In case of several repairmen, if the number of machines inoperative at any
given time exceeds the number of repairmen, then the excess number of machines
will have to wait until repairmen are available. The loss of production due to the
period the machines have to wait for service is termed as machine interference. Thus
the period of time the machine is inoperative before a repairman is available can be
considered as a waiting time in a queue. We shall discuss a classical work of Palm
and Takács on the machine interference problem which is studied in detail in
section 5.1. We also bring forth a discussion by Sharma [57] as a basic queueing
model for machine interference problem in the same section. No matter how efficient
the manufacturing process is, one or more failures may occur in a simple life-testing
instrument of a number of items. In section 5.2, we discuss the Bayesian
approximations and its applications to reliability estimation for different failure
distribution. We briefly discuss a reliability of series and parallel system in classical
Bayesian approach. In many contexts, the instantaneous failure rate plays an
important role in the mathematical theory of reliability. We present some of the
important in IFR and DFR in the study of preservation of monotone failure rate in the
last section.
SECTION 5.1. INTERFERENCE MODEL

Basic queue model

Palm’s model

Assumptions

1. All machines are similar with respect to the average number of breakdowns which each experiences in its unit working or running time.

2. All repairmen are similar with respect to skill and aptitude in servicing the machines, and all machines are similar with respect to the skill needed to restore them to working condition.

3. Uninterrupted running time of a machine is an exponentially distributed nonnegative random variable; i.e., the distribution function of the running time is

\[ A(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } \lambda > 0, \ t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \]

4. Service or repair time of a machine is an exponentially distributed positive random variable, i.e., the distribution function of the service time is

\[ B(\xi) = \begin{cases} 1 - e^{-\mu \xi} & \text{for } \mu > 0, \ \xi \geq 0 \\ 0 & \text{for } \xi < 0 \end{cases} \]

5. All random variables are independently distributed.

6. The queueing system (machines and repairmen) is in a state of statistical equilibrium.
Non-equilibrium behaviour of the queueing system

Let the random variable \( X(t) \) denote the number of machines not working at time \( t \) and let \( P_x(t) = \mathbb{P}(X(t) = x), \ x = 0, 1, \ldots, m. \) If at time \( t \) there are \( x \) machines not working, we will say that the queueing system is in state \( x. \) We observe that a transition from the state \( x \) to state \( x+1 \) is caused by the breakdown of one among the \( m-x \) working machines, while a transition from the state \( x \) to state \( x-1 \) occurs when a machine being serviced is repaired and returned to a working state. In view of assumptions (3) and (4) and the above transitions, the queueing system postulated by Palm is of type M/M/r and can, therefore, be described by a Markov process of the birth-and-death type.

We also observe that in case \( x \leq r, \) the queueing system being in the state \( x \) means that \( x \) machines are being serviced and \( r-x \) repairmen are idle, while in case \( x > r \) the state \( x \) means that \( r \) machines are being serviced and \( x-r \) machines are waiting to be serviced.

We have:

1. If at time \( t \) a machine is working, the probability that in the interval \((t, t + \Delta t)\) it will break down and require service is \( \lambda \Delta t + o(\Delta t). \)
2. If at time \( t \) a machine is being repaired, the probability that in the interval \((t, t + \Delta t)\) it will be repaired and returned to a working state is \( \mu \Delta t + o(\Delta t). \)

Let us now assume that the system is in the state \( x. \) In this case we have

\[
\begin{align*}
\mathbb{P}(x \to x + 1) &= \lambda_x \Delta t + o(\Delta t) \\
\mathbb{P}(x \to x - 1) &= \mu_x \Delta t + o(\Delta t) \\
\mathbb{P}(x \to x) &= 1 - (\lambda_x + \mu_x) \Delta t + o(\Delta t)
\end{align*}
\]

where the birth rates \( \lambda_x \) and death rates \( \mu_x \) are given by
\[
\lambda_x = \begin{cases} 
m \lambda & \text{for } x = 0 \\
(m-x) \lambda & \text{for } 1 \leq x \leq r, \ r \leq x \leq m 
\end{cases}
\]

and \[
\mu_x = \begin{cases} 
0 & \text{for } m = 0 \\
x \mu & \text{for } 1 \leq x \leq r \\
r \mu & \text{for } r \leq x \leq m
\end{cases}
\]

where \( \lambda \) and \( \mu \) are positive constants. Therefore, the Kolmogorov equations describing the non-equilibrium behaviour of the queueing system are

\[
\frac{dP_0(t)}{dt} = -m \lambda P_0(t) + \mu P_1(t)
\]

\[
\frac{dP_x(t)}{dt} = (m - x + 1) \lambda P_{x-1}(t) -[(m-x)\lambda + x \mu]P_x(t) + (x+1) \mu P_{x+1}(t) \quad 1 \leq x < r
\]

\[
\frac{dP_x(t)}{dt} = (m - x + 1) \lambda P_{x-1}(t) -[(m-x)\lambda + r \mu]P_x(t) + r \mu P_{x+1}(t) \quad r \leq x \leq m
\]

Therefore, the equilibrium behaviour of the queueing system is given by the solutions of the difference equations

\[
m \lambda \pi_0 = \mu \pi_1
\]

\[
[(m-x) \lambda - x \mu] \pi_x = (m - x + 1) \lambda \pi_{x-1} + (x+1) \mu \pi_{x+1} \quad 1 \leq x < r
\]

\[
[(m-x) \lambda + r \mu] \pi_x = (m - x + 1) \lambda \pi_{x-1} + r \mu \pi_{x+1} \quad r \leq x \leq m
\]

In the above we have put \( \pi_x = \lim_{t \to \infty} P_x(t) \). From the first of the above equations we obtain the ratio \( \pi_1/\pi_0 \). From the second and third equations induction yields
\[(x + 1) \mu \pi_{x+1} = (m - x) \lambda \pi_x \quad x < r \quad \ldots (1)\]

and
\[r \mu \pi_{x+1} = (m - x) \lambda \pi_x \quad r \leq x \leq m \quad \ldots (2)\]

From (1) and (2) the ratios \( \pi_x / \pi_o \) can be calculated successively, and \( \pi_o \) can be obtained from the condition
\[\sum_{x=0}^{m} \pi_x = 1.\]

The solution of (1) and (2) in terms of the non-cumulative and cumulative Poisson distribution functions are given by
\[P(k, \xi) = \frac{\xi^k}{k!} e^{-\xi}\]
and
\[P(k, \xi) = \sum_{i=k}^{\infty} p(i, \xi)\]

We can also show that the limiting probabilities are
\[\pi_x = \frac{(r^x / x!) p(m-x, r \mu/\lambda)}{S(m, r, r \mu/\lambda)} \quad x < r \quad \ldots (3)\]

\[= \frac{[r^{x-1} / (r-1)!] p(m-x, r \mu/\lambda)}{S(m, r, r \mu/\lambda)} \quad r \leq x \leq m \quad \ldots (4)\]

where
\[S \left\{ \frac{m, r, \mu}{\lambda} \right\} = \sum_{i=0}^{r-1} \frac{r^i}{i!} \left\{ p\left( m - i, \frac{r \mu}{\lambda} \right) \right\} + \frac{r^{r-1}}{(r-1)!} \left( 1 - P\left( m - r + 1, \frac{r \mu}{\lambda} \right) \right) \quad \ldots (5)\]
Further it is to be noted that, when only one repairman serves the m machines (i.e., \( r = 1 \)),

\[
S\left\{ m, 1, \frac{\mu}{\lambda} \right\} = 1 - P\left\{ m + 1, \frac{\mu}{\lambda} \right\}
\]

and the limiting probabilities are given by

\[
\pi_x = \frac{p(m - x, \mu/\lambda)}{1 - P(m+1, \mu/\lambda)}
\]

**Machine interference model**

Let \( a, b \) and \( w \) denote the average number of machines working, being serviced, and waiting to be serviced, respectively. We have the following identities:

\[
a + b + w = m \quad \text{..... (6)}
\]

\[
\frac{a}{b} = \frac{\mu}{\lambda} \quad \text{..... (7)}
\]

\[
b = \sum_{x=0}^{r-1} x \pi_x + x \sum_{x=r}^{m} \pi_x = r - \sum_{x=0}^{r-1} (r - x) \pi_x \quad \text{..... (8)}
\]

Equation (6) is simply an expression of the fact that a machine has to be in one of these states. In (7) the left-hand side is the ratio of the average number of machines working and the average number of machines being serviced, while the right-hand side is the ratio of the average uninterrupted servicing time of a machine and the average repair time. Equality is obvious, since these last two quantities are \( 1/\lambda \) and \( 1/\mu \), respectively. Equation (8) relates to the equality of the number of engaged repairmen and the number of machines being serviced.
From (6) to (8) we obtain

\[ w = m - b - a = m - b \left\{ 1 + \frac{\mu}{\lambda} \right\} \]

\[ = m - \frac{\lambda + \mu}{\lambda} \left[ r - \sum_{x=0}^{r-1} (r-x) \pi_x \right] \quad \ldots \quad (9) \]

For \( r = 1 \), the case of one repairman, (9) becomes

\[ w = m - \frac{\lambda + \mu}{\lambda} (1 - \pi_0) \quad \ldots \quad (10) \]

These two expressions give the average number of machines in the waiting line; hence, they express the interference loss. It is to be noted that, in order to obtain (9) and (10), only the three basic relations (6) to (8) were used; in turn, these relations are based on assumptions (1), (2) and (4). In order to derive the same expressions for interference loss, Palm [47] used all six assumptions. It has been pointed out by Naor [43] that (9) and (10) remain valid in the case when some systematic or semisystematic method of breakdowns and servicing machines replaces the randomness assumptions expressed by (3) to (5). In this case, however, it is necessary to modify assumption (6). We can therefore conclude that (9) and (10) are valid whenever the limiting probability distribution \( \{ \pi_x \} \) is well defined either as the fraction of time spent in the state \( x \) or as probabilities under arbitrary assumptions.

From (8), (3) and (4) we obtain

\[ b = r \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)} \quad \ldots \quad (11) \]
which gives the average number of machines being repaired. By using (6) and (10), we find that the average number of machines working is

\[
a = \frac{r \mu}{\lambda} \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}
\]

Similarly, from (9) and (11), the average number of machines in the waiting line is given by

\[
w = m - \left(1 + \frac{\mu}{\lambda}\right) r \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}
\]

Also the average number of idle repairmen is given by

\[
r - b = r \left(1 - \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}\right)
\]

and the coefficient of loss for repairmen, is given by

\[
\frac{r - b}{r} = 1 - \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}
\]

The operative efficiency is defined as the ratio of the number of machines waiting to be serviced to the number of repairmen; hence,

\[
b = \frac{S(m-1, r, \mu/\lambda)}{r} \frac{S(m, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}
\]

which is one minus the co-efficient of loss for repairmen.

For machines, three co-efficients of loss are defined. The co-efficient of normal loss due to repairs is given by

\[
b = \frac{r}{m} \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)}
\]
The co-efficient of loss due to machine interference is equal to
\[ \frac{w}{m} = 1 - \left( 1 - \frac{\mu}{\lambda} \right) \frac{r}{m} \frac{S(m-1, r, \mu/\lambda)}{S(m, r, \mu/\lambda)} \]

The combined co-efficient of loss equals
\[ \frac{b + w}{m} = 1 - \left( \frac{\mu}{\lambda} \right) \frac{rS(m-1, r, \mu/\lambda)}{mS(m, r, \mu/\lambda)} \]

Finally, the machine efficiency (or machine availability) of the system is given by
\[ \frac{a}{m} = \left( \frac{\mu}{\lambda} \right) \frac{rS(m-1, r, \mu/\lambda)}{mS(m, r, \mu/\lambda)} \]

which is one minus the combined co-efficient of loss.

**Takáč's model**

**Assumptions**

1. The queueing system consists of m machines that work independently and a single repairman.

2. If at time t a machine is in a working state, the probability that it will break down and call for service in the interval \( (t, t + \Delta t) \) is \( \lambda \Delta t + o(\Delta t) \). This probability is the same for each of the m machines. Hence, the distribution function of the running time of a machine is the negative exponential

\[ A(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } \lambda \geq 0, t \geq 0 \\ 0 & \text{for } \lambda < 0 \end{cases} \]
3. If a machine breaks down, it will be serviced immediately unless the repairman is servicing another machine, in which case a queue of machines waiting to be serviced is formed.

4. Machines are serviced in the order of their breakdowns.

5. The service time is a positive random variable with arbitrary distribution function $B(\xi)$, each machine having the same service-time distribution. It is also assumed that the mean and variance, given by

$$\mu = \int_0^\infty \xi \, dB(\xi) \quad \text{and} \quad \sigma^2 = \int_0^\infty (\xi - \mu)^2 \, dB(\xi)$$

exist and may be infinite.

In view of (1), (2) and (5), the queueing system we consider is of type $M/G/1$.

Let the random variable $X(t)$ denote the number of machines working at time $t$. The system is in state $x (x = 0, 1, \ldots, m)$ if $x$ machines are working simultaneously. Now let $t_1, t_2, \ldots, t_n, \ldots$ denote the end points of the consecutive service-time periods, and let $X(t_n - 0) = X_n$. The limiting probability distributions associated with $X(t)$ and $X_n$ are as follows:

$$\pi_x = \lim_{t \to \infty} \mathbb{P}\{X(t) = x\} \quad x = 0, 1, \ldots, m$$

and

$$\pi_x^* = \lim_{n \to \infty} \mathbb{P}\{X_n = x\} \quad x = 0, 1, \ldots, m - 1.$$ 

We also introduce the random variable $Y(t)$, which denotes the period of time required from time $t$ until the servicing of a machine is completed. Finally, we denote by $B_x^*(\xi)$ the conditional limiting distribution of $Y(t)$:
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\[ B_x^*(\xi) = \lim_{t \to \infty} P(Y(t) \leq \xi \mid X(t) = x) \quad x = 0, 1, \ldots, m-1 \]

The stochastic process described is in general non-Markovian: however, it may be considered as a Markov process if the state of the system at time \( t \) is defined in terms of the pair of random variables \( (X(t), Y(t)) \). We note that this process can be applied to the study of telephone traffic in the case of a finite number of trunk lines. In this case we refer to calls and holding times instead of breakdowns and service times, respectively.

**Result 5.1.1**

The limiting distribution \( \{\pi_x^*\} \), which exists independently of the initial distribution of \( X(0) \), is given by

\[ \pi_x^* = \sum_{i=0}^{m-1} (-1)^{i-x} \binom{i}{x} B_i^* \]  \( \quad \ldots (12) \]

which satisfies \( \sum_{x=0}^{m-1} \pi_x^* = 1 \), where \( B_i^* \), the \( i \)th binomial moment of \( \{\pi_x^*\} \), is given by

\[ B_i^* = \frac{\sum_{k=1}^{m-1} \binom{m-1}{k}(1/C_k)}{\sum_{k=0}^{m-1} \binom{m-1}{k}(1/C_k)} \]  \( \quad \ldots (13) \]

where \( C_i = 1 \) for \( i = 0 \)

\[ = \prod_{k=1}^{i} \frac{\varphi(k, \lambda)}{1-\varphi(k, \lambda)} \]  \( \quad \ldots (14) \)

\( \varphi(k, \lambda) \)
and \( \varphi(S) = \int_0^\infty e^{St} dB(\xi) \) \( \Re(S) \geq 0 \)

is the Laplace–Stieltjes transform of \( B(\xi) \).

Result 5.1.2

The limiting distribution \( \{\pi_x\} \), which exists independently of the initial distribution of \( X(0) \), is given by

\[
\pi_x = \sum_{i=x}^m (-1)^{i-x} \binom{i}{x} B_i
\]

which satisfies \( \sum_{x=0}^m \pi_x = 1 \), where \( B_i \), the \( i \)-th binomial moment of \( \pi_x \), is given by

\[
B_i = 1 \quad \text{for } i = 0
\]

\[
= \frac{mC_{i-1}}{i} = \frac{m! \sum_{k=1}^{m-1} \binom{m-1}{k} (1/C_k)}{(1 + m\mu) \sum_{k=0}^{m-1} \binom{m-1}{k} (1/C_k)} \quad \text{for } i = 1, 2, \ldots m
\]

and \( C_k \) is defined by (14).

Result 5.1.3

If the expectation of the service-time distribution is finite, then the limiting distribution

\[
\lim_{t \to \infty} \mathcal{P}(Y(t) \leq \xi \mid X(t) = x) = B_x^*(\xi)
\]

exists, with

\[
B_x^*(\xi) = C_x \sum_{i=0}^{m-1} \left[ \pi_i \int_0^\infty e^{\lambda x} \left( 1 - e^{\lambda x} \right)^{i+1-x} \left[ B(\alpha + \xi) - B(\alpha) \right] d\tau \right]
\]

for \( x = 0, 1, \ldots, (m-2) \)
\[ B^*_n(\xi) = C_{m-1} \pi^*_m \int_0^\infty e^{\lambda(m-1)\tau} [B(\tau + \xi) - B(\tau)]d\tau \]

where

\[ C_x = \frac{m \mu / \pi^*_x}{m \alpha \mu + \pi^*_m} = \frac{x \mu}{\pi^*_x} \quad \text{for } x = 1, 2, \ldots, m-1 \]

\[ = \frac{m \mu \pi^*_0}{m \alpha \mu + \pi^*_m} \quad \text{for } x = 0. \]

We state without Proof the following theorems

**Theorem 5.1.1 (Bharucha-Reid [4])**

The expectation of the production times of the machines in the interval \((0, T)\) is

\[ \mathbb{E} \left\{ \int_0^T X(t) dt \right\} = \frac{m \sum_{i=1}^{m-1} \binom{m-1}{i} \frac{1}{C_i}}{1 + m \mu \lambda \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{C_i}} T \]

**Theorem 5.1.2 (Bharucha-Reid [4])**

Let the random variable \(Z(t)\) be defined as follows:

\[ Z(t) = \begin{cases} 1 & \text{for } X(t) < m \\ 0 & \text{for } X(t) = m \end{cases} \]

The expectation of the busy time of the repairman in the interval \((0, T)\) is

\[ \mathbb{E} \left\{ \int_0^T Z(t) dt \right\} = \frac{m \mu \lambda \sum_{i=1}^{m-1} \binom{m-1}{i} \frac{1}{C_i}}{1 + m \mu \lambda \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{C_i}} T \quad \text{.... (17)} \]
Theorem 5.1.3 (Bharucha-Reid [4])

Let $W_{ni}(\xi)$ denote the distribution function of the waiting time of a machine and let $W_m$ denote the expected waiting time. We have

$$W_m(\xi) = \sum_{i=0}^{m-1} \pi_{x-i}^* B_{i+1}^* (\xi) B_{m-i}^* (\xi) + \pi_{m-1}^* B_0$$

where $B_n(\xi)$ is the $n$-fold convolution of $B(\xi)$ with itself, with $B_0 = 1$ for $\xi \geq 0$ and $B_0 = 0$ for $\xi < 0$.

Furthermore, if $\mu < \infty$, then

$$w_m = (m-1) \mu + \left[ 1 - \frac{1}{\sum_{i=0}^{m-1} \frac{1}{C_i}} \left\{ \frac{\sigma^2 + \mu^2}{2 \mu} - \frac{\mu}{1-\phi(\lambda)} \right\} \right]$$

M/M/r Machine interference model

Consider $N$ identical and automatic machines under the care of $r$ ($1 \leq r \leq N$) operatives; the breakdown of each machine occurring randomly in running time and the repair time of each repairman both have negative exponential distributions with parameters $\lambda$ and $\mu$ respectively. We further assume that $i$ breakdowns initially need attention at $t = 0$. Let

$$p(n, t) = \Pr \text{ (there are } n \text{ machine breakdowns at a given time } t \leq 0)$$

Then $p(n, 0) = \delta_n$ and the forward Chapman-Kolmogorov equations of the system are given by

$$\dot{p}(0, t) = -N \lambda p(0, t) + \mu p(1, t)$$

$$\dot{p}(n, t) = -\{(N-n) \lambda + n \mu\} p(n, t) + (N-n+1) \lambda p(n-1, t)$$

$$+ (n+1) \mu p(n+1, t), \quad 1 \leq n \leq r-1$$
\[ \dot{p}(n, t) = -(N - n) \lambda + r \mu \] 
\[ + r \mu p(n+1, t), \quad r \leq n \leq N \]
\[ \dot{p}(N, t) = -r \mu p(N, t) + \lambda p(N - 1, t). \]

Now taking Laplace transform of (21) and setting

\[ \psi(n, \theta) = \int_0^\infty e^{\theta t} p(n, t) \, dt \]

we get

\[ A \psi = [\delta_{i0}, \delta_{i1} \ldots \delta_{iN}]', \]

where \( A \) is a tridiagonal matrix of order \( N + 1 \) given by

\[
A = \begin{pmatrix}
\theta + N_0 \lambda & -r_1 \mu & 0 & \ldots & 0 & 0 \\
-r_0 \lambda & \theta + N_1 \lambda + r_1 \mu & -r_2 \mu & \ldots & 0 & 0 \\
0 & -N_1 \lambda & \theta + N_2 \lambda + r_2 \mu & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \theta + N_{N-1} \lambda + r_{N-1} \mu & -r_N \mu \\
0 & 0 & 0 & \ldots & -N_{N-1} \lambda & \theta + r_N \mu
\end{pmatrix}
\]

and

\[ \psi = [\psi(0, \theta) \quad \psi(1, \theta) \ldots \psi(N, \theta)]', \]

\[ N_j = N-j, \quad r_j = j \quad \text{for} \quad j \leq r-1, \]

\[ r_j = r \quad \text{for} \quad r \leq j \leq N \quad \forall j = 0, 1, 2, \ldots, N. \]

We also define the j operative (j = 1, 2, ..., r) initial busy period of this M/M/r machine interference model as the time from the instant of the start of the repairing of the first machine breakdown from amongst the arbitrary i (j \leq i \leq N) units initially needing attention that makes the j operatives busy, to the first subsequent moment when one of the j operatives becomes free. That is, we can regard the (j - 1)\textsuperscript{th} state as
an absorbing state for determining the p.d.f. and c.d.f. of the said initial busy period. It may also be noted that an ordinary busy period can be derived from the initial busy period by taking \( i = 1 \). Now let

\[
q_n(t) = \Pr\{N(t) = n / N(0) = i, j < i < N\} \quad \text{(25)}
\]

where \( N(t) \) stands for the number of machine breakdowns needing attention at time \( t \) during the \( r \)-operative initial busy period. Then \( q_n(t) \) satisfies the same set of equations as satisfied by \( p(n, t) \) except that the first equation is replaced by

\[
\dot{q}_{i-1}(t) = j \mu q_i(t) \quad \text{(26)}
\]

and \( (N - n + 1) \lambda q_{n-1}(t) \) is to be excluded from the second equation when \( n = j \) or from the third equation when \( n = j = r \), in which case the second equation becomes redundant. Also if

\[
\psi_n(n, \theta) = \int_0^\theta e^{	heta t} q_n(t) \, dt, \quad \text{(27)}
\]

and taking Laplace transform of the set of equations for \( q_n(t) \), ignoring the first equation, we can write the system as

\[
\Delta \psi = \begin{bmatrix} \delta_j & \delta_{j+1} & \cdots & \delta_N \end{bmatrix}' \quad \text{(28)}
\]

where \( \Delta \) is the \( (N + j + 1) \)th order matrix formed at the bottom right of the matrix \( A \) and

\[
\psi = [\psi_j(\theta) \quad \psi_{j+1}(\theta) \quad \cdots \quad \psi_N(\theta)]'.
\]

Let \( T_j \) denote the length of the \( j \)-operative busy period, \( B_j(t) \) and \( b_j(t) \) be its c.d.f. and p.d.f. respectively, then

\[
L[b_j(t)] = L[q_{j-1}(t)] = j \mu \psi_j(\theta)
\]

\[
L[B_j(t)] = L[q_{j-1}(t)] = j \mu \psi_j(\theta)/\theta. \quad \text{(29)}
\]

Now we can obtain the solutions for \( \psi(n, \theta) \) from (23) and for \( \psi_j(\theta) \) from (28).
SECTION 5.2. RELIABILITY MODEL

Multiple-unit reliability systems

We discuss a general system with the following assumptions:

(i) The system consists of \( n \geq 1 \) units. The system requires \( k \leq n \) units for its successful operation.

(ii) Initially \( k \) units are operative and \( n-k \) units are kept as standbys.

(iii) There is a repair facility with \( r \geq 1 \) repairmen. The repair policy is first in, first out (FIFO).

(iv) The lifetime and repair-time durations of the units are independent random variables with negative exponential distributions with parameters \( \lambda \) and \( \mu \), respectively.

(v) The lifetime of the standby is a negative exponential random variable with parameter \( \lambda_s \).

(vi) On failure an operating unit is taken to the repair facility instantaneously where it is repaired according to the FIFO queue discipline.

(vii) The standby units are also taken to the repair facility on failure to follow the same course of action as the online operating units.

(viii) When the number of operable units is less than \( k \), the system is said to be non-operable.

(ix) On repair completion the unit joins the pile of spares if there are sufficient number of operable units; otherwise it is switched operative.

(x) Repair is assumed to be perfect.

Several special cases of the model can be obtained by specifying the values of the parameters \( k, n \) and \( \lambda_s \). Some of these can be given by
(a) for $k = 1$ and $r = 1$; the system is a $n$-unit warm standby system.

(b) for $k = 1$, $r = 1$ and $\lambda_s = 0$; the resulting system is a cold standby system.

(c) for $k = 1$, $r = 1$ and $\lambda_s = \lambda$; the resulting system is a parallel redundant system.

(d) for $k = 1$, $r > 1$ and $\lambda_s = 0$; the resulting system is a $n$-unit cold standby with a multiple repair facility.

Let $X(t)$ be the number of operable units at the time $t$. Then $\{X(t), t \geq 0\}$ is an integer-valued stochastic process taking values on $(0, 1, 2, .., n)$. $X(t) = 0$ implies that the system is down and $X(t) = n$ implies the system is operable with all $n$ units. The points of discontinuity of the stochastic process are those epochs at which either there is a failure or a repair completion of a unit. In both the cases either the process $X(t)$ decreases or increases by one unit. In view of the exponential duration of the lifetimes and repair times, $X(t)$ is a Markov process and by physical nature it is a birth-death process which will be suitable to discuss the transient behaviour of the system. Because of the multiple units in the models quoted above, the death rates are state-dependent. We use $\lambda_i$ to denote the failure rate when $i$ units are operable and $\mu$ to be the repair rate.

Let

$$p_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}, \ i, j \in \{0, 1, 2, \ldots, n\}$$

and set

$$p_i(t) = p_{nj}(t), \ j \in \{0, 1, 2, \ldots, n\},$$

then the function $p_j(t)$ satisfies the following Chapman-Kolmogrov equations:
\[ \begin{align*}
\dot{p}_0(t) &= -\mu p_0(t) + \lambda_1 p_1(t) \\
\dot{p}_j(t) &= - (\lambda_j + \mu) p_j(t) + \lambda_{j+1} p_{j+1}(t) + \mu p_{j-1}(t), \quad 1 \leq j < n \\
\dot{p}_n(t) &= -\lambda_n p_n(t) + \mu p_{n-1}(t) 
\end{align*} \] .... (30)

with \( p_j(0) = \delta_{jn} \). Now again if \( q_j(t) \) represents the probability that the system is in state \( j \), given the information that the state 0 has not been initiated in \((0, t)\), then the equations satisfied by \( q_j(t) \) are

\[ \begin{align*}
\dot{q}_1(t) &= - (\lambda_1 + \mu) q_1(t) + \lambda_2 q_2(t) \\
\dot{q}_j(t) &= - (\lambda_j + \mu) q_j(t) + \lambda_{j+1} q_{j+1}(t) + \mu q_{j-1}(t), \quad 2 \leq j < n \\
\dot{q}_n(t) &= -\lambda_n q_n(t) + \mu q_{n-1}(t) 
\end{align*} \] .... (31)

with \( q_j(0) = \delta_{jn} \).

The functions \( p_j(t) \) and \( q_j(t) \) adequately describe the important operating characteristics of the system.

**Bayesian approximation in reliability estimation**

Bayes estimators are often obtained as a ratio of two integrals which cannot be expressed in closed forms and numerical approximations are needed. Lindley [38] developed an asymptotic approximation to the ratio

\[ I = \frac{\int_{\Omega} w(\theta) e^{L(\theta)} d\theta}{\int_{\Omega} v(\theta) e^{L(\theta)} d\theta} \]

where \( \theta = (\theta_1, \theta_2, \ldots, \theta_m) \), \( L(\theta) \) is the logarithmic of the likelihood function, \( w(\theta) \) and \( v(\theta) \) are arbitrary functions of \( \theta \) and \( \Omega \) represents the range space of \( \theta \). Clearly, if \( w(\theta) = u(\theta) v(\theta) \) and \( v(\theta) \) is the prior distribution of \( \theta \), then
I = E \{u(\theta) \mid x\} = \text{Posterior expectation of } u(\theta) \text{ given the data } x = (x_1, x_2, \ldots, x_n),
which is the Bayes estimator of } u(\theta) \text{ under the squared-error-loss function.}

Reliability of series systems

Consider a system which consists of k different components \(c_1, c_2, \ldots, c_k\) connected in series. This system fails as soon as any one of the components fails. The reliability of the system \(R_i(s|k)\) is the probability that none of the components fails before time \(t\),
i.e., \(R_i(s|k) = P\{c_1 \geq t, c_2 \geq t, \ldots, c_k \geq t\}\).

On the assumption that the failure time distributions of the components are independent,
\[
R_i(s|k) = P(c_1 \geq t) \cdot P(c_2 \geq t) \ldots P(c_k \geq t)
\]
\[
= R_1(t) \cdot R_2(t) \ldots R_k(t) = \prod_{i=1}^{k} R_i(t)
\]
where \(R_i(t)\) denotes the reliability of the component \(c_i(i = 1, 2, \ldots, k)\). If the failure times of the components are identically distributed,
\[
R_i(s|k) = [R(t)]^k.
\]
We note that since \(R_i(t) \leq 1\), we have \(R_i(s, k) \leq \min_{1 \leq i \leq k} R_i(t)\) and the reliability of a system in which the components are connected in series is much smaller than the reliability of the individual units.

Series system with identical components

Let us consider a system arranged in series and such that each \(c_i\) has failure time distribution given by one-parameter exponential density with p.d.f.
\[ f(x|\sigma) = \frac{1}{\sigma} \exp\left\{ -\frac{x}{\sigma} \right\}, \ x \geq, \ \sigma > 0. \]

Then,

\[ R_t(s|k) = [R_i(t)]^k = \exp \left\{ -\frac{kt}{\sigma} \right\} \]

and if \( S \) denotes the failure time of the system then

\[ P\{S \leq t\} = 1 - \exp\left\{ -\frac{kt}{\sigma} \right\}, \ t > 0, \ \sigma \geq 0. \]

This shows that the failure time distribution of the system \( S \) is itself exponential with p.d.f.

\[ f(y|\theta) = \frac{1}{\theta} \exp\left\{ -\frac{y}{\theta} \right\}, \ y > 0, \ \theta = \frac{\sigma}{k} > 0. \]

If \( n \) such systems are put on test with observed failure times as \((y_1, y_2, \ldots, y_n)\),

We have \[ \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y} \]

and the MLE of the \( R_i(s|k) = R_i(t/\theta) = \exp \{-t/\theta\} \) is given by

\[ \hat{R}_i(s|k) = \exp \left\{ -\frac{t}{\hat{\theta}} \right\} = \exp \left\{ -\frac{nt}{S_n} \right\} \]

where \( S_n = \sum_{i=1}^{n} y_i \)

Next, consider the case where \((c_1, c_2, \ldots, c_k)\) are i.i.d. r.v. with each \( c_i \) having failure time distribution which two-parameter exponential. Thus, the p.d.f. of each \( c_i \) is given by
Here \( R_t(s|k) = \exp \left\{ \frac{k(t - \mu)}{\sigma} \right\}, \quad t > \mu \)

\[ \begin{align*}
R_t(s|k) &= 1, \quad t \leq \mu. \\
R(t/\hat{\mu}, \hat{\theta}) &= \exp \left\{ -\frac{(t - y(i))}{(\bar{y} - y(i))} \right\}, \quad \text{if } t > y(i) \\
&= 1, \quad \text{if } t \leq y(i).
\end{align*} \]

The failure time distribution of \( S \) is itself two-parameter exponential with \( \mu \) and \( \theta = \sigma/k \), i.e., with p.d.f.

\[ f(y|\mu, \sigma) = \frac{1}{\sigma} \exp \left\{ -\frac{y - \mu}{\sigma} \right\}, \quad x > \mu, \ \sigma > 0. \]

It immediately follows that

\[ P\{S \leq t\} = 1 - \exp \left\{ -\frac{k(t - \mu)}{\sigma} \right\}, \quad t > \mu \]

\[ = 0, \quad t \leq \mu. \]

The MLE of \( R_t(s|k) = R(t/\theta, \mu) \) is given by

We note that if \( n \) systems are put on test, we are using \( N = nk \) components and this may be an expensive requirement. We may view the problem in a slightly
different way and obtain estimators for $R_t(s|k)$ with only $m$ number of components put on test, rather than putting $n$ systems on test. Thus, we now assume that $m$ components are put on test with observed failure times $(x_1, x_2, \ldots, x_m)$ and the failure time distribution is one-parameter exponential with mean life $\sigma$. Thus, $(x_1, x_2, \ldots, x_m)$ is a random sample from a population with p.d.f. $f(x|\sigma) = (1/\sigma) \exp(-x/\sigma)$, $x \geq 0$, $\sigma > 0$, and the parametric function that we want to estimate is $R_t(s|k) = \exp\left\{-\left(kt/\sigma\right)\right\}$.

We note that the MLE of $R_t(s|k)$ is given by

$$
\hat{R}_t(s|k) = \exp\left\{-\frac{kt}{\bar{x}}\right\} = \exp\left\{-\frac{mkt}{S_m}\right\}
$$

### Reliability bounds-Bayesian approach

We now consider the Lieberman-Ross [36] model where we have the failure time distribution of components of type $c_1$ and $c_2$ given by

$$
f(x|\lambda_i) = \lambda_i \exp(-\lambda_i x), \; x > 0, \; \lambda_i > 0, \; i = 1, 2,
$$

and we assume the prior

$$
P(\lambda_1, \lambda_2) = \frac{1}{\lambda_1 \lambda_2}.
$$

This is an improper prior and corresponds to assuming $\lambda_1$, $\lambda_2$ independent and $\log \lambda_1$, $\log \lambda_2$ locally uniformly distributed.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ be the observed failure times of the components of type $c_1$ and $c_2$ respectively and let

$$
T = \sum_{i=1}^{n} x_i, \quad S = \sum_{i=1}^{m} y_i
$$
The usual computations lead to the posterior distribution of \((\lambda_1, \lambda_2)\) given by

\[
\pi(\lambda_1, \lambda_2|x, y) = K \exp \left(-\lambda_1 T - \lambda_2 S\right) \lambda_1^{n-1} \lambda_2^{m-1} \quad \ldots \quad (32)
\]

where

\[
K = \frac{T^n S^m}{\Gamma(n) \Gamma(m)}
\]

which implies that given \((T, S)\), \(\lambda_1, \lambda_2\) are independent gamma varieties with shape parameters \(n\) and \(m\) and scale parameters \(1/T\), \(1/S\) respectively.

Bayes estimator of \(R(s|2) = \exp \left\{-(\lambda_1 + \lambda_2) t\right\}\) is given by

\[
\hat{R}_t(B) = E[R_t(s|2)|x, y]
\]

\[
= \int_0^\infty \int_0^\infty \exp \left\{-(\lambda_1 + \lambda_2) t\right\} \prod \left(\lambda_1, \lambda_2 | x, y\right) d\lambda_1 d\lambda_2
\]

\[
= \left\{1 + \frac{t}{T}\right\}^{-n} \left\{1 + \frac{t}{S}\right\}^{-m} \quad \ldots \quad (33)
\]

It is to be noted that for large \(n\) and \(m\) the Bayes estimator \(\hat{R}_t(B)\) will be very close to

the MLE \(\hat{R}(t) = \exp \left\{1 - \frac{1}{x} + \frac{1}{y}\right\}\) and it can be shown that \(|\hat{R}_t(B) - \hat{R}(t)|\) converges

in probability to zero as \(n \rightarrow \infty\) and \(m \rightarrow \infty\). Since \(\lambda = \lambda_1 + \lambda_2\) is the parameter of

interest we obtain the posterior distribution of \(\lambda\) given \(x\) and \(y\). We use the

transformation \(\lambda_1 = \lambda u\) and \(\lambda_2 = \lambda (1-u)\).

The Jacobian of transformation is \(|J| = \lambda\). Substituting in (32) we obtain

\[
\Pi(\lambda, u|x, y) = K \lambda^{m+n-1} \exp \left\{-\lambda \left(T-S\right)\right\} \exp \left\{-\lambda \left(u(T-S)\right)\right\} u^{n-1} (1-u)^{m-1}
\]
Integrating out \( u \), the posterior distribution of \( \lambda \) is given by

\[
\prod \lambda \mid x, y = K \lambda^{m+n-1} \exp (-\lambda S) \int_0^1 \exp \{-\lambda u(T-S)\} u^{n-1} (1-u)^{m-1} du
\]

We can solve for \( \lambda_\alpha \) defined by the equation

\[
\alpha = \int_{\lambda_\alpha}^\infty \prod \lambda \mid x, y \, d\lambda
\]

and use it to obtain the one or two-sided confidence limits for \( R_t(s, 2) = \exp (-\lambda t) \).

Suppose we want to obtain the lower confidence limit \( R_1(t) \) such that

\[
1 - \alpha = P[R_1(t) < R_t(s, 2) = \exp (-t \lambda_\alpha)] = P[\lambda < \log \frac{1}{t} R_1(t)] = \lambda_\alpha.
\]

or equivalently

\[
\alpha = P[\lambda \geq \log \frac{1}{t} R_1(t)] = \lambda_\alpha.
\]

Thus, \( R_1(t) = \exp (-t \lambda_\alpha) \) where \( \lambda_\alpha \) has been defined in (34). Similarly, two-sided confidence limits for \( R_t(s/2) \) would be

\[
R_1(t) = \exp (-t \lambda_{\alpha/2}) \quad \text{and} \quad R_2(t) = \exp (-t \lambda_{1-\alpha/2}).
\]

Illustration 5.2.1

Let independent samples of size 3 and 4 are obtained on each of two components with the following results:
From the above data,

We have $n = 4$, $m = 3$, $r = 3$, $\sum_{i=1}^{4} x_i = 978.48$, $\sum_{i=1}^{3} y_i = 231.22$, $u = 231.22$, $k = 4$, $v_1 = 143.04$, $v_2 = 372.96$, $\sum z_i = 212.54$.

The maximum likelihood and the Bayes estimators of reliability are given by

$$\hat{R}_t = \exp \left\{ -t \left( \frac{4}{978.48} + \frac{3}{231.22} \right) \right\} = \exp (-0.0171 t)$$

$$\hat{R}_t^B = \left( 1 + \frac{t}{978.48} \right)^{-4} \left( 1 + \frac{t}{231.22} \right)^{-3}.$$

Using the method already referred to, $\lambda_{0.05} = 0.03168$ and $\lambda_{0.95} = 0.00695$ and the 95% lower posterior confidence limit of $R_t$ is $\exp (-0.03168 t)$. Substituting in (35), the 90% posterior confidence limits of $R_t$ are $R_1(t) = \exp (-0.03168 t)$ and $R_2(t) = \exp (-0.00695 t)$.

$\{ R_1^*(t), R_1, R_2^*(t) \}$ and $\{ R_1(t), \hat{R}_1^B, R_2(t) \}$ are plotted in Figures 5.2.1. and 5.2.2. on the same scale. We find
Fig. 5.2.1. Graphs of the maximum likelihood estimate $R(t)$ and 90% sampling theory confidence limits ($R_1^*, R_2^*$) for $R(t) = \exp(-\lambda t)$

Fig. 5.2.2. Graphs of the Bayes estimate $R^B(t)$ and 90% posterior confidence limits ($R_1, R_2$) for $R(t) = \exp(-\lambda t)$

(i) $\hat{R}_1 \geq \hat{R}_1^B$

(ii) $R_1^*(t) \leq R_1(t)$ and $R_2^*(t) \geq R_2(t)$.

Numerical calculation supports that $R_2(t) - R_1(t) < R_2^*(t) - R_1^*(t)$ for each $t \leq 500$. 
Parallel systems

Consider a system consisting of components \( c_1, c_2, \ldots, c_k \) arranged not in series but in parallel. Thus, the system will not fail if even one of the components is working, or equivalently the system fails when each of the components fails. Let \( R_t(p|k) \) denote the corresponding reliability for a parallel system. Then

\[
1 - R_t(p|k) = \text{Prob (system fails before t)} = P(c_1 \leq t, c_2 \leq t, \ldots, c_k \leq t).
\]

If the failure time distributions are independent,

\[
1 - R_t(p|k) = \prod_{i=1}^{k} F_i(t) = \prod_{i=1}^{k} [1 - R_i(t)]
\]

or

\[
R_t(p|k) = 1 - \prod_{i=1}^{k} [1 - R_i(t)]
\]

We can show that \( R_t(p|k) \geq \max_{i\leq k} R_i(t) \) by noting that

\[
R_t(p|K) = P(A_1 \cup A_2 \ldots \cup A_k) \geq \max_{i\leq k} P(A_i)
\]

where \( A_i \) is the event that the \( i^{th} \) component does not fail before time \( t \).

Suppose we have a system made up of \( k \) components \( (c_1, c_2, \ldots, c_k) \) connected in parallel. Let the life time \( X \) of each component be independently and identically distributed as \( f(x|\theta) = 1/\theta \exp (-x/\theta), \ x > 0, \ \theta > 0 \).

The system reliability is given by

\[
R_t(p|k) = 1 - P(c_1 \leq t, c_2 \leq t, \ldots, c_k \leq t) = 1 - (1 - R_t)^k
\]

\[
= \sum_{j=0}^{k} (-1)^{j-1} \binom{k}{j} R_t^j
\]
If the life times are independently but not identically distributed, we have

\[ R_t(p|k) = 1 - \sum_{j=1}^{k} (-1)^{j+1} \left( \begin{array}{c} k \\ j \end{array} \right) \exp \left\{ -\frac{jt}{\theta} \right\}. \quad \ldots (36) \]

We note that, in general, the reliability of a parallel system is higher than the reliability of a series system with the same number of similar components since

\[ R_t(s|k) \leq \min_{1 \leq i \leq k} R_i(t) \leq \max_{1 \leq i \leq k} R_i(t) \leq R_t(p|k). \]

**SECTION 5.3. FAILURE DISTRIBUTIONS**

A natural strengthening of the concept of decreasing mean residual life is to assume that the conditional probability of failure given survival to time \( t \) is increasing in \( t \); that is, \( r(t) \) is increasing in \( t \). Distributions with this property will be denoted IFR for increasing failure rate. Distributions for which \( r(t) \) is decreasing will be denoted DFR for decreasing failure rate.

Failure rate is sometimes defined as the probability of failure in a finite interval of time, say of length \( x \), given the age of the component, say \( t \). If \( F \) denotes the failure distribution, then the failure rate by this definition would be

\[
\frac{F(t + x) - F(t)}{\bar{F}(t)}
\]
Definition 5.3.1

A non-discrete distribution $F$ is IFR (DFR) if and only if

$$\frac{F(t + x) - F(t)}{\bar{F}(t)}$$

is increasing (decreasing) in $t$ for $x > 0$, $t \geq 0$ such that $F(t) < 1$.

Definition 5.3.2

A discrete distribution $\{p_k\}_{k=0}^{\infty}$ is IFR (DFR) if and only if

$$\frac{p_k}{\sum_{i=k}^{\infty} p_i}$$

is non-decreasing (non-increasing) in $k = 0, 1, 2, \ldots$.

Definition 5.3.3

A function $p(x)$ defined for $x$ in $(-\infty, \infty)$ is a Polya frequency function of order 2 (PF$_2$) if and only if $p(x) \geq 0$ for all $x$ and

$$\begin{vmatrix} p(x_1 - y_1) & p(x_1 - y_2) \\ p(x_2 - y_1) & p(x_2 - y_2) \end{vmatrix} \geq 0$$

whenever $-\infty < x_1 \leq x_2 < \infty$ and $-\infty < y_1 \leq y_2 < \infty$.

Definition 5.3.4

A function $p(x, y)$ defined for $x \in X$ and $y \in Y$ ($x$ and $y$ linearly ordered sets) is totally positive of order 2 (TP$_2$) if and only if $p(x, y) \geq 0$ for all $x \in X$, $y \in Y$ and

$$\begin{vmatrix} p(x_1, y_1) & p(x_1, y_2) \\ p(x_2, y_1) & p(x_2, y_2) \end{vmatrix} \geq 0$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$ ($x_1, x_2 \in X; y_1, y_2 \in Y$).
Result 5.3.1. (Barlow [2])

The following statements are equivalent. We assume \( F(0') = 0 \).

(a) \( F \) is an IFR (DFR) distribution.

(b) \( \log \tilde{F}(t) \) is concave (convex) for \( t \) in \( \{ t \mid T(t) < 1, t > 0 \} \).

(c) \( \tilde{F}(t) \) is PF2 (\( \tilde{F}(x+y) \) is TP2 in \( x \) and \( y \) for \( x + y \geq 0 \)).

Theorem 5.3.1

If \( F_1 \) and \( F_2 \) are IFR, their convolution \( H \), given by

\[
H(t) = \int_{-\infty}^{\infty} F_1(t-x) \, dF_2(x),
\]

is also IFR.

Proof

Assume \( F_1 \) has density \( f_1 \), \( F_2 \) has density \( f_2 \). For \( t_1 < t_2 \), \( u_1 < u_2 \), form

\[
D = \left| \frac{\partial \tilde{H}(t_i-u_j)}{\partial t_i} \right|_{i=1,2} = \int \tilde{F}_1(t_i-s) \, f_2(s-u_i) \, ds
\]

\[
= \int \int \tilde{F}_1(t_i-s_k) \left\| f_2(s_k-u_i) \right\| |ds_2| \, ds_1.
\]

Integrating the inner integral by parts. We obtain

\[
D = \int \int \begin{vmatrix}
\tilde{F}_1(t_1-s_1) & f_1(t_1-s_2) & f_2(s_1-u_1) & f_1(s_1-u_2) \\
\tilde{F}_1(t_2-s_1) & f_1(t_2-s_2) & f_2(s_2-u_1) & f_1(s_2-u_2)
\end{vmatrix} \, ds_2 \, ds_1
\]

The sign of the first determinant is the same as that of

\[
\begin{pmatrix}
f_1(t_2-s_2) & \tilde{F}_1(t_2-s_2) & f_1(t_1-s_2) & \tilde{F}_1(t_1-s_2) \\
\tilde{F}_1(t_2-s_2) & \tilde{F}_1(t_2-s_1) & \tilde{F}_1(t_1-s_2) & \tilde{F}_1(t_1-s_1)
\end{pmatrix}
\]

assuming nonzero denominators. But
by hypothesis, whereas

\[
\frac{f_1(t_2 - s_2)}{F_1(t_2 - s_2)} \geq \frac{f_1(t_1 - s_2)}{F_1(t_1 - s_2)}
\]

by Result 5.3.1. Thus the first determinant is nonnegative. A similar argument holds for the second determinant, so that \( D \geq 0 \). But by Result 5.3.1, this implies \( H \) is IFR.

If \( F \) and/or \( G \) do not have densities, the theorem may be proved in a similar fashion, using limiting arguments.

Result 5.3.2. (Barlow [2])

If \( F_i(t) \) is a DFR distribution in \( t \) for each \( i = 1, 2, \ldots \), \( a_i \geq 0 \) for each

\[
\sum_1^\infty a_i = 1,
\]

then

\[
G(t) = \sum_{i=1}^\infty a_i F_i(t)
\]

is a DFR distribution.

Result 5.3.3. (Barlow [2])

If \( F_1 \) and \( F_2 \) are log concave, then

\[
H(t) = \int_0^t F_1(t-x) \, dF_2(x)
\]

is also log concave.
Result 5.3.4. (Barlow [2])

If $F_1$ and $F_2$ are IFR with failure rates $r_1(t)$ and $r_2(t)$ respectively and $H$ denotes their convolution with failure rate $r_h(t)$, then

$$r_h(t) \leq \min \{r_1(t), r_2(t)\}.$$

Theorem 5.3.2

Assume $X$ is a random variable with distribution $F$ and density $f$ which is IFR. If $X_1, X_2, ..., X_n$ are $n$ independent observations on $X$, the order statistics

$$U_1 < U_2 < ... < U_n$$

formed from the $X_i$'s are also IFR.

Proof

Let $H$ denote the distribution of $U_k$ and $p = \bar{F}(t)$.

Then

$$\bar{H}(t) = \sum_{i=k}^{n} \left( \frac{n}{i} \right) [\bar{F}(t)]^i [F(t)]^{n-i}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k) \Gamma(n+1-k)} \int_0^p x^{k-1} (1-x)^{n-k} \, dx.$$ 

and

$$\frac{\bar{H}(t)}{H'(t)} = \frac{1}{p} \int_0^p \left( \frac{x}{p} \right)^{k-1} \left( \frac{1-x}{1-p} \right)^{n-k} \, dx.$$ 

Letting $u = x/p$, we have

$$\frac{\bar{H}(t)}{H'(t)} = \frac{p}{f(t)} \int_0^1 u^{k-1} \left( \frac{1-up}{1-p} \right)^{n-k} \, du.$$ 

Because $p/f(t)$ and $(1 - up) / (1-p)$ are each decreasing in $t$, so is $\bar{H}(t)/H'(t)$, or $H'(t)/\bar{H}(t)$ is increasing in $t$. 

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