use of a truncated (i) GC series or (ii) Laguerre representation for \( b(x) \), provided they exist:

(i) \( b(x) = \sum_{k=0}^{M} b_k H_k(x) \alpha(x) \)

\[
b_k = \frac{1}{k!} \int_{y=0}^{\infty} H_k(y) b(y) \, dy = \sum_{m=0}^{[k/2]} \frac{(-1)^m}{m! 2^m} \nu_{k-2m}
\]
where \( \nu_j^* = \nu_j/j! \).

(ii) \( b(x) = \sum_{k=0}^{M} b_k L_k(x) e^x \)

\[
b_k = \int_{y=0}^{\infty} L_k(y) b(y) \, dy = \sum_{m=0}^{k} \frac{(-1)^m}{m!} \left[ \begin{array}{c} k \\ m \end{array} \right] \nu_m
\]
where \( L_n(x) = \sum_{m=0}^{n} (-1)^m \left[ \begin{array}{c} n \\ m \end{array} \right] x^m/m! \)
We obtain

\[ J_{sv} = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S b(y) dy dw \]

\[ J_{sv} = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S \]

\[ \left( \sum_{k=0}^{M} b_k H_k(y) \alpha(y) \right) dy dw \]

\[ = \sum_{k=0}^{M} b_k \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S H_k(y) \alpha(y) dy dw \]

\[ = \sum_{k=0}^{M} b_k K_{s,v,k} \]

where

\[ K_{s,v,k} = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S H_k(y) \alpha(y) dy dw \]

\[ \text{... (58)} \]

and

\[ J_{sv} = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S b(y) dy dw \]

\[ = \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S \sum_{k=0}^{M} b_k L_k(y) e^{-y} dy dw \]

\[ = \sum_{k=0}^{M} b_k \frac{1}{S!} \int_{-\infty}^{\infty} H_v(w) \alpha(w) \int_{y=0}^{\infty} \{\psi(y+\psi^{-1}(w))\}^S L_k(y) e^{-y} dy dw \]

\[ J_{sv} = \sum_{k=0}^{M} b_k K_{s,v,k} \]
where \( K_{s,v,k} \) is defined as
\[
-\frac{1}{s!} \int_{-\infty}^{\infty} H_s(w) \alpha(w) \int_{y=0}^{\infty} \{y+\psi^{-1}(w)\}^s L_k(y)e^y dy dw, \quad ... (59)
\]
so that
\[
L_{q,v} = \sum_{j=0}^{[q/2]} \sum_{i=0}^{q-2j} (-1)^{i+j} \frac{(-1)^{i+j}}{j!} \mu_i \sum_{k=0}^{M} b_k K_{q-2j-i,v,k}.
\]

We observe that \( J_{sv} \) in the form (57) depends on the input size density function \( b(.) \) and on the release rule through \( \chi(.) \) and \( \chi^{-1}(.) \). In the form (58) and (59) of \( K_{s,v,k} \) the dependence on \( b(.) \) has been removed from the integration, but appears in \( J_{sv} \) through moments in an additional combinatorial sum. In our applications we have so far used the form (57).

**An embedded level crossing technique for dams**

Consider a system which is modeled by a continuous parameter stochastic process \( \{W(t), t \geq 0\} \). For some embedded set of points \( \tau_n, n \geq 1 \) in the parameter set, let \( W(\tau_n) = W_n \) and consider the embedded discrete-parameter process \( \{W_n, n \geq 1\} \). The continuous-parameter process has sample functions traced out by the points \( \{(t, w(t)), t \geq 0\} \) in a two-dimensional Euclidean space. For any \( t \) the point \( \langle t, w(t) \rangle \) shall be called the system point. The embedded discrete parameter process has sample functions traced out by the sequence of points \( \{\tau_n, W_n\}, n \geq 1\).

**Definition 4.2.1 Embedded down crossing**

An embedded downcrossing of level \( w \) in the state space occurs in the closed interval \([\tau_n, \tau_{n+1}]\) iff \( W_n > w \geq W_{n+1} \). An embedded upcrossing of level \( w \) in the state space occurs within \([\tau_n, \tau_{n+1}]\) iff \( W_n \leq w < W_{n+1} \).
For some sample function of the embedded process which has an infinite number of time points let $D_n(w)$ and $U_n(w)$, denote respectively the number of embedded downcrossings and embedded upcrossings of level $w$ in the state space during the time interval $[0, \tau_n]$, $n \geq 2$. For every set in the state space, the numbers of entrances and exists by the system point differ by at most one during any time interval. In particular this holds for the intervals $(-\infty, w]$ and $(w, \infty)$, so that

$$\lim_{n \to \infty} D_n(w)/n = \lim_{n \to \infty} U_n(w)/n$$

$$\lim_{n \to \infty} E[D_n(w)/n] = \lim_{n \to \infty} E[U_n(w)/n]$$

where $E[.]$ denotes expectation, whenever these limits exist. The relations in (60) mean that if the system reaches statistical equilibrium, then the relative frequencies of embedded downcrossings and upcrossings tend to be equal. It is assumed that the set of sample functions, which has an infinite number of embedded points, has measure 1.

Model for a dam with general release rule

This can be described in terms of the processes $\{W(t), t \geq 0\}$ and $\{W_n, n \geq 1\}$. The inputs to the dam occur at epochs $\tau_n$, $n \geq 1$, with $\{\tau_n\}$ constituting a renewal process with common distribution $A$, so that the set of sample functions which have an infinite number of embedded points has measure 1. Without loss of generality, let $\tau_1 = 0$. The level in the dam at time $t$, denoted by $W(t)$, $t \geq 0$, increases with a jump of magnitude $S_n$, which is effective at $\tau_n$. $W(t)$ declines deterministically at a rate $r(W(t))$ for $\tau_n < t \leq \tau_{n+1}$, which is dependent on the instantaneous value of $W(t)$. 
Also \( r(x) > 0 \) for \( 0 < x < \infty \), \( r(0) = 0 \), and \( r(x) \) is a non-decreasing piecewise continuous function [5]. The level at time \( \tau_n \) is denoted by \( W_n \) and the distribution function of \( W_n \) is denoted by \( F_n \). Thus \( W(\tau_n^+) = W_n + S_n \), \( n \geq 1 \), and \( W(0) = W_1 = 0 \). Moreover \( F_1(x) = 1 \), \( x \geq 0 \), so that \( dF_1(x)/dx = 0 \) for \( x > 0 \). The random variables \( S_n, n \geq 1 \), have common distribution \( B \) which possesses a differentiable density \( b \) with \( B(0) = 0 \).

The instantaneous decline in level during the time interval \((\tau_n, \tau_{n+1}]\) is described by the differential equation

\[
\frac{dW(t)}{dt} = -r(W(t)) \tag{61}
\]

with initial condition \( W(\tau_n^+) = W_n + S_n \). Sample functions of the process \( \{W(t), t \geq 0\} \) are therefore given by the solution of (61) in the intervals \((\tau_n, \tau_{n+1}]\), \( n \geq 1 \). From (61), it is seen that \( W(t) \) is a decreasing function of \( t \) when \( W(t) > 0 \) in each interval \((\tau_n, \tau_{n+1}]\). Thus the level jumps by positive amounts at the input epochs, and when the level is non-zero it decreases continuously between input epochs. The process \( \{W_n, n \geq 1\} \) is a Markov chain since \( W_{n+1} = [W_n + S_n - \Delta_n]^+ \) where \( \Delta_n \) is the decline in level during \((\tau_n, \tau_{n+1}]\), and \([y]^+ = \max\{0, y\}\).

From (61) it follows that \( \int_{x=u}^{v} \frac{dx}{r(x)} = G(v) - G(u) \) equals the time required for the level to decline from \( v \) to \( u \), \( v > u \), where \( G(x) \) is the antiderivative of \( 1/r(x) \) and is a continuous increasing function defined for \( x > 0 \). Hence a necessary and sufficient condition for the level in the dam to possibly return to zero is [5].

\[
\lim_{u \downarrow 0} \int_{u}^{v} \frac{dx}{r(x)} < \infty \text{ for every } v > 0. \tag{62}
\]
If (62) holds, then \( \lim_{u \to \infty} G(u) = G(0) \) exists and is finite. In pharmacokinetic models where the drug concentration follows the release rule \( r(x) = kx, \ x \geq 0 \) for some constant \( k > 0 \) in every dosing interval, inequality (62) does not hold and a return to the zero level is impossible.

Under reasonable conditions \( \lim_{n \to \infty} F_n = F \) exists and is a proper distribution. Because of the properties of \( B \), the distribution functions \( F_n, \ n \geq 2 \) and \( F \) are absolutely continuous for positive levels so that the density \( f = dF \) exists, and moreover \( \lim_{w \to \infty} f(w) = 0. \)

Assuming (62) holds, an integral equation for \( f = dF \) is now derived using a technique based solely on:

(i) the concept of embedded level crossings.
(ii) the principle of stationary set balance.
(iii) knowledge of the release function \( r(.) \).

Given that \( \tau_{n+1} - \tau_n = y, \ y > 0 \), an embedded downcrossings of level \( w \geq 0 \) occurs within \( [\tau_n, \tau_{n+1}] \) iff \( W_n > w \) and \( W(t) \) descends to level \( w \) at some time point up to \( \tau_n + y \). Since the time required by the system point to drop to level \( w \) measured from \( \tau_n \) is \( \int_w^{w_n+s_n} dx/r(x) = G(W_n + S_n) - G(w) \), this condition is equivalent to

\[
G(W_n + S_n) \leq G(w) + y. \quad \ldots \ (63)
\]

Now \( G(x) \) is an increasing function of \( x \), its inverse denoted by \( G^{-1}(x) \) exists for \( x \geq 0 \) and is also an increasing function of \( x \). Hence (63) may be written as

\[
S_n \leq G^{-1}(G(w) + y) - W_n
\]
The conditional probability of an embedded downcrossings within the interval 
$[\tau_n, \tau_{n+1}]$ given that $\tau_{n+1} - \tau_n = y$, is therefore

$$
\gamma(w, y) \int_{\alpha=w^+}^{\infty} B(\gamma(w, y) - \alpha) \, dF_n(\alpha)
$$

where $\gamma(w, y) = G^{-1}(G(w) + y)$ and generally $dH(\alpha) = h(\alpha) \, d\alpha$, $\alpha > 0$ and $dH(\alpha) = H(0)$, $\alpha = 0$. The unconditional probability of a downcrossing inside $[\tau_n, \tau_{n+1}]$ is

$$
d_n(w) = \int_{y=0}^{\infty} \int_{\alpha=w^+}^{\infty} B(\gamma(w, y) - \alpha) \, dF_n(\alpha) \, dA(y), \quad n \geq 1.
$$

where $\eta(\alpha, w) = G(\alpha) - G(w)$. Let $\delta_i(w)$ equals 1 or 0 as an embedded downcrossing of level $w$ respectively does or does not occur within $[\tau_n, \tau_{n+1}]$. Then

$$
\mathcal{D}_n(w) = \sum_{i=1}^{n} \delta_i(w)
$$

Taking expectations it follows that

$$
E[\mathcal{D}_n(w)] = \sum_{i=1}^{n} E[\delta_i(w)] = \sum_{i=1}^{n} d_i(w)
$$

$$
\lim_{n \to \infty} E(\mathcal{D}_n(w))/n = \lim_{n \to \infty} (1/n) \sum_{i=1}^{n} d_i(w).
$$
Since \( F_n \to F \), then \( d_n(w) \to d(w) \) where

\[
d(w) = \int_{\alpha=w}^{\infty} \int_{y=\eta(\alpha,w)}^{\infty} B(y(w,y)-\alpha) \, dA(y) \, dF(\alpha).
\]

Thus,

\[
\lim_{n \to \infty} E[\mathcal{Z}_n(w)]/n = \int_{\alpha=w}^{\infty} \int_{y=\eta(\alpha,w)}^{\infty} B(y(w,y)-\alpha) \, dA(y) \, dF(\alpha). \quad \text{...(64)}
\]

Similarly an embedded upcrossing of level \( w \geq 0 \) occurs within \([\tau_n, \tau_{n+1}]\) iff \( W_n \leq w \) and the time required for the level to decline from \( W_n + S_n \) to \( w \) exceeds \( \tau_{n+1} - \tau_n = y, y > 0 \). That is,

\[
S_n > G^{-1}(G(w) + y) - W_n.
\]

The conditional probability of an embedded upcrossing of level \( w \) within \([\tau_n, \tau_{n+1}]\) given that \( \tau_{n+1} - \tau_n = y \) is therefore

\[
\int_{\alpha=0}^{w} \int_{\alpha=0}^{w} B(\gamma(w,y) - \alpha) \, dF_n(\alpha)
\]

where \( B(x) = 1 - B(x), x \geq 0 \); and the unconditional probability of an embedded upcrossing within \([\tau_n, \tau_{n+1}]\) is

\[
u_n(w) = \int_{\alpha=0}^{\infty} \int_{y=0}^{w} \bar{B}(\gamma(w,y) - \alpha) \, dF_n(\alpha) \, dA(y)
\]

\[
= \int_{\alpha=0}^{w} \int_{y=0}^{\infty} \bar{B}(\gamma(w,y) - \alpha) \, dA(y) \, dF_n(\alpha).
\]

It then follows in a similar manner as in the derivation of Equation (64) that for embedded upcrossings.
Application of the principle of stationary set balance (60) in order to equate (64) and (65) then results in the desired integral equation for \( f \), namely

\[
\int_{\alpha=0}^{\infty} \int_{y=0}^{\infty} B(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha)
\]

\[
\int_{\alpha=0}^{w} \int_{y=0}^{\infty} B(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha) = 0, \quad w \geq 0, \quad \ldots \ldots (66)
\]

with \( \gamma(w, y) = G^{-1}(G(w) + y) \) and \( \eta(\alpha, w) = G(\alpha) - G(w) \). Using the fact that

\[
F(w) = \int_{0}^{w} dF(\alpha), \quad (66) \text{ reduces to }
\]

\[
F(w) = \int_{\alpha=0}^{w} \int_{y=0}^{\infty} B(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha)
\]

\[
+ \int_{\alpha=W^+}^{w} \int_{y=\eta(\alpha, w)}^{\infty} B(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha), \quad w \geq 0. \quad \ldots \ldots (67)
\]

Differentiation of (67) with respect to \( w > 0 \) yields

\[
f(w) = \int_{\alpha=0}^{w} \int_{y=0}^{\infty} R(w, y) \, b(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha)
\]

\[
+ \int_{\alpha=W^+}^{w} \int_{y=\eta(\alpha, w)}^{\infty} R(w, y) \, b(\gamma(w, y) - \alpha) \, dA(y) \, dF(\alpha), \quad w > 0
\]

\[
\ldots \ldots (68)
\]

where \( R(w, y) = \partial \gamma(w, y) / \partial w = r(\gamma(w, y)) / r(w) \).
Letting \( w = 0 \) in (67) yields

\[
F(0) = \left[ \int_{y=0}^{\infty} B(y(0, y)) \, dA(y) \right]^{-1} \int_{\alpha=0}^{\infty} \int_{y=y(\alpha, 0)}^{\infty} B(y(0, y) - \alpha) \, dA(y) \, dF(\alpha). \quad \cdots (69)
\]

Moreover, we have the normalizing condition

\[
\int_{\alpha=0}^{\infty} dF(\alpha) = F(0) + \int_{\alpha=0}^{\infty} dF(\alpha) = 1. \quad \cdots (70)
\]

If Condition (62) does not hold, a return to zero is impossible. In that case (67) holds for \( w > 0 \), (68) and (70) are unchanged, and \( F(0) = 0 \) in (67)-(70).

Thus we have evolved a simplified method which is accessible to applications-oriented workers who may wish to obtain probability distributions of random variables associated with dams and queues. We also introduce the concept of embedded level crossings which connects the notion of embedded processes with that of level crossings in point processes. A brief review of the principles of stationary set balance in the new setting of embedded level crossings processes is discussed. These ideas are combined and applied to derive an integral equation for the steady-state distribution of the level in a dam with general release rule at moments just before inputs are registered.

**SECTION 4.3. INVENTORY MODEL**

Consider a situation in which a commodity is stocked in order to satisfy a continuing demand. We assume that the replenishing of stock takes place at successive times \( t_1, t_2, \ldots \), and we assume that the cumulative demand for the
commodity over the interval \((t_{n-1}, t_n)\) is a random variable \(\xi_n\) whose distribution function is independent of the time period,

\[
\Pr\{\xi_n = k\} = a_k, \quad k = 0, 1, 2, \ldots, \quad \ldots (71)
\]

where \(a_k > 0\) and \(\sum_{k=0}^{\infty} a_k = 1\). The stock level is examined at the start of each period.

An inventory policy is prescribed by specifying two nonnegative critical values \(s\) and \(S > s\). The implementation of the inventory policy is as follows: If the available stock quantity is not greater than \(s\) then immediate procurement is done so as to bring the quantity of stock on hand to the level \(S\). If, however, the available stock is in excess of \(s\) then no replenishment of stock is undertaken. Let \(X_n\) denote the stock on hand just prior to restocking at \(t_n\). The states of the process \(\{X_n\}\) consist of the possible values of the stock size

\[S, S - 1, \ldots, +1, 0, -1, -2, \ldots,\]

where a negative value is interpreted as an unfulfilled demand for stock, which will be satisfied immediately upon restocking. According to the rules of the inventory policy, the stock levels at two consecutive periods are connected by the relation

\[
X_{n+1} = \begin{cases} 
X_n - \xi_{n+1} & \text{if } s < X_n \leq S, \\
S - \xi_{n+1} & \text{if } X_n \leq s,
\end{cases} \quad \ldots (72)
\]

where \(\xi_n\) is the quantity of demand that arises in the \(n^{th}\) period, based on the probability law \((71)\). If we assume the \(\xi_n\) to be mutually independent, then the stock values \(X_0, X_1, X_2, \ldots\) plainly constitute a Markov chain whose transition probability matrix can be calculated in accordance with the relation \((72)\).

A shopkeeper keeps a certain quantity of stock on hand. When the stock runs low, he places an order to replenish his supplies. The inventory policy in operation is
assumed to be of \((s, S)\) type. Specifically, two levels \(s < S\) are prescribed. Suppose the stock is originally at level \(S\). A period length is also specified, and the stock level at the end of each period is checked. If at the close of a period, the stock level falls below \(s\), a requisition is placed to return the level of stock up to \(S\) ready for dispensation at the start of the next period.

Let \(X_i\) be the quantity of demand accumulated during the \(i^{th}\) period. We assume that \(X_1, X_2, \ldots\), are independent identically distributed positive random variables with distribution function \(F\). Let \(N(t)\) be the corresponding renewal counting process. We find that \((S - s) + 1\) is the number of demand periods elapsed until the first order for refill is placed, at which time the stock level is again \(S\).

Let the number of demand periods between the \((i - 1)^{st}\) and the \(i^{th}\) stock refill to \(\theta_i\). Then \(\{\theta_i\}\) is a discrete renewal process with mean \(E(\theta_i) = 1 + M(S - s)\), and

\[
\Pr \{\theta_i = k\} = F_{k-1} (S-s) - F_k (S-s) \quad \ldots \quad (73)
\]

Let \(W_n\) be the stock level at the end of the \(n^{th}\) demand period. Define \(G_n\) to be the conditional distribution

\[
G_n(x) = \Pr \{ S - x \leq W_n | s \leq W_n \}.
\]

This is the distribution of the stock level at the close of the \(n^{th}\) period, knowing that the level has not fallen below \(s\). This distribution is calculated by conditioning on \(\delta_n\), the number of demand periods since the last stock refill where the stock level was \(S\).

We get

\[
G_n(x) = \sum_{j=1}^{\infty} \Pr \{ \delta_n = j \} \Pr \{ X_1 + \ldots + X_j \leq x / X_1 + \ldots + X_j \leq S-s \}\]
But from (73)

\[ \Pr \{ \theta_1 \leq j \} = 1 - F_j(S - s). \]  

Hence, by appeal to the limit theorem for the excess random variable of the renewal process \( \{ \theta_i \} \) and by virtue of (75), we deduce

\[
\lim_{n \to \infty} \Pr \{ \delta_n = j \} = \frac{1}{E[\theta_1]} \Pr (\theta_1 > j) = \frac{F_j(S-s)}{F_j(S-s)} = \frac{1}{1 + M(S-s)} \]  

Using the above result in (74), we may conclude

\[
\lim_{n \to \infty} \Pr \{ S - x \leq W_n | s \leq W_n \} = \lim_{n \to \infty} \sum_{j=1}^{\infty} \Pr \{ \delta_n = j \} \frac{F_j(x)}{F_j(S-s)} \]

\[
= \frac{M(x)}{1 + M(S-s)} \]

which gives the limiting distribution of stock level in periods in which a requisition order is not pending.

**A cash inventory model**

Let \( \{ X(t), t \geq 0 \} \) be standard Brownian motion whose infinitesimal parameters are \( \mu(x) = 0, \sigma^2(x) = 1 \). Obviously we may take

\[
s(x) = \exp \left\{ -2 \int [\mu(\xi)/\sigma^2(\xi)] d\xi \right\} = 1, \]
and for the scale measure \( S(x) = x \), so that \( u(x) \), the probability of reaching \( b \) prior to \( a \) with initial state \( x \), is

\[
u(x) = \frac{x - a}{b - a}, \quad a \leq x \leq b,
\]

.... (77)

The speed density is

\[
m(\xi) = \frac{1}{s(\xi)} = 1,
\]

and the Green function for the interval \([a, b]\) is

\[
G(x, \xi) = \begin{cases} 
\frac{2(x - a)(b - \xi)}{(b - a)}, & a \leq x \leq \xi \leq b, \\
\frac{2(\xi - a)(b - x)}{(b - a)}, & a \leq \xi \leq x \leq b.
\end{cases}
\]

.... (78)

Then a direct calculation from

\[
v(x) = 2 \left\{ u(x) \int_{\xi}^{b} [S(b) - S(\xi)] m(\xi) \, d\xi + [1 - u(x)] \int_{a}^{x} [S(\xi) - S(a)] m(\xi) \, d\xi \right\}
\]
gives

\[
v(x) = E[T_{a,b} | X(0) = x] = (x-a)(b-x), \quad a \leq x \leq b.
\]

.... (79)

Let \( Z(t) \) be the amount of cash an organization has on hand at time \( t \). We suppose that in the absence of intervention \( \{Z(t), t \geq 0\} \) behaves as a Brownian motion process with zero drift and variance parameter \( \sigma^2 = 1 \).
Holding cash involves an opportunity cost since this cash could be invested. We therefore suppose that holding cash at level \( z \) incurs costs at the rate \( cz \). Since transactions into and out of cash are also costly, we include a cost \( K \) for each transaction.

Consider the following \((s, S)\) type policy for controlling the cash level: “If the cash reaches level \( S \), invest \( S - s \) and reduce the cash level to \( s \). This transaction incurs a cost of \( K \). If the cash ever dips to zero, sell investments, and bring the cash level up to \( s \). This again costs \( K \).”

Consider a cycle to be from one intervention returning the cash level to \( s \) to the next such intervention. The long-run cost per unit time will be the expected cycle time, or

\[
\frac{(K + A)}{B},
\]

where

\[
A = \mathbb{E}[\int_0^T cZ(\tau) \, d\tau | Z(0) = s]
\]

and

\[
B = \mathbb{E}[T | Z(0) = s].
\]

Here

\[
T = \min \{ t > 0 : Z(t) = S \text{ or } Z(t) = 0 \}.
\]

From Eq.\(79\) in the Brownian motion example we have (using \( a = 0 \) and \( b = S \))

\[
B = v(s) = s(S - s).
\]

Taking \( w(x) = \int_a^b G(x, \xi) \, c \, d\xi \) and taking \( G(x, \xi) \) appropriate for the interval \((a, b) = (0, S)\) (as given by \(78\)), we obtain
$$E \left[ \int_0^T g(z(t)) \, dt \mid Z(0) = x \right] = w(x) = \int_0^x G(x, \xi) \, c \, \xi \, d\xi = C(1/3 \times S^2 - 1/3 \times x^3)$$

and in particular

$$A = w(s) = \frac{1}{3} \, cs \, (S^2 - s^2).$$

The average cost is

$$K + A = \frac{1}{3} \, K + \frac{1}{3} \, cs \, (S^2 - s^2) = \frac{K}{s(S-s)} + \frac{c(S+s)}{3s(S-s)}.$$

To minimize, change variables, letting $\bar{S} = S - s$. The average cost is

$$C(s, \bar{S}) = \frac{K}{s \bar{S}} + \frac{c(\bar{S} + 2s)}{3\bar{S}},$$

which we differentiate to get

$$\frac{\partial C}{\partial s} = -\frac{K}{\bar{S}s^2} + \frac{2c}{3s}$$

and

$$\frac{\partial C}{\partial \bar{S}} = -\frac{K}{s \bar{S}^2} + \frac{c}{3\bar{S}}.$$

We equate the derivatives to zero to obtain the cost minimizing $\bar{S} = \bar{S}^*$ and $s = s^*$:

$$(s^*)^2 \bar{S}^* = \frac{3K}{2c}, \quad s^* (\bar{S}^*)^2 = \frac{3K}{c}. \quad \ldots \quad (80)$$

whence

$$\frac{s^*}{\bar{S}^*} = \frac{1}{2}, \quad \ldots \quad (81)$$

or

$$s^* = \frac{1}{2} \, (S^* - s^*) \quad \text{or} \quad s^* = \frac{1}{3} \, S^*. \quad \ldots \quad (82)$$
We substitute (81) in (80) to get

\[(s^*)^3 = \frac{3K}{4c}.
\]

The optimal control parameters are

\[s^* = \left(\frac{3K}{4c}\right)^{\frac{1}{3}} \text{ and } S^* = 3s^*.
\]