CHAPTER - I

CHARACTERIZATION OF STOCHASTIC PROCESSES BY MEANS OF STOCHASTIC INTEGRALS.

Several characterization theorems for Wiener process are now known by means of conditions on either independence or identical distribution of stochastic integrals or through regression properties of them. In this chapter the characterization of wiener process by means of stochastic integrals defined in probability is proved. Also a review of the recent work on the characterization of Wiener and stable processes and connected results through stochastic integrals is presented. The levy canonical representation of the characteristic functions of a stochastic integrals in the sense of convergence in probability is obtained. A different characterization of symmetric stable processes, noticed by Lucaks is derived.

1.1 Some important Definitions & Results:–

Let us now list out all the important definitions and results as follows [56]

1.1.1 Expectation:-

The expectation of a random variable X in symbols E(X) is defined as the integral

\[ E(X) = \int_{-\infty}^{\infty} x \, dF(x) \]

provided that this integral exists.

1.1.2 Characteristic Functions:-

Let X be a random variable and t be an arbitrary real number. Then

\[ e^{itx} = \cos(t \cdot x) + i \sin(t \cdot x) \]

is a complex random variable. We know if \( Z = X + i \, Y \)

Then the expectation of a complex random variable Z is

\[ E(Z) = E(X) + i \, E(Y) \]
$$E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

The expectation \( E(e^{itX}) \) exists for any real \( t \) and is called the characteristic function of this random variable \( X \) or the characteristic function of the distribution function \( F(x) \).

### 1.1.3 Polynomical Regression:

Let us consider two random variables \( X \) and \( Y \) and assume that the conditional expectation \( E(Y/X) \) exists. \( Y \) has polynomical regression of order \( K \) on \( X \) if the relation

$$E(Y/X) = \beta_0 + \beta_1 X + \ldots + \beta_K X^K$$

holds almost everywhere.

Also let us assume that the first moment of \( Y \) and the \( K^{th} \) moment of \( X \) exists. Then it follows from (1.1.1) that

$$E(Y) = \beta_0 + \beta_1 E(X) + \ldots + \beta_K (E(X^K))$$

The co-efficients \( \beta_0, \beta_1, \ldots, \beta_K \) are called the regression co-efficients. If \( \beta_K = 0 \) (ie) if the relation

$$E(Y/X) = E(Y)$$

holds almost everywhere then we say that \( Y \) has constant regression on \( X \). If \( \beta_K = 1 \) and \( \beta_1 \neq 0 \), then \( Y \) has linear regression.

### 1.1.4 Homogeneous Process with independent increments:

Let \( T \) be a set of real numbers and suppose that a random variable \( X(t) \) is given for each \( t \in T \). The set of all random variable \( X(t) \) is then called a stochastic process. In many cases a stochastic process describes the development of a random quantity over time. Therefore one often refers to the parameter \( t \) as time. A stochastic process is said to depend on a continuous parameter if \( T \) is a (finite or infinite) interval.

Let \( X(t) \) be a stochastic process and \( t_1, t_2 \in T \) and \( t_1 < t_2 \), the random variable

\( X(t_2) - X(t_1) \)
$X(t_2) - X(t_1)$ is called the increment of the process $X(t)$ over the interval $[t_1, t_2]$.

A stochastic process $X(t)$ is said to be a process with independent increments if the increments over non-overlapping intervals are stochastically independent. A process $X(t)$ is called a homogeneous process if the distribution of the increment $X(t + \tau) - X(t)$ depends only on $\tau$ but is independent of $t$.

We denote the characteristic function of the distribution of $X(t + \tau) - X(t)$ by $\phi(u, \tau)$.

**1.1.5 Infinitely divisible:**

A distribution function $F$ is called infinitely divisible if for every integer $n > 1$, there exists a distribution function $F_n$ such that

$$F = F_n * F_n * \ldots * F_n = (F_n)^n$$

(or)

$$\phi = \phi_1^n$$

equivalently if its characteristic function (also called infinitely divisible) is the $n$th power of a characteristic function $\phi$, for every integer $n \geq 1$.

**1.1.6 Convolution of distribution functions:**

It is a binary operation denoted by $*$ which associates with every pair $F_1, F_2$ of distribution functions a distribution function $F$ denoted by $F = F_1 * F_2$, and defined by

$$F(x) = (F_1 * F_2)(x) = \int_{-\infty}^{\infty} F_1(x-y) dF_2(y).$$

**1.1.7 Stable distribution:**

A distribution function $F$ or its characteristic function $\phi$ is called stable if for every pair of positive constants $b_1, b_2$ and real constants $a_1, a_2$ there exists $b > 0$ and real $a$ such that

$$F(b_1 x + a_1) * F(b_2 x + a_2) = F(bx + a).$$

**1.1.8 Wiener Process:**

Let $\{X(t), 0 \leq t < \infty\}$ be a stochastic process with the following properites.
1. Suppose $t_0 < t_1 < \ldots < t_n$ then the increments $X_{t_1} - X_{t_0}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ are mutually independent random variables.

2. The probability distribution of $X_{t_2} - X_{t_1}$, $t_2 \geq t_1$ depends only on $(t_2 - t_1)$.

3. \[ P [X_t - X_s \leq x] = \left[ \frac{2\pi (t-s)}{2} \right]^{1/2} \exp \left( -\frac{x^2 (t-s)}{2} \right) \right] \] Then the process is called a Wiener process.

1.1.9 A stochastic process $X(t)$ is said to be of second order if

\[ \mathbb{E}[X(t)]^2 \text{ is finite.} \]

1.1.10 Mean value Function:-

Let $X(t)$ be a stochastic process of second order. The function

\[ M(t) = \mathbb{E}[X(t)] \]

\[ \sigma(t, t') = \mathbb{E}[X(t)X(t')] - \mathbb{E}[X(t)] \mathbb{E}[X(t')] \]

are called the mean value function and the covariance function of the process $X(t)$.

According to Chow and Teicher the notion of infinite divisibility and Levy Kintchine representation is considered as follows.

1.1.11 Support:-

For a distribution function $F$, the set

\[ S(F) = \{ X : F(X + \varepsilon) - F(X - \varepsilon) > 0 \quad \forall \varepsilon > 0 \} \] is called the support or spectrum of $F$. Any point $X \in S(F)$ is called the point of increase of $F$.

1.1.12 Degenerate distribution function:-

A distribution function is degenerate or improper if it has a single point of increase.

1.1.1 Proposition:-

A distribution function $F$ with bounded support is infinitely divisible if and only if it is degenerate.
1.1.2 Proposition:-

An infinitely divisible characteristic function \( \phi(t) \) does not vanish (for real \( t \)).

1.1.1 Lemma:-

If \( f(t) \) is a continuous, non-vanishing complex function on \([-T, T]\) with \( f(0) = 1 \). There is a unique (single valued) continuous function \( \lambda(t) \) on \([-T, T]\) with \( \lambda(0) = 0 \) and \( f(t) = e^{\lambda(t)} \). Moreover \((-T, T)\) is replaceable by \((-\infty, \infty)\).

1.1.13 Definition:-

The function \( \lambda(t) \) designed by \( \lambda(t) = \text{Log} f(t) \) is called the distinguished logarithm of \( f(t) \) and is denoted by \( \text{Log} f(t) \). Also \( \exp \left( \frac{1}{n} \lambda(t) \right) \) is called the distinguished \( n^\text{th} \) root of \( f(t) \) and is denoted by \( f^{1/n}(t) \).

1.1.2 Lemma:-

Let \( f, f_k, k \geq 1 \) be as in Lemma (1.1.1). If \( f_k \rightarrow f \) uniformly in \([-T, T]\).

Then \( \text{Log} f_k \rightarrow \text{Log} f \) uniformly in \([-T, T]\).

1.1.2 Proposition:-

A characteristic function \( \phi \) is infinitely divisible if and only if its distinguished \( n^\text{th} \) root \( \phi^{1/n}(t) = e^{1/n \text{Log} \phi(t)} \) is a characteristic function for all positive integer.

1.1.3 Proposition:-

A finite product of infinitely divisible characteristic functions is infinitely divisible. Moreover if infinitely divisible characteristic function \( \phi_k \rightarrow \phi \), a characteristic function then this limit characteristic function \( \phi \) is infinitely divisible.

1.1.4 Proposition:-

The class of infinitely divisible laws coincides with the class of distribution limits of finite
convolutions of distributions of poisson type

1.1.1 Theorem:--

\[ \phi(t) = \exp(\psi(t, \gamma, G)) \text{ as defined by } \psi(t, \gamma, G) = i \gamma t + \int_{-\infty}^{\infty} \left( e^{iu} - 1 \right) \frac{1 + u^2}{u^2} dG(u) \]  

(1.1.1)

is an infinitely divisible characteristic function for every real \( \gamma \) and \( G \) as stipulated.

Moreover \( \phi \) uniquely determines \( \gamma \) and \( G \).

1.1.2 Theorem:--

Let \( \{ \gamma, \gamma_n, n \geq 1 \} \) be finite real numbers and \( \{ G, G_n, n \geq 1 \} \) non decreasing, left continuous functions of bounded variation which vanish at \(-\infty\). If \( \gamma_n \to \gamma \) and \( G_n \to G \) then

\[ \psi(t, \gamma_n, G_n) \to \psi(t, \gamma, G) \]

for all real \( t \), where \( \psi \) is as in (1.1.1) conversely if \( \psi(t, \gamma_n, G_n) \to \) a continuous function \( g(t) \) as \( n \to \infty \) then necessarily

\[ g(t) = \psi(t, \gamma, G) \text{ and } \gamma_n \to \gamma, G_n \to G. \]

1.1.3 Theorem:--

Levy-Khintchine representation

A characteristic function \( \phi(t) \) is infinitely divisible if and only if

\[ \phi(t) = \exp[i \gamma t + \int_{-\infty}^{\infty} \left( e^{iu} - 1 \right) \frac{1 + u^2}{u^2} dG(u)] \]

(1.1.2)

Where \( \gamma \) and \( G \) are stipulated in (1.1.1).

Stable distributions form a subclass of infinitely divisible Laws and necessary and sufficient condition for a function to be a stably distributions by Chow and Teicher is discussed as follows.
1.1.4 Theorem:-

The class of limit distributions of normed sums \( \sum_{i=1}^{B_n} \frac{X_i - A_n}{B_n} \) of infinitely divisible random variables \( \{X_n, n \geq 1\} \) coincides with the class of stable laws.

1.1.5 Theorem:-

A function \( \phi \) is a stable characteristic function if and only if

\[
\phi(t) = \phi_\alpha(t, \gamma, \beta, C) = \exp[i \gamma t - C |t|^{\alpha} [1+i\beta \frac{1}{|t|} w(t, \alpha)]]
\]

Where \( 0 < \alpha < 2, |\beta| < 1, C > 0 \) and

\[
W(t, \alpha) = \tan \left( \frac{\pi \alpha}{2} \right), \quad \alpha \neq 1
\]

\[
= \frac{\pi}{2} \log |t|, \quad \alpha = 1
\]

\( \beta = 0 = \gamma \implies \) symmetric stable distributions. \( \alpha \) is called the characteristic exponent.

If \( \alpha = 2 \) \( W(t, \alpha) = 0 \). Therefore \( \beta = 0 \) yielding the normal characteristic function.

When \( \alpha < 2 \), absolute moments of order \( \gamma \) are finite if and only if \( \gamma < \alpha \).

1.2. CHARACTERIZATION OF THE WIENER PROCESS THROUGH IDENTICALLY DISTRIBUTED STOCHASTIC INTEGRALS.

1.2.1 Stochastic Integrals:-

First of all let us consider the integrals described by Riedel. Let \( \{X(t), t \in T\} \) be a continuous homogeneous process with independent increments. Suppose \( a(t) \) is a continuous function defined on \([A, B] \subset T\) Stochastic integrals of the form \( \int_{A}^{B} a(t) \, dx(t) \) can be defined either in the sense convergence in probability or in the sense of quadratic mean depending on the properties of the process \( \{X(t), t \in T\} \).

Let \( b(t) \) and \( \omega'(t) \) be functions defined on \([A, B] \subset T = [0, \omega] \) and \( w(t) \) be non negative.
Let \( D_n = A = t_{n,0}, ..., t_{n,n} = B, n \geq 1 \) be a sequence of sub divisions of the interval \([A,B]\) such that
\[
\lim_{n \to \infty} \max_{1 \leq k \leq n} (t_{n,k} - t_{n,k-1}) = 0
\]
Select \( t_{n,k}^* \in [t_{n,k-1}, t_{n,k}] \) and construct the sum
\[
S_n = \sum_{k=1}^{n} b(t_{n,k}^*) [x(w(t_{n,k}^*)) - x(w(t_{n,k-1}^*)))]
\]
If the sequence \( \{S_n\} \) converges in probability to a random variable \( S \) and if this limit is independent of the choice of the sub division and the points \( t_{n,k}^* \) then we say that \( S \) exists in probability and it is denoted by
\[
\int_{A}^{B} b(t) \, dx \, (w(t))
\]
If the limit exists in quadratic mean, then the integral is said to exist in quadratic mean.

1.2.1 Lemma:-

Let \( x(t) \) be a homogeneous stochastic process with independent increments which is defined in a finite closed interval \([A,B]\). Suppose that \( a(t) \) and \( b(t) \) are two continuous functions defined in \([A,B]\) and that the stochastic integrals.
\[
Y = \int_{A}^{B} a(t) \, dx \, (t)
\]
\[
Z = \int_{A}^{B} b(t) \, dx \, (t)
\]
exist in the sense of convergence in probability. Denote the characteristic function of the increment \( X(t+\tau) - X(t) \) and of the random vector \((y, z)\) by \( \Phi(u, \tau) \) and \( \Phi(u, v) \) respectively. Then \( \Phi(u,v) \neq 0 \) for every real \( U \) and \( V \) and \( \psi(u,v) = \ln \Phi(u,v) \)
\[
= \int_{A}^{B} \psi(u, a(t) + v, b(t)) \, dt
\]
Here $\psi(u) = \ln \phi(u,1)$.

**Proof:**

Let us consider a sequence of sub divisions $A = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = B$ of the interval $[A,B]$ such that

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} (t_{n,k} - t_{n,k-1}) = 0$$

and a sequence of points $t_{n,k}^* (k = 1, 2, \ldots, n)$ such that $t_{n,k-1} < t_{n,k}^* < t_{n,k}$ we form the Riemann Stieltjes sums

$$Y_n = \sum_{k=1}^{n} a(t_{n,k}^*) \left[ x(t_{n,k}) - x(t_{n,k-1}) \right]$$

$$Z_n = \sum_{k=1}^{n} b(t_{n,k}^*) \left[ x(t_{n,k}) - x(t_{n,k-1}) \right]$$

Then $\lim_{n \to \infty} Y_n = Y$

$\lim_{n \to \infty} Z_n = z$

Let $\Phi_n(u,v)$ be the characteristic function of random variable $(Y_n, Z_n)$ and

$$\psi_n(u,v) = \ln \Phi_n(u,v)$$

Use the relation $\phi(u, \tau) = [\phi(u,1)]^\tau$ and obtain

$$\Phi_n(u,v) = \prod_{k=1}^{n} (u a_{n,k} + v b_{n,k}) ; 1$$

$$a_{n,k} = a(t_{n,k}^*)$$

$$b_{n,k} = b(t_{n,k}^*)$$

Therefore
\[
\psi_n(u,v) = \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) \psi (u,a_{n,k} + v,b_{n,k})
\]

Right hand side of (1.2.2) is Riemann Stieltjes sum for the integral
\[
\int_{A}^{B} \psi[u(a(t) + v,b(t))] \, dt
\]

Let \( \{X_n\} \) and \( \{Y_n\} \) be two sequences of random variable such that \( p \text{ Lt } X_n = x \) and \( p \text{ Lt } Y_n = y \).

Suppose that \( g(x,y) \) is a function which is single valued, continuous and bounded for all real \( x \) and \( y \) then
\[
\text{Lt}_{n \to \infty} E[g(X_n,Y_n)] = E[g(x,y)]
\]

If \( \{X_n\} \) and \( \{Y_n\} \) be two sequences of random variable such that
\[
p \text{ Lt } X_n = x \quad p \text{ Lt } Y_n = y.
\]

Suppose that \( g(x,y) \) is a single valued continuous function then
\[
p \text{ Lt } g(X_n,Y_n) = g(X,Y)
\]

Let us denote the distribution of \( X_n \) and \( X \) by \( F_n(x) \) and \( F(x) \) respectively, then
\[
\{F_n(x)\} \to F(x) \text{ in all continuity points of } F(x).
\]

\( f_n(u,v) \) and \( f(u,v) \) be characteristic function of \( (X_n, Y_n) \) and \( (X,Y) \)

Therefore
\[
\text{Lt}_{n \to \infty} E[\cos (uX_n + vY_n)] = E[\cos (uX + vY)]
\]

Therefore
\[
\text{Lt}_{n \to \infty} E[\sin (uX_n + vY_n)] = E[\sin (uX + vY)]
\]

So that
\[
\text{Lt}_{n \to \infty} f_n(u,v) = f(u,v)
\]

Therefore
\[
\psi_n(u,v) = \psi(u,v) = \int_{A}^{B} \psi (u(a(t) + v,b(t))) \, dt
\]

Therefore
\[
\psi(u,v) = \ln \phi(u,v) = \int_{A}^{B} \psi [ua(t) + vb(t)] \, dt
\]
1.2.1 Theorem:

Let \( x(t) \) be a homogeneous stochastic process with independent increments which is defined in a finite closed interval \([A,B]\). Suppose that \( a(t) \) and \( b(t) \) are two continuous functions defined in \([A,B]\) such that either \( a(t) \) or \( b(t) \) does not vanish in \([A,B]\). Assume further that the stochastic integrals \( Y = \int_A^B a(t) \, dx(t) \) and \( Z = \int_A^B b(t) \, dx(t) \) exist in the sense of convergence in probability. The process \( x(t) \) is a Wiener Process if and only if

(i) \( Y \) and \( Z \) are stochastically independent.

(ii) \( \int_A^B a(t) b(t) \, dt = 0 \)

Proof:

Assume that \( a(t) \neq 0 \) in \([A,B]\) and conditions (i) and (ii) are satisfied.

since \( Y \) and \( Z \) are stochastically independent from the previous Lemma 1.2.1

\[
\int_A^B \psi \left[ u a(t) + v b(t) \right] \, dt = \int_A^B \psi \left[ u a(t) \right] \, dt + \int_A^B \psi(v b(t)) \, dt
\]

Multiply both sides of (1.2.3) by \((x-u)\) and integrate with respect to \( u \) from 0 to \( x \).

Therefore we set

\[
\int_0^x [(x-u) \int_A^B \psi \left[ u a(t) + v b(t) \right] dt ] \, du
\]

\[
= \int_0^x [(x-u) \int_A^B \psi \left[ u a(t) \right] dt ] \, du + \int_0^x \psi(v b(t)) \, dt
\]

Reverse the order of the two integrations in left hand side

Put \( U = \frac{L \rightarrow V}{a(t)} b(t) \)
\[ du = \frac{dw}{a(t)} \]

\[
\int_A^B \int_a(t) \left[ x - \frac{w - vb(t)}{a(t)} \right] dw \, dt = \int_A^B \int_a(t) \left[ \frac{x - w}{a(t)} \right] \psi(ua(t)) \, dt \, du + \frac{x^2}{2} \int_A^B \psi(vb(t)) \, dt
\]

\[
\int_A^B \int_a(t) \left[ x - \frac{w - vb(t)}{a(t)} \right] \psi(w) \, dw \, dt = \int_A^B \int_a(t) \left[ \frac{x - w}{a(t)} \right] \psi(ua(t)) \, dt \, du + \frac{x^2}{2} \int_A^B \psi(vb(t)) \, dt \Rightarrow (1.2.4)
\]

According to H. Carslaw [10] We can integrate both Left hand side and Right hand side of (1.2.4) as follows.

Let \( F(x,y) \) either be a continuous function of \((x,y)\) in the region \( a \leq x \leq a', \, b \leq y \leq b' \), \( a' \) being arbitrary or be of the form \( \phi(x,y) \, \psi(x) \) where \( \phi(x,y) \) is continuous and \( \psi(x) \) is bounded and integrable in the arbitrary interval \((a,a')\). Also let \( F(x,y) \) have a partial differential \( \frac{\partial F}{\partial y} \) co-efficient \( \cdots \) which satisfies the same condition.
Hence if the integral

\[
\int_{a}^{\infty} F(x, y) \, dx \text{ converges to } f(y) \text{ and }
\]

\[
\int_{a}^{\infty} \frac{\partial F}{\partial y} \, dx \text{ converges uniformly in } (b, b')
\]

\( f(y) \) has a differential co-efficient at every point in \((b,b')\) and

\[
f'(y) = \int_{a}^{\infty} \frac{\partial F}{\partial y} \, dx
\]

Therefore the above integrals, can be differentiated twice with respect to \( v \).

\[
v_b(t) \frac{\partial}{\partial v} \int_{a(t)}^{\infty} F(v, w) \, dw
\]

\[
f(v) = \int_{a}^{\infty} F(v, w) \, dw
\]

\[
f'(v) = \int_{0}^{v_b(t)} \frac{\partial F}{\partial v} \, dw
\]

\[
= \int_{0}^{v_b(t)} \frac{b(t)}{a(t)} \psi(v_b(t)) \, dw
\]

\[
f''(v) = \int_{a(t)}^{b^2(t)} \frac{v_b(t)}{a(t)} \psi'(w) \, dw
\]

\[
= \frac{b^2(t)}{a(t)} \psi(v_b(t))
\]
\[
\begin{align*}
&\int_{A}^{B} \frac{b^2(t)}{a^2(t)} \psi(a x(t) + v b(t)) \, dt = \\
&\int_{A}^{B} \frac{b^2(t)}{a^2(t)} \psi(v b(t)) \, dt \\
&+ x \int_{A}^{B} \frac{b(t)}{a(t)} \psi(v b(t)) \, dt \\
&+ \frac{x^2}{2} \int_{A}^{B} \psi(v b(t)) \, dt \\
&\Rightarrow (1.25)
\end{align*}
\]

\[
\begin{align*}
&\int_{A}^{B} b(t) \psi(v b(t)) \, dt = C_1 \int_{A}^{B} \frac{b^2(t)}{a(t)} \, dt \\
&\psi(0) = \ln \phi(0,1)
\end{align*}
\]

\[
\begin{align*}
&\int_{A}^{B} \frac{d^2}{dv^2} \psi(v b(t)) \, dt = -C_2 \int_{A}^{B} \frac{b^2(t)}{a(t)} \, dt
\end{align*}
\]

Put \( v = 0 \) in (1.2.5), we get

\[
\int_{A}^{B} b^2(t) \psi(a x(t)) \, dt = x C_1 \int_{A}^{B} \frac{b^2(t)}{a(t)} \, dt - \frac{x^2}{2} C_2 \int_{A}^{B} b^2(t) \, dt
\]

Assume that the distribution of

\[
X(t+1) - X(t)
\]

is symmetric so that

\[
\psi(u) = \psi(-u) \text{ for all real } u. \text{ Therefore } C_1 = 0
\]

Since \( \psi(u) \) is real and negative \( C_2 > 0 \)

Therefore

\[
\int_{A}^{B} \frac{d^2}{dv^2} \psi(a x(t)) \, dt + \frac{x^2}{2} C_2 \int_{A}^{B} a^2(t) \, dt = 0 \Rightarrow (1.2.6)
\]
\(\psi(u)\) is a continuous even function which has its maximum at \(a = b\)

Therefore \(\psi [xa(t)] = \frac{-x^2}{2} c_2 a^2(t)\)

Therefore \(\phi(u,1) = e^{c_2 c^2/2}, c_2 > 0\) in a certain neighbourhood of the point \(u = 0\).

Therefore \(X(t + \tau) - X(t)\) are normally distributed.

Therefore \(X(t)\) is a Wiener Process.

Their necessity follows from the expression for \(\psi(u,v)\).

1.2.2 Theorem:-

Let \(X(t)\) be a stochastic process, defined in a finite closed interval \([A,B]\) which satisfies the following conditions.

(i) \(X(t)\) is homogeneous and has independent increments.

(ii) \(X(t)\) is of second order and its mean value function and co-variance function are of bounded variation in \([A,B]\).

Suppose that \(a(t)\) and \(b(t)\) are two continuous functions defined in \([A,B]\) such that \(a(t)b(t) \neq 0\) for all \(t \in [A,B]\). Where \(A \leq A_i \leq B_i \leq B\). Suppose further that \(a(t)\) is not proportional to \(b(t)\). Let

\[
Y = \int_{A}^{B} a(t) \, dx(t)
\]

\[
Z = \int_{A}^{B} b(t) \, dx(t)
\]

be two stochastic integrals, defined as limits in the quadratic mean. The process \(X(t)\) is a Wiener Process if and only if
(i) Y has linear regression on Z

(ii) The conditional variance of Y, given X does not depend on Z.

**Proof:**

Let \( a(t) \) be a function which is continuous in the finite closed interval \([A,B]\). Suppose that \( X(t) \) is a stochastic process of second order such that its mean value \( m(t) \) and its co-variance function \( \sigma(t, t') \) are of bounded variation in \([A,B]\) then the stochastic integral

\[
\int_A^B a(t) \, dx(t) \text{ exists as a limit in quadratic mean}
\]

Let \( \phi(u,v) \) be the characteristic functions of random variables \((Y,Z)\) and

\[
\phi(u,v) = \ln \phi(u,v).
\]

To prove that (i) and (ii) are sufficient

Therefore \( E[Y/Z] = \alpha + \beta Z \)

\[
\text{Var} \left[ Y/Z \right] = \gamma \quad \text{hold almost everywhere} \quad \ldots \quad 1.2.7
\]

According to the following lemma, we have \( X \) and \( Y \) two random variables with finite second moments. The random variable \( Y \) has linear regression and constant scatter on \( X \) (ie)

\[
E[Y/X] = \alpha + \beta X
\]

\[
\text{Var} \left[ Y/X \right] = \delta \quad \text{almost everywhere if and only if}
\]

\[
\frac{\partial^2 g(u,v)}{\partial u \partial v} \bigg|_{v=0} = - (\sigma^2 + \alpha^2) g(u,0) + 2i \alpha \beta \frac{d}{du} g(u,0) + \frac{d^2}{du^2} g(u,0)
\]

and

\[
\frac{\partial^2 g(u,v)}{\partial u \partial v} \bigg|_{v=0} = - (\sigma^2 + \alpha^2) g(u,0) + 2i \alpha \beta \frac{d}{du} g(u,0) + \frac{d^2}{du^2} g(u,0)
\]

hold for all real \( u \). Where \( g(u,v) \) is the characteristic function of random
variable \( y, z \).

\[
\frac{\partial^2 \psi(u,v)}{\partial u^2} \bigg|_{u=0} = -\gamma + \beta^2 \frac{d^2 \psi(0,v)}{dv^2}
\]

we know that

\[
\psi(u,v) = \int_A \psi \left( (ua(t) + vb(t)) \right) dt
\]

we have

\[
\psi^{\|} (vb(t)) a^2(t) dt = -\gamma + \beta \int_A \psi^{\|} (vb(t) b(t)) dt
\]

Differentiating (1.2.10) with respect to \( V \) we get

\[
\int_A \psi^{\| [vb(t) a(t) b(t)]} dt = \beta \int_A \psi^{\|} (vb(t) b^2(t)) dt
\]

From (1.2.11) and (1.2.12) we have

\[
\int_A \psi^{\| [vb(t)] (a(t) - \beta b(t))^2} dt = -\gamma
\]

Put \( v = 0 \) in (1.2.13)

\[
\gamma = -\psi^{\|} (0) \int_A (a(t) - \beta b(t))^2 dt
\]
\[ \int_\mathbb{R} \left( \psi''(v(t)) - \psi''(0) \right) \left( a(t) - b(t)^2 \right) dt = 0 \]

The function \( \theta'(u) = \psi(u) + \psi(-u) \). Therefore satisfies the relation

\[ \theta'(u) = \psi'(u) - \psi'(-u) \]
\[ \theta''(u) = 2 \psi''(u) \]
\[ \theta''(0) = 2 \psi''(0) \]

\[ \int_\mathbb{R} \left( 1 - \theta''(v(t)) \right) \left[ a(t) - b(t)^2 \right] dt = 0 \]

Since \( 1 - \theta''(v(t)) \) is real and non-negative for all real \( v \),

\( \theta''(0) \)

(1.2.15) can hold only if \( \theta''(v(t)) = \theta''(0) \)

Then \( \theta(u) \) is a quadratic polynomial. Using Cramer's theorem

If the sum of two independent random variables is normally distributed then each summand is also normally distributed.

Therefore the increments of the process are normally distributed. The necessity follows from (1.2.9)

**1.2.2 Lemma:**

Let \{ \phi_n(t) \} be a sequence of functions and suppose that \( f_n(t) = \exp \left[ \phi_n(t) \right] \) is determined by some constant \( a_n \) and some function \( \phi_n(x) \) according to (1.2.16). Assume that the sequence
\( \phi_n(t) \) converges to some function \( \phi(t) \) which is continuous at \( t = 0 \). Then there exists a constant \( a \) and a bounded and non-decreasing function \( \theta(x) \) such that

(i) \( \lim \theta_n = a \)

(ii) \( \lim \theta_n(x) = \theta(x) \)

(iii) \( \lim \int_{-\infty}^{\infty} \theta_n(x) \, dx = \int_{-\infty}^{\infty} \theta(x) \, dx \)

The function \( \theta(x) \), together with \( a \), determines \( f(t) = \exp[\phi(t)] \) according to (1.2.16).

The functions \( f_n(t) \) are characteristic functions. Therefore \( f(t) = \exp[\phi(t)] \) is also a characteristic function and \( \theta(t) \) is everywhere continuous.

\[
\lambda_n(t) = \left( \frac{\phi_n(t) - \phi_n(t+h) + \phi_n(t-h)}{2} \right)
\]

\[
\Lambda_n(x) = \left( 1 - \sin y \right) \left( 1 + y^2 \right) \int_{-\infty}^{\infty} \theta(y) \, dy
\]

So that

\[
\lambda_n(t) = \int_{-\infty}^{\infty} e^{iax} \, d\Lambda_n(x)
\]

From the continuity of \( \phi(t) \) we conclude that the sequence \( \lambda_n(t) \) converges to a continuous function.

\( \Lambda_n(x) \) converges weakly to a bounded non-decreasing function \( \Lambda(x) \) and

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \Lambda_n(x) \, dx = \int_{-\infty}^{\infty} \Lambda(x) \, dx
\]

We have

\[
\theta_n(x) = \int_{-\infty}^{\infty} \left( \frac{1 - \sin y \sqrt{1 + y^2}}{y} \right) \, d\Lambda_n(y)
\]
\[ \lim_{n \to \infty} \theta_n(x) = \int_{-\infty}^{\infty} \left( \frac{1 - \sin y}{y} \right)^{\frac{1}{2}} \left( \frac{\gamma}{1 + \gamma^2} \right) \text{d} \Lambda(y) = \theta(x) \]

\[ \lim \ln(t) = I(t) \]

Where \( I(t) = \int_{-\infty}^{\infty} \left( e^{ix} - 1 - itx \right) \left( \frac{1}{1 + x^2} \right) \text{d} \theta(x) \)

From the convergence of \( \phi_n(t) \) and \( I_n(t) \), \( \{ a_n \} \) must converge and that \( \phi(t) \) is determined by \( a = \lim a_n \) and \( \theta(x) \) according to (1.2.16)

**1.2.3 Lemma:**

Let \( f(t) \) be an infinitely divisible characteristic function. Then there exists a sequence of functions \( \phi_n(t) \) which have the form (1.2.16) such that

\[ \lim \phi_n(t) = \phi(t) = \log f(t) \]

We have as \( n \to \infty \)

\[ n \left[ (f(t))^{\frac{1}{n}} - 1 \right] = n \left[ e^{\frac{1}{n} \phi(t)} - 1 \right] = n \left[ (1/n) \phi(t) + O(1/n) \right] \]

So that \( \phi(t) = \lim n[f(t)]^{\frac{1}{n}} - 1 \)

(1.2.17)

Since by assumption \( f(t) \) is infinitely divisible.

\[ [f(t)]^{\frac{1}{n}} \] is a characteristic function. Denote the corresponding distribution function by \( F_n(x) \) then

\[ n[f(t)]^{\frac{1}{n}} - 1 = n \int_{-\infty}^{\infty} \left( e^{ix} - 1 \right) \text{d} F_n(x) = n \int_{-\infty}^{\infty} \left( e^{ix} - 1 - itx \right) \text{d} F_n(x) \]

\[ \phi_n(x) = n \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \text{d} F_n(x) \]

(1.2.18)
\[ \phi_n(t) = \text{anit} + \int_{-\infty}^{\infty} \frac{e^{-itx} - 1 - itx}{1 + x^2} \, dx \]

From (1.2.17) and (1.2.18) \( \phi_n(t) \) has the form of (1.2.16)

\[ \lim_{n \to \infty} \phi_n(t) = \phi(t) \]

Now \( f(t) \) is an infinitely divisible characteristic function. Since \( f(t) \) cannot vanish, \( \phi(t) = \log f(t) \) is defined for all values of \( t \). By lemma 1.2.3 there exists a sequence of functions \( \phi_n(t) \) which has the following property.

Each \( \phi_n(t) \) has the form 1.2.16 and the sequence \( \phi_n(t) \) converges to \( \phi(t) \) as \( n \to \infty \). From lemma (1.2.2) that \( \phi(t) \) has also the form (1.2.16). If we combine this with the result of Lemma 1.2.1 we obtain the following theorem.

**The Levy - Khinchine canonical Representation:**

The function \( f(t) \) is an infinitely divisible characteristic function if and only if it can be written in the canonical form

\[ \log f(t) = ita + \int_{-\infty}^{\infty} \frac{(e^{itx} - 1 - itx)}{1 + x^2} \, dx \, \theta(x) \]

Where \( a \) is real and where \( \theta(x) \) is a non-decreasing and bounded function such that \( \theta(-\infty) = 0 \).

The integrand is defined for \( x = 0 \) by continuity to be equal to \( -t^2 \).

We define two functions \( M(u) \) and \( N(u) \) and constant \( \sigma^2 \) by
The functions $M(u)$ and $N(u)$ are non-decreasing in the intervals $(-\infty, 0)$ and $(0, +\infty)$ respectively.

The integrals $\int_{-\infty}^{0} u^2 dM(u)$ and $\int_{0}^{\infty} u^2 dN(u)$ are finite conversely any two functions $M(u)$ and $N(u)$ are finite conversely any two functions $M(u)$ and $N(u)$ and any constant $\sigma^2$ satisfying these conditions determine (1.2.19) and (1.2.16) an infinitely divisible characteristic function. We have therefore obtained a second canonical form.

The Levy canonical Representation:-

The function $f(t)$ is an infinitely divisible characteristic function if and only if it can be written in the form.

\[
\log f(t) = \frac{-\sigma^2}{2} t^2 + \int_{-\infty}^{0} (e^{iu} - 1 - itu) \frac{dM(u)}{1+u^2} + \int_{0}^{\infty} (e^{iu} - 1 - itu) \frac{dN(u)}{1+u^2}
\]

Where $M(u)$ and $N(u)$ and $\sigma^2$ satisfy the following conditions.

(1) $M(u)$ and $N(u)$ are non-decreasing in $(\infty, 0)$ and $(0, +\infty)$ respectively
(2) \( M(-\infty) = N(+\infty) = 0 \)

\[
\int_0^t u^2 \, dM(u) \quad \text{and} \quad \int_0^t u^2 \, dN(u) \quad \text{are finite for all} \ t > 0
\]

(4) The constant \( \sigma^2 \) is real and non-negative. The representation (1.2.20) is unique.

**1.2.4 Lemma:-**

The characteristic function of a stable distribution has the canonical representation.

\[
\log f(t) = \frac{-\sigma^2}{2} t^2 + \int_{-\infty}^{\infty} \frac{(e^{iu} - 1)}{1 + u^2} \, dM(u)
\]

\[
+ \int_{0}^{\infty} \frac{(e^{iu} - 1)}{1 + u^2} \, dN(u) \quad \text{where either} \ \sigma^2 \neq 0 \ \text{and} \ M(u) = 0, \ N(u) = 0 \ \text{or}
\]

\[
\sigma^2 = 0 \ M(u) = C_1 |u|^\alpha (u < 0) \ N(u) = -C_2 |u|^\alpha (u > 0)
\]

The parameters are here subject to the restrictions \( 0 < \alpha < 2, \)

\( C_1 > 0, \ C_2 \geq 0 \ \text{and} \ C_1 + C_2 > 0. \ \alpha \ \text{is known as the exponent of the stable distributions.} \)

**Proof:-**

A distribution function \( F(x) \) is said to be stable if to every \( b_1 > 0, \ b_2 > 0 \) and real \( C_1, C_2 \) there corresponds a positive number \( b \) and a real number \( c \) such that the relation

\[
\frac{F(x-C_1)}{b_1} \ast \frac{F(x-C_2)}{b_2} = \frac{F(x-C)}{b} \quad \longrightarrow (1.2.21)
\]

holds. The characteristic function of a stable distribution is called a stable characteristic function.
The defining relation (1.2.21) can be expressed in terms of characteristic function as
\[ f(b_1 t) f(b_2 t) = f(bt) e^{\gamma t} \]  
\[ \rightarrow (1.2.22) \]

Where \( \gamma = c - c_1 - c_2 \)

Let \( b_1', b_2', \ldots, b_n' \) be \( n \) positive real numbers it follows then from (1.2.22) that
\[ f(b_1't) \cdots f(b_n't) = f(b't) e^{\gamma t} \]

\( \gamma \) is some real number and \( b' \) is positive number.

Put \( b^j_1 = 1 \) \( (j = 1, 2, \ldots, n) \)
\[ b_n = b' \]
\[ \left[ f(t) \right]^n = f(b_n t) e^{\gamma n t} \]
\[ f(t) = \exp \left( \frac{-\gamma t}{nb_n} \right) \]

Therefore a stable characteristic function is infinitely divisible. Therefore a stable characteristic function has real zeros.

Take Log in (1.2.22) and express this equation in terms of the second characteristic \( \phi(t) \).
\[ \phi(b_1 t) + \phi(b_2 t) = \phi(bt) + i \gamma t \]  
\[ \rightarrow (1.2.23) \]

Since \( \phi(t) \) is the logarithm of an infinitely divisible characteristic function, we can write
\[ \log f(t) = ita + \int_{-\infty}^{\infty} \left( e^{ix} - itx \right) \frac{(1+x^2)}{1+x^2} \theta(x) \]

\[ \phi(t) = ita + \int_{-\infty}^{\infty} \left( e^{ix} - itx \right) \frac{(1+x^2)}{1+x^2} \theta(x) \]

\[ \text{as} \]

\[ \phi(t) = ita + \int_{-\infty}^{\infty} \left( e^{ix} - itx \right) \frac{(1+x^2)}{1+x^2} \theta(x) \]

\[ \text{Put } t = bt \text{ bx } = y \]
\[
\phi(bt) = iab + \int_{-\infty}^{\infty} \frac{e^{iy(t)} - i\theta}{1 + by^2} \cdot \frac{(1 + b^2y^2)}{b^2y^2} \, d\theta(y)
\]

is bounded, we have

\[
b \int_{-\infty}^{\infty} \frac{z}{1 + by^2} \, d\theta(z) = \int_{-\infty}^{\infty} \frac{y}{1 + y^2} \, d\theta(b'y) \text{ exists}
\]

\[
ab = ba + (1 - b^2) \int_{-\infty}^{\infty} \frac{y}{1 + y^2} \, d\theta(b'y)
\]

\[
\phi(bt) = iab + \int_{-\infty}^{\infty} \frac{e^{iy(t)} - i\theta}{1 + by^2} \cdot \frac{(1 + b^2y^2)}{b^2y^2} \, d\theta(y)
\]

Put \( M(u) = \int_{-\infty}^{\infty} \frac{1 + y^2}{y^2} \, d\theta(y), \) \( u < 0 \)

\[
N(u) = -\int_{u}^{\infty} \frac{(1 + y^2)}{y^2} \, d\theta(y), \quad u > 0
\]

\[
\sigma = \theta(0) - \theta(-0)
\]

Therefore (1.2.23) and (1.2.24) =>

\[
\frac{\text{ita} b_1 - b_1 \sigma^2 t^2}{2} + \int_{-\infty}^{0} \frac{(e^{i\eta(t)} - i\theta)}{1 + y^2} \, d\theta(b_1'y) + \int_{0}^{\infty} \frac{(e^{i\eta(t)} - i\theta)}{1 + y^2} \, d\theta(b_1'y + \text{ita} b_2 \sigma^2 t^2)
\]

\[
+ \int_{0}^{\infty} \frac{d\theta(b_1'y + \text{ita} b_2 \sigma^2 t^2)}{1 + y^2}
\]

\[
= \theta(0) - \theta(-0)
\]
\[ + \int_{-\infty}^{0} \left( e^{-\beta x^2} \right) \frac{dM}{b_2 y} + \int_{0}^{\infty} \left( e^{-\beta x^2} \right) \frac{dN}{b_2 y} \]

\[ = \text{it} \frac{ab - b^2 \sigma^2 t^2}{2} + \int_{-\infty}^{0} \left( e^{-\beta x^2} \right) \frac{dN}{b_2 y} + (j \gamma t) \]

From the uniqueness of the Canonical representation

We see that \( \sigma^2 (b_2^2 - b_1^2 - b_2^2) = 0 \) \( \rightarrow (1.2.25) \)

\[ M(b_2^2y) = M(b_1^2y) + M(b_2^2y) \text{ if } y < 0 \] \( \rightarrow (1.2.26) \)

\[ N(b_2^2y) = N(b_1^2y) + N(b_2^2y) \text{ if } y > 0 \] \( \rightarrow (1.2.27) \)

To find \( M(u), u < 0 \)

Let \( \beta_1, \beta_2, \ldots, \beta_n \) be \( n \) positive real numbers.

From (1.2.26) there exists a positive number \( \beta \)

\[ \beta(\beta_1, \beta_2, \ldots, \beta_n) \text{ such that} \]

\[ M(\beta_1^2y) + M(\beta_2^2y) + \ldots + M(\beta_n^2y) = M(\beta y) \]

Put \( \beta_j = 1, j = 1, 2, \ldots, n \)

\[ \beta(1, 1, \ldots, 1) = A_n \]

\[ nM(y) = M(y A_n) \]

\[ 1/n M(y) = M(y/A_n) \]

Here \( y < 0 \) \( A_n > 0 \)

Using this reasoning, we see that to every positive rational number \( r = m/n \).

\( (m, n \text{ positive integers}) \) there corresponds a positive real number \( A = A(r) = Am/A_n \) such that

\[ \gamma M(y) = M(Ay) \text{ (} y < 0 \text{)} \] \( \rightarrow (1.2.28) \)
The function $A = A(r)$ is defined for all rational $r > 0$.

To show that $A(r)$ is non increasing for rational values

Let $r_1$ and $r_2$ be two rational numbers and suppose that $r_1 < r_2$.

Since $M(u) > 0$ we see that

$$r_1 M(u) < r_2 M(u) \implies M[A(r_1)u] < M[A(r_2)u] \text{ by (1.2.28)}$$

Since $M(u)$ is non decreasing and $u < 0$, we conclude that $A(r_1) \geq A(r_2)$. By the same reasoning. We can show that $A(r)$ is strictly decreasing provided $M(u) \neq 0$.

We define a function for all positive real values of $x$ by means of $B(x) = A(x)$ if $x$ is a positive rational number

$$B(x) = \begin{cases} \max \{A(r) \mid r > x \text{ is irrational} \} & \text{if } x > 0 \text{ is irrational} \\ \max \{A(r) \mid r > x \text{ is rational} \} & \text{if } x > 0 \text{ is rational} \end{cases} \quad (1.2.29)$$

It follows from the definition that $B(x)$ is non increasing and is strictly decreasing.

Let $X$ be an arbitrary positive real number then there exists two sequences ${r_1}$ and ${r_1'}$ of rational numbers such that $r_1 \to X$ from below while $r_1' \to X$ from above.

Since $r_1 < x < r_1'$, we have $B(r_1) > B(x) > B(r_1')$ and hence $yB(r_1) < yB(x) < yB(r_1')$ for any $y < 0$. Since $M(u)$ is non decreasing, we see that

$$M[yB(r_1)] \leq M[yB(x)] \leq M[yB(r_1')]$$

From (1.2.28) and (1.2.29) $\implies r_1 M(y) < M[yB(x)] < r_1' M(y)$

Let $v \to \infty$ for all real positive $x$, there exists $B(x) > 0$ such that

$$x M(y) = M(y B(x)), y < 0 \quad \implies (1.2.30)$$

Since $M(u)$ is non decreasing and $M(\infty) = 0$ we have
The strictly decreasing function \( Z = B(x) \) has an inverse function \( x = \beta(z) \). This function is defined for \( z > 0 \) and is single valued and nonnegative.

Therefore (1.2.30) can be written in terms of \( \beta(z) \) for all real \( z > 0 \),

there exists \( \beta(z) > 0 \) such that

\[
\beta(z) \ M(y) = M(yz) \quad \text{is satisfied} \quad \therefore (1.2.31)
\]

Let \( M_1(y) \) and \( m_2(y) \) be two solutions of (1.2.31) and suppose that \( m_1(y) \neq 0 \)

Put \( m(y) = \frac{m_2(y)}{m_1(y)} \)

\[
m(zy) = \frac{m_2(zy)}{m_1(zy)} = \frac{\beta(z) m_2(y)}{\beta(z) m_1(y)} = \frac{m_2(y)}{m_1(y)} = m(y)
\]

Therefore quotient of two solutions of (1.2.31) is constant.

\[
m_1(y) = |Y|^\alpha \quad \text{is a solution}
\]

\[
\beta(z) = |Z|^\alpha
\]

Therefore the general solution of (1.2.31) is

\[
m(y) = C_1 |Y|^\alpha
\]

Since \( M(\infty) = 0 \), we must have \( \alpha_1 > 0 \) and since \( M(y) \) is nondecreasing,

we see that \( C_1 > 0 \).

Also \( \int_{-1}^{0} u^2 dM(u) \) is finite. This permits the conclusion, \( \alpha_1 < 2 \).
Therefore \( M(u) = C_1 |u|^\alpha \) \( (C_1 > 0, 0 < \alpha < 2, u < 0) \) --- (1.2.32)

The Solution of (1.2.32) includes \( M(y) = 0 \) since \( C_1 = 0 \)

Using (1.2.32) in (1.2.26)

\[ C_1 [b^{\alpha_1} - b_1^{\alpha_1} - b_2^{\alpha_1}] = 0 \]  --- (1.2.33)

From (1.2.27) \( N(u) \) can be found

\[ N(u) = -C_2 u^{\alpha_2} \] \( C_2 \geq 0 \) \( 0 < \alpha_2 < 2, u > 0 \)  --- (1.2.34)

\[ C_2 [b^{\alpha_2} - b_1^{\alpha_2} - b_2^{\alpha_2}] = 0 \]  --- (1.2.35)

To show that

\[ \sigma^2 \neq 0 \implies C_1 = C_2 = 0 \]

Therefore \( M(u) = 0 \) and \( N(u) = 0 \)

From (1.2.25) \( b_1^2 - b_2^2 = 0, \alpha_1 < 2 \) and

\[ \alpha_2 < 2 \implies c_1 = c_2 = 0 \] [Using (1.2.33 and 1.2.35)]

If \( M(u) \neq 0, c_1 > 0, c_2 > 0 \)

Put \( b_1 = b_2 = 1 \). Therefore \( b^{\alpha_1} = b^2 \) or \( \alpha = 2 \).

Therefore \( b^2 / 2 \implies \alpha = 0 \)

To show that \( \alpha_1 = \alpha_2 \)

Suppose that \( c_1 > 0, c_2 > 0 \). Put \( b_1 = b_2 = 1 \).

Therefore (1.2.33) and (1.2.35) \( \implies b^{\alpha_1} = 2 = b^{\alpha_2} \implies \alpha_1 = \alpha_2 \)

1.2.2 Theorem:-

Let \( W(t) \) be as in the definition 1.2.1. Let the canonical representation for the characteristic function of \( X(t+1) - x(t) \) be given by \( a, \delta, \mathcal{M} \) and \( \mathcal{N} \) as defined above. Then the Levy Kintchin
Canonical representation for the characteristic function of the stochastic integral

\[
\int_{A}^{B} t \, dx(\mathcal{M}(t)) \quad \text{is given by}
\]

\[
a_w, \sigma_w, M_w \text{ and } N_w \text{ where}
\]

\[
a_w = \int_{A}^{B} \left\{ t a + t(l-t^2) \right\} \frac{x^1}{1+(tx^2)(1+x^2)} \, d[M(-x)+N(x)] \, dw(t)
\]

\[
\sigma_w^2 = \sigma^2 \int_{A}^{B} t^2 \, dw(t)
\]

\[
M_w(x) = \begin{cases} 
\min(B,0) & \text{if } x < 0 \\
\min(A,0) & \text{if } x = 0 \\
\max(A,0) & \text{if } x > 0
\end{cases} - N(x/t) \, dw(t) + \int_{A}^{B} M(x/t) \, dw(t), x < 0
\]

\[
N_w(x) = \begin{cases} 
\min(B,0) & \text{if } x < 0 \\
\max(A,0) & \text{if } x > 0
\end{cases} - M(x/t) \, dw(t) + \int_{A}^{B} N(x/t) \, dw(t), x > 0
\]

1.2.3 Theorem:-

Let \( b(t) \) be a continuous function on \([A,B]\) and \( W(t) \) be a non decreasing, non negative and left continuous function on \([A,B]\).

Define \( C = \min b(t) \)

\[
A \leq t \leq B
\]

\[
D = \max \left( b \left( t \right) \right) \quad \text{and} \quad A \leq t \leq B
\]

Then the integrals

\[
Y = \int_{A}^{B} b(t) \, dx(\mathcal{M}(t)) \quad \text{and} \quad Z = \int_{C}^{D} t \, dx(\mathcal{M}(t))
\]
exist in the sense of convergence in probability and they are identically distributed. Further more
the characteristic function $\Phi$ of the random variable $Y$ is given by

$$\log \Phi(u) = \int_{A} \log \psi(u\,b(t)) \, dw(t)$$

$$= \int_{C} \log \psi(ut) \, dw(t)$$

Where $\psi(u)$ is the logarithm of the characteristic function of $x(t+1) - x(t)$, $t$, $t+1 \in [A,B]$.

Proof of Theorem:-

We have already proved the Levy Canonical Representation given

Without loss of generality, we assume $A \leq 0 \leq B$

(i) First let us assume that there exists a number $t_0 > 0$ such that $t_0$ is a point of continuity of $W$
and denote by $h$ the characteristic function of the

$$\int_{A} \int_{t_0}^{t} \, dw(t)$$

Then by theorem

$$\log h(u) = \int_{A} \log \psi(u\,t) \, dw(t)$$

Now we define a function $s$ by

$$s(u,x,t) = r(ut,x) - r(u,tx)$$

Where in view of $r(u,x) = e^{ux} - 1 - (iux)$
We have,\[ s(u,x,t) = \left( e^{\frac{-\alpha t u}{1+x^2}} - \frac{i\alpha t x}{1+x^2} \right) \]

\[ \frac{(i\alpha t x)}{(1+(tx)^2)(1+x^2)} = \frac{(i\alpha t x^2(1-t^2))}{(1+(tx)^2)(1+x^2)} \]

Since \( s(u,x,t) = 0(x^2) \) as \( x \to 0 \)

and \( s(u,x,t) = 0(1) \) as \( x \to \infty \). The function \( s \) is integrable with respect to \( M \) and \( N \).

By the Canonical representation and the definition of \( s \) we have

\[ \log V(ut) = \alpha \frac{\int s(u,x,t) dM(x) + (ut)^2}{2} \]

Applying \( \text{Euler's Hospital's rule twice} \) we get

\[ \lim_{x \to 0} \frac{r(u,t^2)}{x^2} = \frac{(ut)^2}{2} \]

Hence there is a constant \( C \) such that for fixed \( u \), and \( t \in [A,t_0] \)

\[ r(u,t^2) < C (ut)^2 x^2 \]

Therefore we get for \( I_1 \), the estimation \( (t \to 0^+) \)

\[ I_1 \leq C_1 u^2 \int_A^t t^2 dw(t) \int x^2 dM(x) \]

\[ = 0(1) \]
Further we can transform \( I_1 \) in the following way

\[
I_1 = -\int_0^A \int_0^t r(u, x, t) \, d(-M(x/t)) \, dw(t) - \int_0^A r(u, x) \, d\int_0^t -M(x/t) \, dw(t)
\]

As \( t \to 0 \)

\[
1 = \int_0^A \int_0^t r(u, x) \, d\int_0^t -M(x/t) \, dw(t)
\]

Similarly transforming the fourth, fifth and sixth terms in a similar manner we get

\[
\log \mathcal{J}(u) = iau \left( -\frac{\sigma_x^2}{2} u^2 - \int_0^\infty r(u, x) \left( \int_0^t -M(x/t) \, dw(t) \right) \right)
\]

Therefore

\[
\log \mathcal{J}(u) = iau \left( \int_0^A \int_0^t s(u, x, t) \, dM(x) \, dw(t) \right) - \int_0^\infty r(u, x) \left( \int_0^t -M(x/t) \, dw(t) \right)
\]

Using the definitions of \( \sigma^2 \) and \( \sigma_x^2 \), we can write this relation in the form

\[
\log \mathcal{J}(u) = iau \left( -\frac{\sigma^2}{2} u^2 - \int_0^\infty r(u, x) \left( \int_0^t -M(x/t) \, dw(t) \right) \right)
\]
By the above result, we get

\[
\log f_i(u) = iau - \frac{1}{2} \int_0^t \int_{-\infty}^\infty r(u, t, x) \, dM(x) \, dw(t)
\]

Decomposing the third term of above result

Therefore

\[
I = \int_0^t \int_{-\infty}^\infty r(u, t, x) \, dM(x) \, dw(t)
\]

Using the definition of \( M_u \) and \( N_u \), we have

\[
\log f_i(u) = iau - \frac{1}{2} \sigma_u^2 u^2 + \int_{-\infty}^0 r(u, x) \, dM_u(x) + \int_0^\infty r(u, x) \, dN_u(x)
\]

Let us complete this proof by showing that \( a_u, \sigma_u, M_u \), and \( N_u \) satisfy the condition of lemma. Obviously \( a_u \) and \( \sigma_u^2 > 0 \)

Also \( M_u \) and \( N_u \) are non-decreasing in the intervals \((-\infty, 0)\) and \((0, \infty)\) having the properties

\[
M_u(-\infty) = N_u(\infty) = 0
\]
for all \( \epsilon > 0 \), we obtain the inequality

\[
\int_0^\infty x^2dN_w(x) - \int_0^{\epsilon / t} x^2dW(x)dW(t) - \int_0^{t_o} x^2dM(x)dW(t) < \infty
\]

Similarly

\[
\int_0^\infty x^2dM(x) < \infty
\]

1.3 Characterization of a Wiener Process taking values in a Hilbert Space:

Rényi [1] discussed the characterization of a Wiener process taking values in a Hilbert space as follows:

Let \( \Lambda \) be the interval \([0,1]\) and \( \mathcal{B} \) denote the \( \sigma \) - algebra of Borel subsets of \([0,1]\).

For each \( \Lambda, \epsilon \in \mathcal{B} \), let \( \phi(\Lambda) \) be a random element taking values in a real separable Hilbert space \( \mathcal{H} \). Suppose \( \phi(\Lambda) \) satisfies the following properties.

(i) If \( \Lambda \) and \( \Lambda' \) are disjoint Borel subsets of \([0,1]\), then \( \phi(\Lambda) \) and \( \phi(\Lambda') \) are independent.

\[
\phi(\Lambda \cup \Lambda') = \phi(\Lambda) + \phi(\Lambda')
\]

(ii) \( \phi(\Lambda) \) has stationary increments (ie) \( \phi(\Lambda) \) and \( \phi(\Lambda') \) are identically distributed if \( \Lambda \) and \( \Lambda' \) have the same Lebesgue measure.

(iii) If \( \mu_t \) denotes the probability measure of \( \phi([0,t]) \) then \( \mu_t \) converges weakly to the distribution degenerate at the origin as \( t \to 0 \).

For any two \( \bar{x}, \bar{y} \) in \( \mathbb{R}^k \)

\[
< \bar{x}, \bar{y} > = \sum_{j=1}^{k} x_j y_j
\]
The complex valued function \( \hat{\mu} \) on \( \mathbb{R}^n \) is called the Fourier transform or characteristic function of the probability measure or distribution \( \mu \).

If \( f \) is an \( \mathbb{R}^n \) valued random variable on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( \mu = \mathbb{P} \) is the distribution of \( f \), its characteristic function \( \hat{\mu} \) is given by

\[
\hat{\mu}(t) = \int e^{i\langle \tilde{\lambda}, \tilde{v} \rangle} \ d\mu(\tilde{v}), \quad \tilde{v} \in \mathbb{R}^n
\]

\[
= \int e^{i\langle \tilde{\lambda}, \tilde{v} \rangle} \ dp(\tilde{v}) = \mathbb{E} e^{i\langle \tilde{\lambda}, \tilde{v} \rangle}
\]

\( \hat{\mu} \) is the characteristic function of the random variable \( f \).

1.3.1 Proposition:--

The multivariate normal distribution in \( \mathbb{R}^n \) with mean vector \( \tilde{m} \) and co-variance matrix \( \Sigma \) has characteristic function \( e^{i\langle \tilde{z}, \tilde{m} \rangle} \sqrt{\Sigma} \).

Proof:--

Let \( \psi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\alpha^2} e^{\alpha z} \ dx \).

if \( z = t \), \( \psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\alpha^2} \ e^{\sqrt{2} \alpha t} \ dx \).

\( \psi \) is an analytic function defined in the entire complex plane.

\( \psi(t) = e^{iv/2} \).

Thus the analytic function \( \psi(z) \) and \( e^{iv/2} \) agree on the entire real axis.

\( f(it) = e^{iv/2} \).
In other words the standard normal variate distribution on $\mathbb{R}$ has characteristic function:

$$\exp(-t^2/2)$$

Multivariate normal distribution has characteristic function:

$$\exp\left(-\frac{1}{2} \xi^T \Sigma^{-1} \xi\right)$$

$$= \exp\left(-\frac{1}{2} \langle \xi, \Sigma^{-1} \xi \rangle\right)$$

Let $\xi$ be a random variable with $\{\xi_1, \xi_2, \ldots, \xi_k\}$ being independently distributed with standard normal distribution.

Let $\tilde{\eta} = A \xi + m$ where $A$ is $(k \times k)$ matrix $m$ constant vector. The characteristic function of $\tilde{\eta}$ is

$$e^{\frac{1}{2} \langle \tilde{\eta}, \Sigma^{-1} \tilde{\eta} \rangle} = e^{\frac{1}{2} \langle \xi, A^T A \xi \rangle}$$

If $m = 0$ then characteristic function is $\exp\left(-\frac{1}{2} t(sy, y)\right)$

$\xi$ is of the form $AA^T$ where $A^T$ is the transpose of $A$.

Let $\phi$ be a homogeneous process with independent increments on $\mathbb{R}$ with mean 0 and with

$$E_x[|x|^2] < \infty$$

where $\mu$ is the distribution of $x = \phi[0,1]$

Let $S$ denote the $S$ operators with $\phi$ For any bounded linear operator $A$, Define

$$n(\phi) = \left[\text{Tr}(A^T A)\right]^{1/2} + \left[\text{Tr}\left(A^T A\right)\right]^{1/2}$$

The trace of $A$ is the sum of the elements on the main diagonal of $A$.

$$\text{tr}A = \sum_{i=1}^{n} a_{ii}$$

1. $\text{tr}(\lambda A) = \lambda \text{tr}(A)$
2. $\text{tr}(A + B) = \text{tr}A + \text{tr}B$
3. \( \text{tr}(AB) = \text{tr}(BA) \)

Then the set \( \{ A : n(A) = 0 \} \) is a linear semigroup in the linear group of all bounded linear operators \( A \).

The function \( n \) is a norm in the corresponding factor group.

We shall not distinguish between a coset and the individual operator in the coset. In this sense, \( n \) is a norm in the linear set of all bounded linear operators. Let \( \mathcal{A} \) denote the completion of this set in the norm. Consider the space \( L_2 = L_2(\Lambda, B, m, \mathcal{A}) \) of functions \( A(\cdot) \) with values in \( \mathcal{A} \), which are strongly measurable and such that

\[
|A|^2 = \int n^2(A(\cdot)) \, dm < \infty
\]

where \( m \) is the Lebesgue measure on \( \Lambda \). The stochastic integrals of the form

\[
J = \int A(\lambda) \cdot \phi(\lambda) \, d\lambda
\]

for functions \( A(\cdot) \) is \( L_2 \).

Let \( \{ \xi_n(w) \} \) be a sequence of independent random variables on a probability space \( S = (\Omega, \mathcal{A}, p) \) such that each \( \xi_n(w) \) has a standard normal distribution.

Let \( \{ W_n(x) \} \) denote the system of Walsh functions and define the random variables \( \eta(t) \) as follows.

\[
\eta(t) = \sum_{n=1}^{t} \xi_n(w) \int_{0}^{t} W_n(x) \, dx
\]

To prove that

The system \( \eta(t) \) of random variables \((0 \leq t \leq 1)\) is Wiener process or Brownian movement process. It is enough to prove the following statements for Wiener process:

1. For each \( s \) and \( t, \quad 0 \leq s < t < 1 \), \( \eta(t) - \eta(s) \) has the normal distribution \( N(0, \sqrt{t-s}) \).

2. If \( 0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < s_k, t_k < 1 \), the random variables \( \eta(t_j) - \eta(s_j) \) \((j=1,2,\ldots,k)\) are independent.
3. One has $E(\eta(s), \eta(t)) = s$ if $0 < s < t < 1$

4. With probability 1, $\eta(t)$ is a continuous function of $t$.

1.3.1 Definition :-

Orthonormal System:

A sequence $\xi_n (n = 1, 2, \ldots)$ of random variables on a probability space $S = (\mathbb{R}, A, P)$ is called an orthonormal system, if $\xi_n$ belongs to the Hilbert space $L_2(S)$.

(ie)

$$E(\xi_n^2)$$ exists and has

$$E(\xi_n^2) = 1, n = 1, 2 \ldots$$

$$E[\xi_n \xi_m] = 0, n \neq m$$

1.3.2 Hilbert Space:

A Banach space is called a Hilbert space, if the function $(x, y)$ inner product of $x$ and $y$ has the following properties.

1. $(x, y) = (y, x)$

2. $(x, x) = ||x||^2$

3. For fixed $y$, $A(x) = (x, y)$ is a linear functional.

(i.e)

$$[A(ax + by) = aA(x) + b(A(y))]$$

1.3.3 Complete orthonormal system:

The orthonormal system $\{\xi_n\}$ is complete if $\eta \in L_2(s)$, $E(\eta \xi_n) = 0, n = 1, 2, \ldots$ it follows that $\eta = 0$ almost surely.
Fourier Co-efficient:-

If \{ \xi_n \} is an orthonormal system on \( S \) and \( \eta \) is an arbitrary random variable in \( L_2(\mathbb{S}) \), the sequence \( C_n = \mathbb{E}[\eta \xi_n] \) is called the sequence of fourier co-efficient of \( \eta \) and the series \( \sum C_n \xi_n \) is fourier series of \( \eta \) with respect to \{ \xi_n \}

\[ \sum_{n=1}^{\infty} C_n^2 = \mathbb{E}(\eta^2) \]

is Parseval's relation.

1.3.4 Rademacher Functions:-

Let \( S \) be Lebesgue probability space and consider the Rademacher functions

\[ R_n(x) = \text{sgn} \left( \sin \frac{n \pi x}{2} \right) \quad (0 \leq x \leq 1, \ n=1,2,\ldots) \]

1.3.4 Walsh Functions:-

Let us now define the functions \( W_n(x), \ 0 \leq x \leq 1, \ n=0,1,2,\ldots \) as follows.

\[ W_0(x) = 1 \quad 0 \leq x \leq 1 \]

Further if the representation of \( n \geq 1 \) in the binary system is

\[ n = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_r} \quad \text{where} \ 0 \leq k_1 < k_2 < \ldots < k_r \ \text{are integers} \]

Put \( W_n(x) = R_{k_1}(x) R_{k_2+1}(x) \ldots R_{k_r+1}(x) \)

The function \( W_n(x) [n=0,1,\ldots] \) are called the walsh functions.

To prove that \{ \{ W_n(x) \} \} form an orthonormal system on the Lebesgue probability space

if \( \xi_1, \xi_2, \ldots, \xi_n \) are independent random variables with finite expectation then \( \xi_1, \xi_2, \ldots, \xi_n \) also has finite expectation and

\[ \mathbb{E}(\xi_1 \xi_2 \ldots \xi_n) = \prod_{k=1}^{n} \mathbb{E}(\xi_k) \]

1.3.1 Theorem:-

If the random variables \( \xi_n \) are independent \( (n=1,2,\ldots) \) \( \mathbb{E}(\xi_n^2) = 1 \)
and $E(\xi, n) = 0$ for $n = 1, 2, \ldots$ then all the products $\xi_{k_1} \xi_{k_2} \ldots \xi_{k_r}$

\[(1 \leq k_1 < k_2 \ldots < k_r, r = 1, 2, \ldots)\]

belong to $L_2(s)$ and they form together with the constant 1 and orthonormal system.

**Proof:-**

We have $E(\prod_{j=1}^{r} \xi_{k_j}) = \prod_{j=1}^{r} E(\xi_{k_j}) = 1$

If we take any two non identical products $\xi_{k_1} \xi_{k_2} \ldots \xi_{k_r}$

\[(k_1 < k_2 \ldots < k_r) \text{ and } (l_1 < l_2 \ldots < l_r) \text{ (w.l.o.g.)} \text{ and } E(\xi_{k_1} \xi_{k_2} \ldots \xi_{k_r}) = 0\]

Then we have $E(1 \cdot \xi_{k_1} \xi_{k_2} \ldots \xi_{k_r}) = 0$

Next to prove that this system is complete

Let $x$ be a real number $0 \leq x < 1$ which is not a binary rational number.

Let the binary expansion of $x$ be

\[x = \sum_{k=1}^{\infty} \frac{\xi_{k(x)}}{2^k} \quad (\xi_{k(x)} = 0 \text{ or } 1)\]

Then $\xi_{k(x)} = \frac{1 - R_{k}(x)}{2}$

$R_{k}(x) = 1 \text{ or } -1 \text{ when } \xi_{k(x)} = 0 \text{ or } 1$

Let $i^m_{\infty}(x)$ denote the indicator function of the interval $[m, m + \delta]$ where $m$ and $n$ are non negative integers and $0 \leq m < 2^n$

Let the binary expansion of $m/2^n$ be

\[m \quad \sum_{k=1}^{n} \frac{\delta_k}{2^k} \quad (\delta_k = 0 \text{ or } 1, k = 1, 2, \ldots, n)\]
Then \( i_{n', m}(x) \) can be written in the form

\[
i_{n', m}(x) = \pi \left( \frac{1 + (1-2^{\delta})}{2} \right)
\]

If \( x \) is not a binary rational number.

Multiplying out, we see that \( i_{n', m}(x) \) is a linear combination of certain Walsh function

\[
i_{n', m}(x) = \sum_{0 \leq m < 2^n} a_{n', m} W(x)
\]

It follows that if \( f = f(x)e^{i \lambda x} \in L^2(\mathbb{R}) \) is such a function that

\[
E(fw_n) = 0, \quad n = 0, 1, \ldots...
\]

Then we have

\[
\int_{m/2^n}^{2m/2^n} f(x) \, dx = 0, \quad 0 \leq m < 2^n \quad n = 1, 2, \ldots
\]

\[
\int_{0}^{2^n} f(x) \, dx = 0, \quad 0 \leq m < 2^n \quad n = 1, 2, \ldots
\]

Thus putting \( F(x) = \int_{0}^{x} f(t) \, dt \)

We get \( F(r) = 0 \) for every binary rational number \( r \) in \((0,1)\). The function \( F(x) \) being the definite integral of an integrable function is continuous thus \( F(x) = 0 \) for all \( x \) in \([0,1]\)

Therefore \( F(x) = 0 \) for almost all \( x \)

Therefore, the system of Walsh functions is complete.
The series defining \( \eta(t) \) is almost surely convergent, because denoting by
\[
e_i(x) = \begin{cases} 0 < x < 1 \\ 0 < t < 1 
\end{cases}
\]
the indicator of the interval \((0, t)\) and taking into account that
\[
\int_0^t W_n(x) \, dx = \int_0^1 e_i(x) w_n(x) \, dx
\]
and Fourier Walsh co-efficients of
\[
e_i(x)
\]
since \( \{W_n(x)\} \) is a complete orthonormal system
we get from Parseval relation
\[
\sum_{n=0}^{\infty} \xi_n^2 = 1
\]
Therefore
\[
E[\eta^2(t)] = \sum_{n=0}^{\infty} \xi_n^2 \cdot (\int_0^t W_n(x) \, dx)^2 = t [\text{since } E[\eta(\infty)] = 1]
\]
The Parseval relation \( \Rightarrow \) \( 0 < s < t < 1 \)
\[
E[\eta(s) \eta(t)] = E[\sum_{n=0}^{\infty} \xi_n^2 \cdot (\int_0^t W_n(x) \, dx)^2]
\]
\[
= \sum_{n=0}^{t} W_n(x) \, dx \cdot (\int_0^s W_n(x) \, dx)^2 = \sum_{n=0}^{t} W_n(x)^2
\]
\[
= \int_0^t e_i(x) e_i(t) \, dx = \mathcal{A}
\]
\[
E[\eta(t_1) - \eta(s_1)] [:] [\eta(t_2) - \eta(s_2)]
\]
\[
= t_1 - s_1 - t_1 + s_1 = 0
\]
\[
E[\eta(t) - \eta(s)]^2 = t + s - 2s = t - s \text{ if } s < t
\]
Now as the sum of independent normally distributed random variables also has a normal
Similarly we get that the joint distribution of \((\eta_{i(j)} - \eta_{j(s)})\) for \(1 \leq j \leq k\) is a \(k\)-dimensional normal distribution. As the components of a \(k\)-dimensional normally distributed vector are independent if and only if they are uncorrelated.

It follows that \(\eta_{i(j)} - \eta_{j(s)}\) for \(j = 1 \ldots k\) are independent. The almost sure continuity of \(\eta(t)\) as a function of \(t\) can be proved as follows:

If \(2^s \leq n < 2^{s+1}\)

\[ W_n(x) = R_{s+1}(x) \cdot W_n, \text{ where } W_n, \text{ is a product of the Rademacher functions} \]

\[ R_k(x), \quad k \leq s \quad \text{and thus is constant on every interval of the form} \left(\frac{r}{2^s}, \frac{r+1}{2^s}\right) \]

On the other hand, the indefinite integral of \(R_{s+1}(x)\) over \(\left(\frac{r}{2^s}, \frac{r+1}{2^s}\right)\) increases linearly from \(1\) increases linearly from zero to \(\frac{r+1}{2^s}\) and then decreases linearly to zero.

It follows that

\[ \sum_{n=2^s}^{2^{s+1}-1} \int_0^1 W_n(x) dx \text{ is for } \omega \in \Omega \text{ a continuous function of } t \text{ such that} \]

on every interval \(\left(\frac{r}{2^s}, \frac{r+1}{2^s}\right)\) \(t\) varies between

\[ \pm \sum_{n=2^s}^{2^{s+1}-1} \xi_n e_n W_{n,2} s \left(\frac{r+1/2}{2^s}\right) \]

and thus \(e_n = +1\) or \(-1\)
Now the sum $\sum \xi_n e^{x_n}$ is normally distributed with variance $2^x$.

Thus put $\delta = \sum \xi_n e^{x_n}$

$$P[\delta > s2^{x_2}] < e^{sx^2}$$

It follows that

$$P[\max | \delta_x | > s2^{x_2}] < 2^x e^{sx^2}$$

Borel Cantelli Lemma $\sum 2^x e^{sx^2}$ is convergent for almost all $\omega \in \Omega$ $\max | \delta_x | < s2^{x_2}$ for all but a finite number of values of $s$.

This implies that for almost all values of $\omega$, one has uniform $\chi^2$ for $0 < t < 1$

$$\int_0^\infty \left| \sum_{i=1}^{\infty} \xi_n(\omega) \right| W_s(x)dx \leq \sum_{s=1}^{\infty} s < \infty$$

Therefore $\eta (t)$ is for almost all $\omega$ the sum of a uniformly convergent series of continuous function. (ie) it is almost surely a continuous function of $t$.

### 1.4 Characterization of the Wiener process by constant Regression.

We know that the following theorem which characterizes normal Distribution by constant regression of one linear statistic on another linear statistic.

#### 1.4.1 Theorem:

Let $X_i$, $1 < i < n$ be a random sample from a univariate population with non degenerate distribution function $F(x)$ and assume that $F(x)$ has moments of every order. Consider the linear statistics.
\[ U = a_1 x_1 \cdots + a_n x_n \]
\[ V = b_1 x_1 \cdots + b_n x_n \]

Where \( a_i, 1 \leq i \leq n, b_i, 1 \leq i \leq n \) have the property that \( \Sigma a_i b_i = 0 \) implies that \( \Sigma a_i b_i = 0 \) for all \( k > 1 \). Then \( U \) has constant regression on \( V \).

(i) \( E \left[ \frac{U}{Y} \right] = E[U] \) a.e

if and only if (i) the population distribution \( F \) is normal

(ii) \( \sum a_i b_i = 0 \)

Let us now derive a similar result for a characterization of the Wiener process \([33]\)

1.4.1 Lemma:-

A random variable \( Y \) with finite expectation has constant regression on a random variable \( Z \).

(i) \( \frac{E(Y|Z)}{E(Y)} \) almost everywhere if and only if

\[ E[Y e^{aZ}] = E[Y] E[e^{aZ}] \]

1.4.2 Theorem:-

Let \( X \) and \( Y \) be two random variables and assume that the expectations \( E(y) \) and \( E(x^k) \) exist where \( k \) is a non-negative integer. The random variable \( Y \) has polynomial regression of order \( k \) on \( X \) if and only if the relation

\[ E[Y e^{ax}] = \sum_{j=0}^{k} \beta_j E[X^j e^{ax}] \]

holds for all real \( t \), where \( \beta_j \) are real constants.

**Proof:-**

Let us consider two random variables \( X \) and \( Y \) and assume that \( E(y|x) \) exists.
If $Y$ has polynomial regression of order $k$ then $E[y/x] = \beta_0 + \beta_1 x + \ldots + \beta_k x^k$ \quad (1.4.2)

Also $E(y) = \beta_0 + \beta_1 E(X) + \ldots + \beta_k E(X^k)$ \quad (1.4.3)

If we multiply (1.4.2) by $e^{xt}$ and taking the expectation then condition (1.4.1) is necessary.

To prove the sufficient condition, assume that (1.4.1) holds for all real $t$.

Then $E \{e^{xt}[Y - \sum \beta_j x^j]\} = 0$

or $e^{xt} E[Y - \sum \beta_j x^j] dF(x) = 0$

Where $F_1(x)$ is the marginal distribution of $X$. We introduce the probability function $P_x(A)$ of the random variable $X$ instead of distribution function $F_1(x)$. This is a set function defined on all Borel sets of $R_1$.

Therefore $\int_{R_1} e^{xt} E[Y - \sum \beta_j x^j] d\lambda_x = 0$

Let $\mu(A) = \int_A E[Y - \sum \beta_j x^j] d\lambda_x$

this is a function bounded variation which is defined on all Borel sets $A$ of $R_1$ and we see that

$\int_{R_1} e^{xt} d\mu = 0$

It is known that a function of bounded variation is just like a distribution function, uniquely determined by its fourier transform.

Therefore $\mu(A) = \mu(R_1) = 0$ for all Borel sets $A$. This is possible if

$E[Y - \sum \beta_j x^j/X = x] = 0$ almost everywhere

Therefore Theorem is proved.

**Proof of Lemma 1.4.1:**

In the above theorem, as a special case $Y$ has constant regression on $X$ if and only if the
relation $E[Ye^{\alpha x}] = E(y)E(e^{\alpha x})$ holds

1.4.2 Lemma:–

Let $\{X(t) / t \in T\}$ be a continuous homogeneous stochastic process with independent increments on $T = [A,B]$. Further suppose that the process is a second order process and its mean function and co-variance function are of bounded variation on $[A,B]$. Let $g(.)$ and $h(.)$ be continuous functions on $[A,B]$.

Denote $Y = \int_A^B g(t)dX(t)$ and $Z = \int_A^B h(t)dX(t)$ then for any real number $V.$

$$E[Ye^{\alpha V}] = -i \int_A^B g(t) \psi[Vh(t)] dt \ e^{i\psi(A)^{\theta(t)}}$$

Where $\psi(u) = \log \phi(u; 1)$ is the logarithm of the characteristic function of $x(t+1) - x(t)$

Proof:–

Let $\theta(u,v)$ denote the characteristic function of the bivariate random variable $(Y, Z)$.

By lemma $\log \theta(u,v)$ is well defined and

$$\log \theta(u,v) = \int_A^B \psi [ug(t) + vh(t)] dt$$

(ie) $E[e^{uy+vj}] = e^{\int_A^B (ug(t)+vh(t)) dt}$

Differentiating on both sides with respect to $u,$ we get

$$E[iy e^{(uy+vj)}] = \exp \int_A^B \psi (ug(t)+vh(t)) dt. \int_A^B \psi' (ug(t)+vh(t))g(t)dt \quad \rightarrow (1.4.4)$$
Put \( u = 0 \) in (1.4.4)

\[
E[\psi^m] = \exp \int_\rho^\beta \psi'(vh(t)) g(t) \, dt
\]

Therefore \( E[\psi^m] = -i \exp \int_\rho^\beta \psi'(vh(t)) dt \int_\rho^\beta \psi'(vh(t)) g(t) \, dt \)

1.4.2 Theorem:-

Let \( \{X(t) \mid t \in T\} \) be a continuous homogeneous process with independent increments and suppose that the increments have non-degenerate distributions. Further suppose that the process has moments of all orders and its mean function as well as its covariance function are of bounded variation in \( T = [A,B] \). Let \( a(\cdot), b(\cdot) \) be continuous functions defined on \([A,B]\) with the property that

\[
\int_A^B a(t)b(t) \, dt = 0 \quad \Rightarrow \quad \int_A^B (a(t)b(t))^k \, dt \neq 0 \text{ for all } k > 1
\]

Let \( U = \int_A^B a(t) \, dX(t) \; \quad V = \int_A^B b(t) \, dX(t) \)

Then \( U \) has constant regression on \( V \) (ie)

\[
E[U/V] = E[U] \text{ almost everywhere if and only if}
\]

(i) \( \{X(t), t \in T\} \) is a Wiener Process with the linear mean function,

(ii) \( \int_A^B a(t)b(t) \, dt = 0 \)

Proof:-

Suppose \( \{X(t), t \in T\} \) is a Wiener Process with mean \( \lambda \) and covariance function.

\( r(s,t) = \sigma^2 \min(s,t) \). Where \(-\infty < \lambda < \infty, \sigma^2 > 0 \)

Let \( a(\cdot) \) and \( b(\cdot) \) be continuous functions.
on $[A,B]$ and defined $U$ and $V$ as in theorem

Also $\int_A^B a(t) b(t) \, dt = 0$ \rightarrow (1.4.5)$

Since $\psi(\cdot)$ is the logarithm of the characteristic function of

$X(t+1)-X(t)$, it is well known that $\psi(t) = i \lambda t - \frac{1}{2} \sigma^2 t^2$

In order to show that $E[U/V] = E(U)$ almost everywhere it is enough to prove that

$E[U e^{iv}] = E(U) \mathcal{L}'(e^{i\delta v}) \mathcal{L} \psi(\delta v)$ \text{ lemma (1.4.1)}$

By lemma (1.4.2) and (1.4.3) and the condition (1.4.5) it follows that

\begin{align*}
E[U e^{iv}] &= -i \int_A^B [a(t) \psi'(sb(t))] \mathcal{L} \psi(\delta v) dt \\
&= -i \int_A^B a(t) \psi'(sb(t)) \, dt \, E[e^{iv}] \\
&= -i \int_A^B a(t) [i \lambda - \sigma^2 b(t)] \, dt \, E[e^{iv}] \\
&= -i \int_A^B i \lambda a(t) \, dt \, E[e^{iv}] \\
&= \int_A^B \lambda \, a(t) \, dt \, E[e^{iv}]
\end{align*}$

We can show that

$E[U] = \lambda \int_A^B a(t) \, dt$ which completes the proof.

Conversely

Let $U$ and $V$ be as defined in this theorem. Let $\psi(\cdot)$ denote the logarithm of the
characteristic function of $X(t+1)-x(t)$ Further suppose that $u$ has constant regression on

$V(\text{ie}) \ E[U/V] = E[U]$ almost everywhere. This implies

$E \ [Ue^{iv}] = E[U] \ E[e^{iv}]$ by lemma (14.1)

By lemma (14.2) and (1.43), we have

$$\int_{A}^{-i} \psi'(sb(t)) \ a(t) \ dt \ e^{v(t)} \ dt$$

$$= E(U) \ e^{v(t)}$$

$$= E(U) \ \int_{A}^{v(t)} \ e^{v(t)} \ dt$$

Therefore $iE(U) = \int_{A}^{B} \psi' \ (sb(t)) \ a(t) \ dt$ for any real number $s$.

Differentiating with respect to $s$.

$$\int_{A}^{B} \psi'' \ (sb(t)) \ a(t) \ b(t) \ dt = 0$$

Let $s = 0$. Then $\psi''(0) \int_{A}^{B} \ a(t) \ b(t) \ dt = 0$

$\psi''(0) \neq 0$ implies $\int_{A}^{B} \ a(t) \ b(t) \ dt = 0$. Differentiating $k$ times with respect to $s$.

$$\int_{A}^{B} \psi^k \ (sb(t)) \ a(t) \ (b(t))^{k-1} \ dt = 0$$

Put $s = 0$

$$\psi^k(0) \int_{A}^{B} \ a(t) \ (b(t))^{k-1} \ dt = 0$$

Since the function $a(\cdot), b(\cdot)$ have the property that
\[
\int_{A}^{B} a(t)b(t) \, dt = 0 \implies \int_{A}^{B} a(t)(b(t))^k \, dt \neq 0
\]

\[
\Rightarrow \psi^k(0) = 0, \quad k \geq 3
\]

\[
\Rightarrow \psi(t) = i\lambda t - \frac{1}{2} \sigma^2 t^2, \quad -\infty < \lambda, < \sigma^2 > 0 \quad \text{for some } \lambda \text{ and } \sigma^2
\]

Therefore \{x(t), t \in T\} is a Wiener Process.

Laha studied the characterization of symmetric stable laws through regression properties and he proved the following theorem.

1.4.3 Theorem:-

Let \(X\) and \(Y\) two independent non-degenerate random variables whose expectations exist and are zero. Suppose that a structure is given by

\[
U = aX + bY
\]
\[
V = cX + dY
\]

\(a, b, c, d \neq 0\) and \(ad \neq bc\). Then \(X\) and \(Y\) have symmetric stable distribution with the same exponent \(\lambda > 1\) if and only if:

(i) There exists a constant \(\delta > 0\) such that the relation \(E[V/U] = \beta\) holds for all \(a\) such that \(0 < |a| < \delta\) and

\[
\beta = (ca^{-1} \alpha_1 |a|^{-\lambda} + db^{-1} \alpha_2 |b|^\lambda) \left( \alpha_1 + \alpha_2 \right)^{-1}
\]

(ii) \(\beta = (ca^{-1} \alpha_1 |a|^{-\lambda} + db^{-1} \alpha_2 |b|^\lambda) \left( \alpha_1 + \alpha_2 \right)^{-1}\)

Where \(\alpha_1\) and \(\alpha_2\) are the scale parameters of the distributions of \(X\) and \(Y\) respectively.

1.4.3 Lemma:-

Let \(U\) and \(V\) be two random variables and assume that the expectations exist and that \(E(U) = E(V) = 0\). Denote the characteristic function of the random vector \((U, V)\) by \(h(u, v)\) then \(E(U/V) = \beta U\).
Almost everywhere if and only if
\[
\frac{\partial h(u,v)}{\partial v} \neq 0
\]
\[
\frac{d}{du} h(u,0) \text{ holds for all real } u \quad \quad \quad (1.4.6)
\]

**Proof:**

The proof follows from Theorem (1.4.2)

### 1.4.4 Theorem:

Let \( X \) and \( Y \) be two independent random variables which have symmetric stable distributions with common exponent \( \lambda > 1 \), and scale parameters \( \alpha_1 \) and \( \alpha_2 \) respectively. Suppose that a stochastic structure is given by

\[
U = aX + bY \\
V = CX + dY, \quad a \neq 0, \quad b \neq 0, \quad c \neq 0, \quad d \neq 0, \quad ad \neq bc
\]

Then \( E(U/V) = B U \) where

\[
B = \frac{C/a \cdot \alpha_1 |a|^\lambda + d/b \cdot \alpha_2 |b|^\lambda}{\alpha_1 |a|^\lambda + \alpha_2 |b|^\lambda} \quad \quad \quad \rightarrow (1.4.7)
\]

**Proof:**

Let the characteristic function of the random variables \( X, Y \) and \( U/V \) be \( f(t) \), \( g(t) \) and \( h(u,v) \) respectively.

Then

\[
f(t) = \exp (- \alpha_1 \ |t| ^{\lambda} )
\]
\[
g(t) = \exp (- \alpha_2 \ |t| ^{\lambda} )
\]
\[
h(u,v) = f(aU+cV) \ g(bu+dv)
\]

Since \( \lambda > 1 \) we have
\[
\frac{d}{du} |u|^{\lambda} = \lambda \frac{u}{|u|} \quad |u|^{\lambda-1}
\]

Also \( \frac{\partial}{\partial v} \ln h(u,v) \bigg|_{v=0} = C/\alpha_1 |a|^{\lambda} + d/b \alpha_2 |b|^{\lambda} \frac{\partial \ln h(u,0)}{\partial u} \)

\[
\frac{d}{du} \lnh(u,0) = \frac{d}{du} \frac{c/a \alpha_1 |a|^{\lambda} + d/b \alpha_2 |b|^{\lambda}}{\alpha_1 |a|^{\lambda} + \alpha_2 |b|^{\lambda}} \frac{d}{du} \lnh(u,0)
\]

cancelling \(d/du\) \(\lnh(u,0)\) we get the result.

**Proof of theorem (1.4.3)**

From theorem (1.4.4) the condition is necessary.

To prove the sufficient condition -

Assume that condition (i) holds

Let the characteristic function of \(X, Y\) and \((U,V)\) by \(f(t), g(t)\) and \(h(u,v)\) respectively then

\[h(u,v) = f(au + cv) g(bu + bv)\]

From condition (i) and lemma (1.4.3)

\[(c-\alpha \beta) f'(au) g(bu) + (d-\beta \alpha) f(au) g'(bu) = 0 \quad \Rightarrow (1.4.8)\]

We show first by means of an indirect proof that neither of the functions \(f(u)\) and \(g(u)\) has real zeros. If this is not the case, then at least one of these functions has real zero.

Let \(\rho_1\) and \(\rho_2\) be absolute value of the zeros of \(f(u)\) and \(g(u)\). Where \(\rho_1(\rho_2) = \infty\)

if \(f(u)\) \([g(u)]\) has no real zeros. Let us take
\[ p = \min \left( \frac{P_1}{\delta}, \frac{P_2}{|b|} \right) \]

and restrict the values of \( u \) to the interval \( |u| < p \) and \( a \) to a closed interval \([a_1, a_2]\)

where either \( 0 < a_1 < a_2 < \delta \) or \( -\delta < a_1 < a_2 < 0 \). Then

\[ f(au) g(bu) \neq 0 \]

We can divide both sides of (1.4.8) by \( f(au) g(bu) \) and obtain,

\[ \frac{f'(au)}{f(au)} + \frac{g'(bu)}{g(bu)} = 0 \rightarrow (1.4.9) \]

If \( (C - a \beta) = 0 \) or if \( d - b \beta = 0 \) then

\[ g(u) \text{ (or } f(u) \text{)} \text{ belongs to degenerate distribution so that} \]

\[ C - a \beta \neq 0 \text{ and } d - b \beta \neq 0. \] Let us introduce

\[ \Theta(a) = \frac{-a(d-b \beta)}{b(C-a \beta)} \]

and obtain from (1.4.9)

\[ \inf (au) = \Theta(a) \ln g(bu) \rightarrow (1.4.10) \]

Since the first moment of \( x \) exists, we can differentiate equation (1.4.10) with respect to \( a \) and get

\[ \frac{f'(au)}{f(au)} = \Theta'(a) \ln g(bu) \rightarrow (1.4.11) \]

\( \psi(u) = \ln g(bu) \) from (1.4.9) and (1.4.11) we get

\[ \inf \psi(u) = \frac{a \Theta'(a)}{\Theta(a)} \psi(u) \rightarrow (1.4.12) \]
We note that \( \psi(0) = 0 \) and that \( \psi(u) \) cannot vanish identically in a neighbourhood of the origin. We next show that \( \psi(u) \) has no zeros in the interval \( |u| < p \) except at the point \( u = 0 \). we give an indirect proof of the statement and assume that

\[ p^* \quad (0 < |p^*| < p) \text{ is the zero of } \psi(u) \]

Which is nearest to the origin.

Therefore \( \psi(u) = 0 \) for \( 0 < |u| < |p^*| \)

Divide (1.4.12) by \( (u) \) we get

\[
\frac{u \ \psi(u)}{\psi(u)} = \frac{a \theta'(a)}{\theta(a)}
\]

L.H.S. of the equation is independent of \( a \) while the R.H.S. in independent of \( u \).

Hence both sides are equal to some constant and we get the following differential equations.

\[
\frac{u \ \psi'(u)}{\psi(u)} = \lambda \quad 0 < |u| < |p^*| \quad \rightarrow (1.4.13)
\]

\[
\frac{a \ \theta'(a)}{\theta(a)} = \lambda \quad 0 < |a| < \quad \rightarrow (1.4.14)
\]

Integrating (1.4.13) and get

\[
(u) = \begin{cases} 
    k_1 \ |u|^{\lambda} & \text{if } 0 < u < |p^*| \\
    k_2 \ |u|^{\lambda} & \text{if } -|p^*| < u < 0 
\end{cases} \quad \rightarrow (1.4.15)
\]

Since \( \psi(u) \) is continuous it follows from (1.4.15) that \( \psi(p^*) = 0 \) This is in contradiction with the definition of \( p^* \) and \( \psi(u) \). Therefore has no zeros in the region

\[ 0 < |u| < p \]
from (1.4.15) we get

\[ g(bu) = e^{\lambda b} |u|^{\lambda} \text{ if } 0 < u < \rho \]

\[ e^{\lambda |b|^{\lambda}} \text{ if } -\rho < u < 0 \]

Since \( g(-u) = \overline{g(u)} \), \( k_1, k_2 \)

and \( k_1 = \alpha + i\gamma \)

\( k_2 = \alpha - i\gamma \)

\[ g(bu) = e^{[\alpha + i\gamma] |u|^{\lambda}}, |u| < \rho \]

Here \( \alpha \neq 0 \) since \( \alpha = 0 \) implies \( g(u) \) belongs degenerate distribution.

Integrating (1.4.14) and obtain

\[ f(au) = e^{(\alpha u^2 + (u^{\lambda}|u|^{\lambda})} |u|^{\lambda} \text{ if } |u| < \rho \]

\[ \theta(a) = \mu |a|^\lambda \text{ if } 0 < a < \delta \]

\[ = \mu |a|^\lambda \text{ if } -\delta < a < 0 \]

Therefore (1.4.16) becomes

\[ f(au) = e^{(\alpha u^2 + (u^{\lambda}|u|^{\lambda})} |au|^{\lambda} \text{ if } 0 < a < \delta \]

\[ \gamma \mu_1 = -\gamma \mu_2 \]

Change of sings of \( \alpha \) and \( u \) leaves \( f(au) \) unchanged. Since \( \alpha \neq 0 \Longrightarrow \gamma = 0 \) while

\[ \mu_1 = \mu_2 = \mu \]

Thus \( f(au) = e^{\alpha u} |au|^{\lambda} \)

\[ g(bu) = e^{\alpha |u|^\lambda} \]

\[ \text{If } 0 < |a| < \delta \text{ and } |u| < \rho \]

Since \( f(-u) \) and \( \overline{f(u)} \Longrightarrow \mu \) is real. To show that (1.4..17) and (1.4.18) holds for all real \( u \).
If \( \frac{\rho_i}{\delta} \) then from (1.4.17) implies \( f(u) = e^{a \cdot u^\lambda} \) for \( |u| < \rho \delta = \rho_i \).

Since \( f(u) \) is continuous neither \( -\rho_i \) nor \( +\rho_i \) can be zeros of \( f(u) \). Similarly -\( \rho_2 \) and \( \rho_2 \) not the zeros of \( g(u) \).

Therefore \( f(u) \) and \( g(u) \) have no real zeros.

Also \( \alpha_i = -\mu \alpha \) and \( \alpha_j = -\frac{\alpha}{2|\beta|} \).

From (1.4.17) and (1.4.18)

\[
\begin{align*}
  f(u) &= e^{\alpha_{i+}u^\lambda} \\
  g(u) &= e^{\alpha_{j+}u^\lambda}.
\end{align*}
\]

Since a characteristic function is bounded \( \alpha_i > 0 \) and \( \alpha_j > 0 \).

Also the first moments of random variable \( X \) and \( Y \) exist implies \( 1 < \lambda < 2 \).

Therefore we can get the \( \beta \) as in the previous theorem.

1.4.6 Theorem:-

Characterization:-

Let \( \{x(t), t \in T\} \) be a homogeneous process with independent increments. Suppose that

1. \( X(0) = 0 \)

2. \( E[X(t)] = 0 \) for all \( t \).

3. The increments of the process have non-degenerate symmetric distributions

Let \( Y_\lambda = \int_0^1 t^\lambda \, dx(t) \) for any \( \lambda > 0 \).
Then the process is a symmetric stable process with exponent \( \alpha > 1 \), if and only if for some positive numbers \( \lambda \) and \( \mu \), \( \lambda \neq \mu \)

\[
e^\left( \frac{Y_\lambda}{Y_\mu} \right) \delta y_\mu
\]

for some constant \( B \) depending on \( \lambda \), \( \mu \). Further more \( \alpha, \lambda, \mu \) and \( B \) are connected by the relation 

\[
(\mu \alpha + 1) = B (\lambda - \mu + \mu \alpha + 1)
\]

\( \rightarrow (1.4.19) \)

**Proof:**

Let \( \{x(t), t \in T \} \) be a symmetric stable process with \( x(0) = 0 \) and \( E(x(t)) = 0 \) for all \( t \).

Let \( \theta(u,v) \) denote the log of the characteristic functions of \((Y_\lambda, Y_\mu)\).

Since \( \{X(t), t \in T\} \) is infinitely divisible \( \theta(u,v) \) is well defined.

Let \( \psi(u) \) denote the logarithm of the characteristic function of \((x(t+1) - x(t))\).

Since \( E(Y_\lambda) = E(Y_\mu) = 0 \),

From Lemma (1.4.3)

\[
\theta(u,v) = \int \psi(ut^\lambda + vt^\mu) \, dt
\]

Hence

\[
\frac{\partial \theta(u,v)}{\partial u} \bigg|_{u=0} = \int t^\lambda \psi'(vt^\mu) \, dt \rightarrow (1.4.20)
\]

and

\[
\frac{\partial \theta(0,v)}{\partial v} = \int t^\mu \psi'(vt^\mu) \, dt \rightarrow (1.4.21)
\]

Since the process is symmetric stable with mean zero,

\[
\psi(u) = -c |u|^{\alpha} \text{ for some real number } c \text{ and } \alpha > 1
\]

\[
\frac{\partial \theta(u,v)}{\partial u} \bigg|_{u=0} = B \frac{\partial \theta(0,v)}{\partial v}
\]
Where \( \beta \) satisfies (1.4.19)

This relation inturn shows that \( \mathbb{E}[Y_{\lambda} / Y_{\mu}] = \beta Y_{\mu} \) a.e. By lemma (1.4.1)

If part, let us define \( \theta(u,v) \) and \( \psi(u) \) as before.

Since \( \mathbb{E}[Y_{\lambda} / Y_{\mu}] = \beta Y_{\mu} \) a.e.

\[
\frac{\partial \theta(u,v)}{\partial u} = \frac{\beta d \theta(0,v)}{dv}
\]

\[
\frac{\partial}{\partial u} \int_0^1 \psi(ut^\lambda + vt^\mu) \, dt \bigg|_{u=0} = \beta d \int_0^1 \psi(vt^\mu) \, dt \quad \longrightarrow (1.4.22)
\]

Since the process is infinitely divisible with finite mean. We have from 1.4.22

\[
\int_0^1 t^\lambda \psi'(vt^\mu) \, dt = \beta \int_0^1 t^\mu \psi'(vt^\mu) \, dt \quad \text{for all } v
\]

Intergrating both sides with respect to \( v \)

\[
\int_0^1 t^\lambda \psi'(vt^\mu) \, dt = \beta \int_0^1 \psi'(vt^\mu) \, dt \quad \text{for all } v
\]

Since \( \psi(0) = 0 \)

\[
\int_0^\lambda u^{\lambda-2} \psi(u) \, du = \beta \int_0^{\lambda-\mu} \psi'(s) \, ds
\]

differentiating with respect to \( v \) and simplifying.

\[
V^{\mu-1} \psi(v) [1-\beta] = \beta (\lambda-\mu) \mu^{\mu-1} \int_0^\lambda \psi(s) \, ds
\]

Differentiating with respect to \( V \)

\[
(1-\beta) \mu \psi'(v) = \psi(v) [\beta (\lambda - \mu) - (1-\beta)]
\]

\[
\longrightarrow (1.4.23)
\]

Since \( X(1) \) has a non degenerate distributions we have \( \beta \neq 1 \)

Therefore \( \psi'(v) \psi[ (v)^{-1} ] = [\beta (\lambda - \mu) - (1-\beta)] \mu(1-\beta) \)

\[
\longrightarrow (1.4.24)
\]

Let \( \alpha = \beta (\lambda - \mu) - (1-\beta) [\mu(1-\beta)]^{-1} \)
Solving (1.4.23) with \( \psi(\cdot) \) continuous at the origin

We get \( \psi(v) = -cv^n \), \( c \) constant \( \neq 0 \). Since \( \psi \) is the logarithm of the characteristic function of a symmetric distribution with finite mean, we have

\[
\psi(v) = -c |v|^\alpha \text{ for all } v.
\]

Therefore \( \psi(\cdot) \) is the characteristic function of a symmetric stable law with exponent \( \alpha \).

Therefore \( \{x(t), t \in T\} \) is a symmetric stable process with finite mean.

1.5 Characterization of stable process by identically distributed stochastic integrals.

Let \( x(t) \) be a homogenous and continuous stochastic process with independent increments. Reidel [41] obtained characterization the stable process by two identically distributed stochastic integrals formed by means of \( x(t) \) in the sense of convergence in probability as follows.

The stochastic integral is denoted by \( \int_{\mathcal{A}}^\beta \mathcal{X}(t) \, dx(t) \) (1.5.1)

If the random variable \( X \) and \( Y \) are identically distributed.

We write \( X \equiv Y \)

1.5.1 Theorem:-

The stochastic integral (1.5.1) exists and its characteristic function is given by

\[
\log h(u) = \int_{\mathcal{A}}^\beta \log f(u \mathcal{X}(t)) \, dv(t)
\]

1.5.1 Definition:-

It is well known that there exists a finite Borel measure \( V \) with support contained in \([A,B]\) such that

\[
V(\varnothing, t) = 0 \text{ if } t < A,
\]

\[
= V(t) - V(A) \text{ if } A \leq t \leq B
\]
\[ V(B) - V(A) \] if \( t > B \)

Put \( V_a(t) = V(a^t, t) \)

\[ C = \min a(t) \]

\[ D = \max a(t) \quad \lambda \leq t \leq B \]

then \( V_a \) as well as \( V \) is non decreasing non negative and left continuous function

1.5.2 Theorem:-

The integral (1.5.1) and \( \int_a^b t \, dV(t) \) exist and are identically distributed (i.e)

\[ \int_a^b t \, dV(t) \equiv \int_c^d t \, d(V_a(t)) \]

Let \( a_j \) and \( V_j \) \( (j = 1, 2) \) be defined in the intervals \([A_j, B_j] \) for \( \text{Re} \, Z > 0 \).

\[ s(z) = \int_{B_j}^{A_j} |a_j(t)|^z dv(t) - \int_{B_j}^{A_j} |a_j(t)|^z dv(t) \]

\[ \hat{s}(z) = \int_{B_j}^{A_j} |a_j(t)|^{-z} a_j(t) dt - \int_{B_j}^{A_j} |a_j(t)|^{-z} a_j(t) dt \]

\( S \) and \( \hat{S} \) are analytic in \( \text{Re} \, Z > 0 \) and continuous in \( \text{Re} \, Z > 0 \)

1.5.3 Theorem:-

Assume that \( z = 0 \) is not an accumulation point of zeroes of \( S, \hat{S} \)

and \( \lim_{x \to 0} x \log |S(x)| = 0 \), \( (z \neq x + iy) \)

\[ \longrightarrow (1.5.1) \]

for the equivalence of the two assertions.

(A) \( X(t) \) is a stable process with characteristic exponent \( Z_0 \)

\[ \int_{B_j}^{A_j} a_j(t) \, dV_j(t) \equiv \int_{B_j}^{A_j} a_j(t) \, dV_j(t) + q \text{ for a certain real } q. \]

It is necessary and sufficient that the following condition be satisfied.

(i) There exists an unique real zero \( Z_0 \) of \( s \) \( (0 < z_0 < Z) \)

In case that \( Z_0 < 2 \) its multiplicity is not higher than 2.
(ii) \( S(Z_0) = 0 \)

(iii) If \( Z_0 < Z^- \) then \( S(Z) \neq 0 \) for \( \text{Re} \ Z = Z_0 \)

1.5.1 Corollary:

Assume (1.5.1.) and (1.5.2). For the equivalent of (A) and (B')

\[
\int_{A_1} a_1(t) dx(v_1(t)) = \int_{A_2} a_2(t) dx(v_2(t))
\]

It is necessary and sufficient that the following conditions be satisfied

(i) and (ii) and (iii) and

(iv) \( s''(1) = 0 \) if \( Z_0 \neq 1 \)

\( S(1) = S'(1) = 0 \) if \( Z_0 = 1 \)

1.5.6 Theorem:

Let \( X(t) \) be a continuous homogenous process with independent and symmetric increments satisfying \( x(0) = 0 \)

Assume that \( z = 0 \) is not an accumulation point of zeros of \( S \) (1.5.3)

\[
\lim_{x \to 0^+} x \log |s(x)| = 0 \quad \rightarrow (1.5.4)
\]

For the equivalence of the two assertions.

(A') \( X(t) \) is a symmetric stable process with characteristic exponent \( Z_0 \)

(B') It is necessary and sufficient that the following conditions be satisfied.

(i), (iii)' If \( z_c < 2 \) then \( s(Z) \neq 0 \) for \( \text{Re} \ Z = Z_0 \)

(iv)' \( S(1) = 0 \)
1.5.7 Theorem:-

Suppose (1.5.3) & (1.5.4) for the equivalence of (A\"') x(t) is a Wiener process with 
linear mean value function m(t) and (B').

It is necessary and sufficient that

(i)' S(2) = 0

(iii)' S(z) \neq 0 for 0 < Z < 2, \int m Z = 0

(iv)' S(1) = 0 or \ m(t) = 0

Investigation of a functional equation:-

Put Q = \max (-A_1, -A_2, B_1, B_2) continue V in [-Q, Q] by the formulas.

v_j (t) = v_j (A_j) if -Q \leq t < A_j

= V_j (t) if A_j \leq t < B_j

= V_j (B_j) if B_j \leq t \leq Q

W(t) = V_j(t) - V_j(t)

From theorem (1.5.4) condition B is equivalent to (\because \gamma)

\int_{-Q}^{Q} \int_{0+}^{\infty} \frac{t^2}{\theta^4} \, \frac{d(M(x)+N(x)) \, dw(t)}{(1+(tx)^4)(1+x^2)} = -q \rightarrow (1.5.5)

\sigma^2 \int_{-Q}^{Q} t^2 \, dw(t) = 0 \rightarrow (1.5.5)

L_1(x) = \int_{-Q}^{0} -N(x/t) \, dw(t) + \int_{0+}^{Q} M(x/t, dw(t)) = 0 \quad (x < 0) \rightarrow (1.5.6)

L_2(x) = \int_{-Q}^{0} M(x/t) \, dw(t) + \int_{0+}^{Q} N(x/t) \, dw(t) = 0 \quad (x > 0) \rightarrow (1.5.7)
Next, let us derive a functional equation for the function

\[ \Pi_1(z) = \oint_{r} x^{-r} dN(x) \quad (\text{Re } Z \geq 0) \]

\[ \Pi_2(z) = \oint_{-r} |x|^{-r} dM(x) \quad (\text{Re } z > 0) \]

Where \( r \) is an arbitrary continuity point of \( N(x) \) and \( M(-x) \).

The functional \( \Pi_1 \) and \( \Pi_2 \) are analytic in \( \text{Re } Z > 0 \) and bounded in \( \text{Re } Z \geq 0 \).

Set \( S_1(z) = Q^r \int_{0}^{Q} |t|^r \, dt \)

\[ S_2(z) = Q^r \int_{0}^{Q} t^r \, dt \]

\[ k_{11}(z) = (rQ)^r \oint_{0}^{Q} \oint_{0}^{Q} x^{-r} dN(x) dt \]

\[ k_{22}(z) = (rQ)^r \oint_{0}^{Q} \oint_{0}^{Q} |x|^{-r} dM(x) dt \]

\[ k_{12}(z) = (rQ)^r \oint_{0}^{Q} \oint_{0}^{Q} |x|^{-r} dN(x) dt \]

\[ S_1(z) + S_2(z) = Q^r \left[ \oint_{0}^{Q} |t|^r \, dt + \oint_{0}^{Q} t^r \, dt \right] \rightarrow (1.5.8) \]

\[ S_2(z) - S_1(z) = Q^r \oint_{0}^{Q} t^r \, dt - \oint_{0}^{Q} |t|^r \, dt \rightarrow (1.5.9) \]

The functions \( k_{ij} \) are analytic in \( \text{Re } Z < 0 \) continuous and bounded in \( \text{Re } Z \leq 0 \).
\[ K_i(z) = S_2(-z)[k_{i,2}(z) + k_{2,1}(z) - S_2(-z)(k_{1,1}(z) + k_{2,2}(z))] \]
\[ k_2(z) = S_2(-z)(k_{1,1}(z) + k_{2,2}(z)) - S_1(-z)(k_{1,1}(z) + k_{2,2}(z)) \]

\[ k_1 \text{ and } k_2 \text{ have same properties as } k_{1,1} \]

1.5.1 LEMMA :-

Let (1 5 6) and (1 5 7) satisfied then we have

\[ H_1(z) = S(-Z) \left( -S(-Z) k_{1,1}(z) \right) \]

Using the known result we have

\[ (rQ)' \left( \int_{\mathbb{C}} |x|^2 \, d \int_{\mathbb{C}} -\mathbb{N}(x/t) \, dw(t) \right) \]
\[ = (rQ)' \left( \int_{\mathbb{C}} |x|^2 \, dx \int_{\mathbb{C}} -\mathbb{N}(x/t) \, dw(t) \right) \]
\[ = I \]

Put \( y = x/t \)

\[ \int_{\mathbb{C}} = (rQ)' \left( \int_{\mathbb{C}} |y|^2 \, \mathbb{N}(y) \, dw(t) \right) \]
\[ = S_1(-z) H_1(z) - k_{1,1}(z) \]

Similarly we obtain the following relation

\[ (rQ)' \left( \int_{\mathbb{C}} |x|^2 \, d \int_{\mathbb{C}} M(x/t) \, dw(t) \right) \]
\[ = S_2(-Z) H_2(z) - k_{2,2}(z) \]

\[ (rQ)' \left( \int_{\mathbb{C}} x^2 \, d \int_{\mathbb{C}} -M(x/t) \, dw(t) \right) \]
\[ = S_1(-z) H_2(z) - k_{1,2}(z) \]
\[
\begin{align*}
(rQ)' \int x^+ \ dQ &= S_2 (-z) H_1 (z) - k_2 (z) \quad \rightarrow (1.5.14) \\
\int \frac{dL_1}{Q} (x) &= -N(x/t) + M(x/t) \\
\int \frac{dL_2}{Q} (x) &= -M(x/t) + N(x/t) \\
0 &= \left( \frac{rQ}{Q} \right) \int \left| x \right| \ dL_1 (x) \\
S_1 (-z) H_1 (z) + S_2 (-z) H_2 (z) - (k_{1i} (z) + k_{2i} (z)) \\
= (rQ)' \int_x^0 x^+ |t| \ dM (x) \\
+ (rQ)' \int x^+ |t| \ dN (x) \\
+ (rQ)' \int x^+ |t| \ dw (t) \\
0 &= \left( \frac{rQ}{Q} \right) \int \left| x \right| \ dM (x) \\
&S_1 (-z) H_1 (z) + S_2 (-z) H_2 (z) - (k_{1i} (z) + k_{2i} (z)) \quad \rightarrow (1.5.15) \\
0 &= \left( \frac{rQ}{Q} \right) \int \left| x \right| \ dM (x) \\
&\rightarrow (1.5.16) \\
\text{Multiply (1.5.15) by } S_1 (-z) \text{ and (1.5.16) by } S_2 (-z) \text{ and} \\
(S_1 (-z))^2 H_1 (z) + S_2 (-z) S_1 (-z) H_2 (z) - S_1 (-z) (k_{1i} (z) + k_{2i} (z)) = 0 \quad \rightarrow (1.5.17) \\
S_2 (-z) S_1 (-z) H_2 (z) + (S_2 (-z))^2 H_1 (z) - S_2 (-z) (k_{1i} (z) + k_{2i} (z)) = 0 \quad \rightarrow (1.5.18) \\
(s_i (-z)^2 - S_2 (-z)^2) H_i (z) = k_i (z) \\
\text{Using (1.5.8) and (1.5.9)} \\
Q^2 \left( S (-z) (S (-z) H_1 (z)) = k_i (z) \right) \text{ Similarly we can prove for } j=2.
\end{align*}
\]