CHAPTER 5
APPLICATION OF MARTINGALES TO SOME EPIDEMIC MODELS.

5.1 INTRODUCTION:-

The considerable literature now existing on stochastic epidemic models is mainly concerned with closed population epidemics, such as the general stochastic epidemic. Such models can provide a good description of the short term behaviour of an epidemic. They can also provide an useful indication of the effects of some parameters in complicated models, in incorporating immigration and deaths. A major part of the work on stochastic epidemic models has been on the general stochastic epidemic a name given by Bailey [2]. We give a simpler proofs for the threshold theorem due to Williams and Whittle using Rajarsi [37] technique.

A major complication of many diseases is the existence of so-called carriers (ie) individuals who although apparently healthy themselves are already infected and are capable of transmitting the infection to others. Philippe Picard [30] gives some simple applications of martingale to epidemics. All the results are in connection with stopping times T and include the expression of the joint generating function Laplace transform of $X_t$, $\sum x_u y_u$ du and $\sum y_u$ du and relation between moments of these three variables. The relation between Downstom's model and the general epidemic is discussed and finally a generalisation of one of Daniel's classical results is given.

In a real epidemic the increase of the number of infectives usually generates sanitary measures in order to isolate infectives and prevent contacts with susceptibles. Therefore Picard [31] in the general epidemic model considered the parameters are as functions of
infectives which gives a better approximation. In these cases the martingale approach proves very valuable and gives explicit results quite easily.

The considerable literature now existing on stochastic epidemic models is mainly concerned with closed population epidemics, such as the general stochastic epidemic and thus is of limited direct use in modelling most AIDS epidemics where immigrations into and deaths from the class of susceptibles can be an important feature.

AIDS needs no introduction. AIDS (Acquired Immune Deficiency Syndrome) which breaks down the body’s natural immune system is transmitted primarily by sexual contact or by bodily fluids exchanged between drug addicts, who share needles. AIDS epidemic an explosive spread of disease was first diagnosed almost 12 years ago. To understand AIDS one must know a little about the functioning of the human body and its resistance power, or capacity of the body or immune system.

The AIDS is caused by a germ HIV which enters the body’s white blood cells and makes it impossible for the body to defend itself against illness. The HIV doesn’t kill the people directly. But it weakens the body’s resistance power and finally destroys the body’s immune system. The major signs of AIDS are loss of more than 10% of normal body weight and severe diarrhoea for more than a month and continuous fever for more than one month. However, standard stochastic epidemic theory is often still not applicable because the infection process is modelled slightly differently. The usual term \( \beta_{xy} \) (where \( x \) and \( y \) are the number of susceptibles and infectives respectively) for the rate of new infections is replaced by \( \frac{\beta_{xy}}{x + y} \).

The justification for the new term is that AIDS spread by individuals changing sexual partners. So if we removed individuals are no longer available as sexual partners, then the probability that
a new partner of a given susceptibles infected is $y / (x + y)$.

John A. Jacquez Philiponville [19] compare the threshold results for the deterministic and stochastic versions of the homogeneous $SI$ model with recruitment death due to the disease a background death rate and transmission rate. A fundamental concept that has come out of the deterministic mathematical theory of epidemics is that of the basic reproduction number.

Let us examine the deterministic and stochastic formulation for the SI, SIS, SIR and SIRS models for homogeneous populations. Finally Frankball Philpó’neill [14] consider a model for the spread of an epidemic in a closed, homogeneously mixing population in which new infection occur at rate $\frac{\beta xy}{x + y}$ where $x$ and $y$ are the number of susceptibles and infectious individuals respectively and $\beta$ is the infection rate. This differs with the standards general epidemic in which new infectious occur at rate $\beta xy$. Both the deterministic and stochastic versions of the modified epidemic are analysed.

5.2 EPIDEMIC MODELS AND THRESHOLD THEOREMS:-

It is just about 85 years ago that the mathematical theory of epidemics, in the modern sense of phrase was first started by the work of William Hamer and Ronald Ross. The use of mathematical and statistical methods primarily intended in understanding the mechanism underlying the spread of infectious diseases. The great scourages of history may now seems to be more than unpleasant memories there are no grounds for optimism. It is true that many previously widespread diseases are gradually disappearing from the modern world. Tuberculosis has constantly declined. Smallpox on the otherhand seems very close to virtual extinction. Malaria has been eradicated from many areas of the world where it was previously endemic, by the
simplex way of draining swamps and marshes.

The total load of human misery and suffering from communicable disease in the world today is incalculable and presents a formidable challenge to public health authorities, epidemiologists, parasitologists, entomologists, bio mathematicians. Modern medicine now can do much to cure many infectious diseases once they have been contracted but by for the most spectacular results have been in the field of prevention.

The eradication of poverty and the provision of sufficient social and public health measures such as isolation of infectious cases, provision of clean water supplies, vaccination and inoculation etc have provided the main contributions to the fight against disease.

Let us develop some mathematical models to study the large scale population phenomena of immediate relevance to any social and public health measures that might be undertaken. In particular, let us like to know more about the transmission and spread of infectious disease, about trying to predict the course of an epidemic and about the recognition of threshold densities of population.

In the context of endemic disease, let us require to know more about how the endemic level is related to factors which can be controlled by public health intervention. It is necessary to develop models that will assist the decision making process by helping to evaluate the consequences of choosing one of the alternative strategies available. Thus mathematical models of the dynamics of communicable disease can have a direct bearing on the choice of an immunization program, the optimal allocation of scarce resources or the best combination of control or eradication techniques. In all these matters, mathematical and statistical investigations have an essential part to play.
Epidemiological Principles:

Let us present a short account of the main epidemiological concepts that appear to be of importance in discussing mathematical models. Let us call an actively infectious individual an infective and the period during which organisms are discharged, the infectious period. The time elapsing between the receipt of infection and the appearance of symptoms is called the incubation period, and the period from the observations of symptoms in one case to the observation of symptoms in a second case directly infected from the first is the serial interval. When symptoms occur either during or immediately after the end of the infectious period, the incubation period is exactly equal to the sum of the latent period and the part of the infectious period during which the patient is still a danger to other.

It is supposed for the group of individuals that at any given instant there is a chance of contact between any two individuals sufficient for the transmission of disease if one is an infective and one a susceptible. When there are several infectives in a group, a given susceptible will remain free of disease only if he happens to escape adequate contact with any of them. With continuous infection models a suitable analogous assumption is that the chance of one new case in a very short interval of time is jointly proportional to the length of the interval, the number of susceptibles and the number of infectives. Such ideas readily lead to mathematical equations describing the whole process. Carriers are the individuals who although apparently healthy themselves, harbour infection which can be transmitted to others.

Methodological aspects:

Let us consider briefly a number of methodological aspects concentrating more on the general philosophical implications.
Let us first look at the simplest type of epidemic model in which infection spreads by contact between the members of a community but in which there is no removal from circulation by death, recovery or isolation. Ultimately, all susceptibles therefore become infected. When dealing with large number of both susceptibles and infectives we should expect the effect of statistical fluctuations on large scale phenomena to be much reduced. In such circumstances it is not unreasonable to use as a first approximation a deterministic model in which we assume that for given numbers of susceptibles and infectives and for a given attack rate, certain definite numbers of new infectives will occur in any specified time. In stochastic models probability distributions of the numbers of susceptibles or infectives occurring at any instant replace the point values of deterministic treatments. In general, the form of behaviour predicted by a stochastic model is likely to be very similar by the corresponding deterministic version when the numbers of susceptibles and infectives are both sufficiently large but in other situations there may be important differences. Moreover, there is a good reason to suppose that the assumption of homogeneous mixing is approximately valid for epidemic in only comparatively small as individual household where statistical fluctuations may be large. We can use the concept of epidemic curve, defined by the rate at which new cases are recognized (ie) the rate of change with respect to time of the total number of removals. When working with simple epidemics having no removal, it is convenient to define the epidemic curve in terms of the total number of cases instead of total number of removals.

**Deterministic model:**

In the simplest deterministic formulation, we suppose that we have homogeneously mixing group of individuals of total size $n+1$ and that the epidemic is started off at time $t = 0$, by
just one individual becoming infectious, the remaining \( n \) individuals all being susceptible, but as yet uninfected. In general, at time \( t \), let us write \( x + y = n + 1 \), where \( x \) denotes the number of susceptibles and \( y \) the number of infectives. Also, let us consider the rate of occurrence of new infections is proportional to both the number of infectives and the number of susceptibles. Thus the actual number of new infections in the time interval \( \Delta t \) can be written as \( \beta xy \Delta t \) where \( \beta \) is the infection rate. It follows that \( \Delta x = -\beta xy \Delta t \).

Suppose, we have a community of total size \( n \), comprising at time \( t \), \( x \) susceptibles, \( y \) infectives in circulation and \( z \) individuals who are isolated, dead or recovered and immune. Thus \( x + y + z = n \). In time \( \Delta t \), there are \( \beta xy \Delta t \) new infections and \( \gamma \Delta t \) removals where \( \gamma \) is the removal rate. Also, \( \rho = \gamma / \beta \) is known as the relative removal rate.

**Stochastic model:**

Let us now consider the simplest probability version of the deterministic model. As before, we shall assume a homogeneously mixing group of \( n+1 \) individuals and suppose for simplicity that the epidemic starts at time \( t = 0 \) with one infective and \( n \) susceptibles. This time let us take the random variables \( X(t) \) and \( Y(t) \) to represent the number of susceptibles and infectives respectively at time \( t \), where \( x(t) + y(t) = n+1 \). Then the chance of a contact between any two specified individuals in an interval \( \Delta t \) is \( \beta \Delta t + O(\Delta t) \), where, \( \beta \) is the contact rate, and \( \beta = \text{constant} \). It follows that the chance of one new infection in the whole group in \( \Delta t \) is \( \beta x y \Delta t \) to order \( \Delta t \). When this transition occurs \( X \) decreases by one unit and \( Y \) increases by one unit. Suppose if we take the possibility of removal, then the chance of one removal in \( \Delta t \) can be taken as \( \gamma y \Delta t \) where \( \gamma \) is the removal rate. The variable \( Y \) decreases by one unit after the transition, but \( X \) remains unchanged.
We suppose that at time $t = 0$, there are $n$ susceptibles and $a$ infectives. Let $P_n(t)$ be denoted as the probability that at time $t$, there are $r$ susceptibles still uninfected and $s$ infectives in circulation. The chance of one new infection in time $\Delta t$ is taken to be $\beta r s \Delta t$ and the change of one removal $\gamma s \Delta t$. Also the time interval from the infection of any given susceptible to his eventual removal has a negative exponential distribution. Also the time scale is given by $t - \beta t$, instead of $t$, and $\gamma / \beta = \rho$, the ratio of removal rate to infection rate which we shall call the relative removal rate.

**Carrier Models:**

Carriers are individuals who, although apparently healthy themselves, harbour infection which can be transmitted to others. Diseases such as poliomyelitis, tuberculosis or typhoid are typical examples. Some carriers may be infectious for a very long time, others may become clear of infection more quickly. In either case, the carriers are effectively removed from circulation, but as they are not ill and exhibit no normal symptoms of disease, they are not themselves usually recognized as actual cases. On the other hand, carriers may be suspected because of the existence of an otherwise inexplicable occurrence of scattered cases. This may lead to a deliberate search for carriers and some or all of them may be identified through the use of special tests.

**Basic deterministic Model:**

In this basic model, let us concentrate attention on only two types of individuals, susceptibles and carriers. It is assumed that only carriers are responsible for actual spread of infection. When a susceptible is infected, he is supposed to exhibit symptoms sufficiently quickly
to be effectively recognized and removed from circulation before he can transmit the disease to others. The elimination of carriers proceeds at some finite rate which depends on both spontaneous recovery and public health detection. Let us suppose at time $t$, we have $x$ susceptibles and $y$ carriers. The number of new infections in time $\Delta t$ is $\beta xy \Delta t$ where $\beta$ is the infection rate and the number of carriers removed in $\Delta t$ is assumed to be $\gamma y \Delta t$ where $\gamma$ is the removal rate.

**Basic Stochastic Model:**

In this model, let us take the number of susceptibles and carriers at time $t$ are represented by the random variables $x(t)$ and $U(t)$. The chance of one new infection occurring in time $\Delta t$ is taken to be $\beta x U \Delta t$. When this event happens, $x$ decreases by one unit and $U$ remains unchanged. Again, we assume that the chance of one carrier being removed in $\Delta t$ is $\gamma U \Delta t$. In this case, $U$ is decreased by one unit but $x$ is unchanged. Let us take $\rho = \alpha / \beta$ and $\tau = \beta t$, and let the probability of $r$ susceptibles and $u$ carriers at time $\tau$ be $P_n(\tau)$.

**Epidemic Models:**

**Simple deterministic Model:**

Let $x(t)$ be the number of susceptibles (i.e.) the number of people who can be infected. $Y(t)$ the number infected people in the population and $Z(t)$ the number of people removed from the population by recovery, immunity, death, hospitalization or removal from the scene of infection.
If \( N \) is the total population size, we have that
\[
X(t) + Y(t) + Z(t) = N = \text{constant}.
\]

In the simple deterministic model there is no removal. If \( n \) is the initial number of susceptibles into which one infected person is introduced, we have that
\[
\frac{dx}{dt} = -\beta xy
\]
\[
\frac{dy}{dt} = \beta xy
\]
\[
X(t) + Y(t) = n + 1.
\]

Integrating and taking the limit as \( t \to \infty \), we find that
\[
x(t) \to 0
\]
\[
y(t) \to n+1
\]

**THRESHOLD THEOREMS:**

Rajarshi [37] gives a simpler proofs of two threshold theorems for a general stochastic epidemic using reflection principle.

A fairly elementary proof for the threshold theorems due to Williams and Whittle, simpler than Bailey [02] is given. The proofs are based on an application of the reflection principle through the ballot problem.

**Some important Definitions:**

5.2.1 One dimensional Random Walk:

It is a Markov Chain whose state space is a finite or infinite subset \( a, a+1 \ldots b \) of the integers in which the particle if it is in state \( i \) can in a single transition either stay in \( i \) or move
5.2.2 Total size of the epidemic:-

It is the total number of removals after the elapse of a very long, ideally infinite period of time.

5.2.3 Intensity of the epidemic:-

It is the proportion of the total number of susceptibles that finally contracts the disease and is denoted by $i$.

5.2.4 Relative Removal Rate:-

The ratio of removal rate to infection rate is known as the relative removal rate and is denoted by $\rho = \gamma / \beta$.

5.2.5 Relative removal rate per susceptible:-

The ratio of relative removal rate to the number of susceptibles is called the relative removal rate per susceptible and is denoted by $\theta = \rho / n$.

5.2.6 Reflection principle:-

This principle relates to the fact that there is a one to one correspondence between all paths from $A(a_1,a_2)$ to $B(b_1,b_2)$ which touch or cross the $x$-axis and all paths from $A'(a_1,-a_2)$ to $B$.

5.2.7 Ballot problem:-

Suppose that in a ballot candidate $P$ scores $p$ votes and candidate $Q$ scores $q$ votes where $p > q$. The probability that through the counting there are always more votes for $P$ than for $Q$ is $\frac{p-q}{p+q}$.
Ballot Theorem:-

Let \( n \) and \( x \) be positive integers. There are exactly \( x/n \) paths \((S_1, \ldots, S_n)\) from the origin to \((n, x)\) such that \( S_1 > 0 \), \( \ldots \), \( S_n > 0 \).

Proof:-

clearly there exists exactly as many admissible paths as there are paths from \((1, 1)\) to \((n, x)\) which neither touch or cross the \( t \) axis.

The number of such paths equals

\[
N_{n+1, x+1} - N_{n, x+1} = \binom{p + q - 1}{p - 1} - \binom{p + q - 1}{p}
\]

\[
\text{R.H.S} = N_{n,x}(\frac{p - q}{p + q})
\]

Recently a more direct algebraic proof of the threshold theorem for large \( n \) has been obtained by William. Also an ingenious method of investigating limiting behaviour more fully has been found by Whittle.

5.2.1 William Threshold theorem:-

If \( n \) is sufficiently large then \( P(0) \) the probability of a finite epidemic size is given by

\[
P(0) = \begin{cases} 
0 & \text{if } 0 < 1 \\
1 & \text{if } 0 \geq 1 
\end{cases}
\]

Proof:-

Let \( \{X(t), Y(t), Z(t), t \geq 0\} \) be a general stochastic epidemic with \( \beta \) and \( \gamma \) as the infection and removal rates respectively. Where \( X(t) \) is the number of susceptible at \( t \), \( Y(t) \) is the number of infectious persons at \( t \) and \( Z(t) \) the number of removal in \((0, t)\) we assume that
Let the intensity of an epidemic be denoted by $i$.

Let $W = \lim_{t \to \infty} Z(t) - a$ be the size of an epidemic.

Then with $P[\omega = \omega_j] = \pi_j = \sum_{\omega \subseteq \omega_j} P(\omega)$ gives the probability of an epidemic with intensity not greater than $i$.

We regard the progress of the epidemic in terms of the succession of population states represented by the points $(r, s)$. The process is thus seen as a random walk starting from the point $(n, a)$ and ending at one of the points $(n-w, 0)$ with $0 \leq w \leq n$, with an absorbing barrier along the line $s = 0$.

The transitional probabilities are

$$P_i[(r, s) \to (r-1, s+1)] = \frac{r}{r+i}$$

and

$$P_i[(r, s) \to (r, s-1)] = \frac{p}{r+i}$$

The formula required can be written down more or less directly by considering the sum of the probabilities of all possible paths from $(n, a)$ to $(n-w, 0)$. One way of doing this is to take all paths to the point $(n-w, 1)$ which do not go below the line $s = 1$ followed by the final step to $(n-w, 0)$ with probability $\frac{p}{n+p-w}$.

We thus obtain

$$P(w) = \frac{\sum (n+p)^{a_0} (n+p-w)^{a_1} \ldots (n+p-w)^{a_w} \alpha}{\prod_{n-w} (n+p-w)}$$

where the summation is over all compositions $a+w-1$ into $w+1$ parts such that
\[ \sum_{j=0}^{i} \alpha_j \leq n + i - 1 \]

for \( 0 \leq i < w - 1 \) and \( 1 \leq \alpha_w \leq n + w - 1 \)

Let \( \mathcal{A} \) be the collection of all \( \alpha = (\alpha_n, \alpha_1, ..., \alpha_w) \) such that \( \alpha_i \) is a non-negative integer for every \( i \)

\[ \alpha_n + \alpha_1 + ... + \alpha_j < a + j \] for \( j = 0, 1, 2, ..., w - 1 \)

\[ \alpha_w \geq 1 \text{ and } \alpha_1 + \alpha_2 + ... + \alpha_w = a + w \]

then

\[
P(w) = \frac{n(n-1)...(n-(w-1))}{(n+p)(n+p-1)...(n+p-(w-1))} \cdot \frac{a+w}{\tilde{\alpha}^w} 
\]

For a fixed \( w \) (and a sufficiently large \( n \)),

\[
n(n-1)...(n-(w-1)) \quad \frac{n^w}{(n+p)(n+p-1)...(n+p-(w-1))} = \frac{n^w}{(n+p)^w}
\]

\[
\pi \left( \frac{\tilde{\alpha}^w}{\pi} \right) = \frac{1}{(n+p-r)^w} \quad \frac{1}{(n+p-r)^w} 
\]

\[
\frac{1}{(n+p-r)^{\alpha_0}} \cdot \frac{1}{(n+p-r)^{\alpha_1}} \cdot \frac{1}{(n+p-r)^{\alpha_2}} \cdot \cdots \cdot \frac{1}{(n+p-r)^{\alpha_w}} 
\]

\[
\frac{1}{(n+p)^{\alpha_0}} \cdot \frac{1}{(n+p)^{\alpha_1}} \cdot \frac{1}{(n+p)^{\alpha_2}} \cdot \cdots \cdot \frac{1}{(n+p)^{\alpha_w}} 
\]

\[
\frac{1}{(n+p)^{\alpha_0}} \cdot \frac{1}{(n+p)^{\alpha_1}} \cdot \frac{1}{(n+p)^{\alpha_2}} \cdot \cdots \cdot \frac{1}{(n+p)^{\alpha_w}} 
\]
5.2.1 Lemma:

Let \( x \) and \( y \) be two positive integers. Suppose on a two dimensional plane, we allow only the two types of transition (\( x', y' \)) to \((x'-1, y'-1)\) (to the north west) and

\((x', y') \) to \((x'+1, y')\) (to the south). Where \( x' \) and \( y' \) are non negative integers. Then the total number of ways of reaching \((0,0)\) from \((x,y)\) without touching or crossing the \( x \) axis is given by

\[
\binom{2x + y - 1}{x} \frac{y}{x + y}
\]

Proof:

To arrive at \((0,0)\) we first observe that there have to be \( x \) transitions of the type I and \( x+y \) transitions of the type II. To every transition of the north west type we associate a south east transition and to every transition to the south we associate a north east transition. In this new walk, the condition of not touching or crossing the \( x \) axis is naturally retained. The situation is similar to the ballot problem.

Out of \(2x+y\) total votes, the winner wins by \( y \) votes. The total number of ways in which he maintains the lead throughout the counting is

\[
\frac{Y}{2x + y}
\]

where \( N_m, r \) is

\[
\binom{m}{\frac{1}{2} (m - r)}
\]
Going back to the general stochastic epidemic we imbed a discrete time process and plot the number of susceptibles and the number of infectives on the \((x, y)\) plane. Then we notice that the north west transition is a new infection and the transition to the south corresponds to a removal of a case. Thus in a two dimensional random walk, subjected to the above conditions, we are looking for the number of paths reaching \((n-w, 0)\) from \((n, a)\).

\[
N_{_{2x+y, y}} = \left( \frac{2x+y}{y} \right) x \frac{y}{2x+y}
\]

\[
P(w) = \left( \frac{n}{n+p} \right)^{w} \frac{a+w}{\rho} \left( \frac{1}{n+p} \right) a+w
\]

Taking \(x = w, \ y = a\).

\[
= \left( \frac{n}{n(1+p/n)} \right)^{w} \frac{p^{a+w}}{\rho^w (1+p/n)^{a+w}} \left( \frac{2x+y}{x} \right)^{y} \frac{y}{2x+y}
\]

\[
= \left( \frac{1}{1+0} \right) \frac{1}{\alpha+w} \left( \frac{2w+a}{w} \right)^{a} \frac{a}{2w+a}
\]

\[
= \frac{2w+a}{w} \frac{a}{2w+a} \left( \frac{1}{1+0} \right) \frac{0}{1+0} \frac{a}{2w+a}
\]
\[ P(w) = \frac{\binom{2w+a-1}{w} \cdot \binom{a+w}{p} \cdot \binom{w}{q}}{\binom{w}{1+w}}. \]

\[ p = \frac{\theta}{1+\theta}, \quad q = \frac{1-p}{1+\theta} = \frac{1}{1+\theta} \]

\[ P(w) \text{ is } w^{th} \text{ term in the expansion } \left(1 - \frac{|p-q|}{2q}\right)^a. \]

\[ \sum_{\lambda=0}^{\infty} p(w) = \frac{\langle 1 - \frac{\theta}{1+\theta} \rangle^a}{2 \left(\frac{1}{1+\theta}\right)}. \]

\[ = \frac{\left[1+\theta - |\theta - 1|\right]^a}{2} = \left[\min\left(\theta, 1\right)\right]^a. \]

\[ = \frac{(1+\theta) + (\theta-1)}{2} = 2 \quad \text{if } \theta < 1 \Rightarrow \theta-1 < 0 \]
\[ = \frac{(1+\theta) - (\theta-1)}{2} = 2/2 = 1. \]
Therefore $P(\theta) = 0$ if $0 < \theta < 1 = 1$ if $0 < 1$

If the relative removal rate per susceptible $\theta$ is greater than or equal to unity, there is no true epidemic, while if it is less than unity, a true epidemic can occur with probability $1 - \theta$.

### 5.2.2 Whittle's threshold theorem:

For an epidemic with intensity $i$, we have

$$[\min(\theta, 1)] < \pi_i < [\min\left(\frac{\theta}{\theta - 1}, 1\right)]$$

**Proof:**

Let us consider an epidemic

$\{X(t), Y(t), Z(t); \lambda, \mu, t > 0\}$ for which the conditional probability of a new infection in $(t, t + h)$ is $\lambda y(t) h + o(h)$ and the same for a removal is $\mu y(t) h + o(h)$.

Then the epidemic with intensity $\lambda = \beta n (1 - i)$

$\mu = \gamma$ and it is slower than the epidemic with $\lambda = \beta n (1 - i)$ and $\mu = \gamma$.

Let $P(w, \lambda, \mu)$ denotes the probability that $W = w$

$$P(w, \lambda, \mu) = \frac{\lambda^{w+a-1}}{\lambda^{w+a}} \frac{\mu^{w+a}}{\mu^{w+a}} \frac{1}{1 + \theta} \left(\frac{1}{1 + \theta}\right)^{\lambda}$$

$$= \frac{\lambda^{w+a-1}}{\lambda^{w+a}} \frac{\mu^{w+a}}{\mu^{w+a}} \frac{1}{1 + \theta} \left(\frac{1}{1 + \theta}\right)^{\lambda}$$

$$= \frac{\lambda^{w+a-1}}{\lambda^{w+a}} \frac{\mu^{w+a}}{\mu^{w+a}} \frac{1}{1 + \theta} \left(\frac{1}{1 + \theta}\right)^{\lambda}$$

$$= \frac{\lambda^{w+a-1}}{\lambda^{w+a}} \frac{\mu^{w+a}}{\mu^{w+a}} \frac{1}{1 + \theta} \left(\frac{1}{1 + \theta}\right)^{\lambda}$$

$$= \frac{\lambda^{w+a-1}}{\lambda^{w+a}} \frac{\mu^{w+a}}{\mu^{w+a}} \frac{1}{1 + \theta} \left(\frac{1}{1 + \theta}\right)^{\lambda}$$
\[
\begin{align*}
\int_{w+1}^{2w+a-1} & \quad a \left( \frac{\mu/\lambda}{1 + \mu/\lambda} \right)^{w} \left( \frac{1}{1 + \mu/\lambda} \right)^{l} \\
\int_{w}^{w+a} & \quad a \left( \frac{\mu/\lambda}{\lambda + \mu} \right)^{w} \left( \frac{1}{\lambda + \mu} \right)^{l} \\
\int_{w+1}^{2w+a-1} & \quad a \left( \frac{\mu/\lambda}{\lambda + \mu} \right)^{w} \left( \frac{1}{\lambda + \mu} \right)^{l}
\end{align*}
\]

\[w = 0, 1, 2, \ldots, n - 1\]

\[
\begin{align*}
P(n, \lambda, \mu) &= 1 - \sum_{w=0}^{n-1} P(w, \lambda, \mu) \\
\sum_{w=0}^{\infty} P(w, \lambda, \mu) &= \left( \min \left( \frac{\lambda}{\mu}, 1 \right) \right)^n \text{ and}
\end{align*}
\]

comparing the epidemic with intensity \(i\)
\[
\rho = \frac{\mu}{\lambda}, \quad \pi_i = \sum_{w=0}^{\infty} P(w, \lambda, \mu)
\]

(i) \(\rho < n (1-i) \left( \frac{\rho}{n} \right) \rho < \rho_i < \left( \frac{\rho}{n} \right)^i\)

(ii) \(n(1-i) < \rho < n, \left( \frac{\rho}{n} \right)^i \leq \pi_i < 1\)

(iii) \(n_i \leq \rho, \pi_i = 1\).

Therefore, \(\rho \geq n\), there is zero probability of an epidemic exceeding any pre-assigned intensity \(i\).

While if \(\rho < n\), the probability of an epidemic is approximately \(1 - \left( \frac{\rho}{n} \right)^i\) for small \(i\).

Returning to the fast and slow processes with \(\lambda = \beta n\) and \(\lambda = \beta n (1-i)\) respectively.
\[ \sum P_w (\beta n) \leq \pi_i \leq \sum P_w (\beta n - \beta n) \]

For sufficiently large \( n \).

\[ \min [\min (\frac{\rho}{n}, 1)), 1] \leq \pi_i \leq \min [\frac{\rho}{n (1 - i)}, 1] \]

where \( \rho = \frac{\mu}{\lambda} \)

Slower process \( \frac{\mu}{\lambda} = \gamma / \beta n (1 - i) = \rho / n (1 - i) \)

Faster process \( \frac{\mu}{\lambda} = \gamma / \beta n = \rho / n \)

Three main cases follow from (5.2.1)

(i) \( \rho < n (1 - i), \) \( \frac{\rho}{n} \leq \pi_i \leq \frac{\rho}{n (1 - i)} \)

(ii) \( n (1 - i) \leq \rho < n, \) \( \frac{\rho}{n} \leq \pi_i \leq 1 \)

(iii) \( n < \rho, \pi_i = 1. \)

Application of Martingale theory to some epidemic models:

Picard [30] gives some very simple applications of martingales to epidemics. The results are all connected with stopping times \( T \) and include the expression of the joint generating function Laplace transform of \( X_T \),

\[ \mathcal{L} \left( e^{Tu} X_T \right) = \int_0^T X_u Y_u du \]

and \( \int_0^T Y_u du \) and simple relations between moments of these three variables. Several relations between different types of epidemics are derived at the end.

5.3.1 Stopping Time:-

Let \((\mathcal{F}_n, \mathbb{P})\) be a probability space and \([ \mathcal{F}_n, n \geq 1] \) an increasing sequence of
sub σ algebras of : smash C 1 smash C 2 smash C 3 : A measurable function T = T(w)

taking values 1, 2, ... is called a stopping time relative to {T_n} if {T = j} e T_j, j=1,2,

Downtons classical model:-

Let us consider Downton's classical model (ie) a time homogeneous two type birth and
depth process (X_t, Y_t) t ≥ 0 such that P[x_{t+At} = r, Y_{t+At} = s / x_t = i, Y_t = j]

= α (Δ t + 0(Δ t) if r = i-1, s = j+1
= α (1-π)j Δ t + 0(Δ t) if r = i-1, s = j
= βj Δ t + 0(Δ t) if r = i, s = j-1
= 1 - (α_i + β_j) Δ t + 0(Δ t) if r = i, s = j
= 0(Δ t) in all other cases.

where α > 0, β > 0, 0 < π < 1 are parameters and x_0, y_0 the initial data.

(X_t, Y_t), t ≥ 0 will have its values in

D = {(i, j) ∈ N^2 / 0 ≤ i+j ≤ x_0 + y_0}

where X_t - the number of susceptibles at time t and

Y_t - the number of carriers at the same time.

Any susceptible may be removed or changed into carrier while a carrier may be removed

only.

5.3.1 Theorem:-

is the σ field generated by x_u, y_u for 0 ≤ u ≤ t.

put V_t = a(x_t, Y_t) e^{-it}

Z(t) = \int_0^t h(x_u, y_u) \, du
where $a$ and $h$ are functions from $D$ into $\mathbb{R}$ then prove that $(V_n, \mathcal{F}_t^N), \ t \geq 0$ is a martingale.

**Proof:-**

To prove that

(i) $E\{|a(x, y) e^{-zt}|\} < \infty$

(ii) $E[a(x, y) e^{-zt} \mathbb{1}_{\{h(x, y) \leq t\}}] < \infty$, $0 \leq t \leq 1$.

Since $0 \leq a(x, y) < \max a(i, j)$

Therefore $(x, y)$ will have its values in $D$.

Therefore (i) is true.

To prove (ii)

Let us take $m(t) = E_{w_0}[a(x, y) e^{-zt}]$ for $t \leq 1$

$\mathcal{F}_{w_0} = E_{w_0} E_t$

For $\Delta t > 0$,

$m(t + \Delta t) = E_{w_0} \left[ a \left( X_{t+\Delta t}, Y_{t+\Delta t} \right) e^{-z(t+\Delta t)} \right]$  

Since $D$ is finite and $E_{w_0} = E_{w_0} E_t$

$m(t + \Delta t) = E_{w_0} \left[ a \left( X_t, Y_t \right) e^{-zt} \mathbb{1}_{\{h(x, y) \leq t\}} \right] e^{-z\Delta t}$

$+ E_{w_0} \left[ a \left( X_t, Y_t \right) \frac{\beta Y_t \Delta t}{(1-h)} e^{-z\Delta t} \right] e^{-z\Delta t}$

$+ E_{w_0} \left[ a \left( X_t, Y_t \right) \left( 1 - \alpha X_t Y_t + \beta Y_t \Delta t \right) e^{-z\Delta t} \right] e^{-z\Delta t} [l_h(x_t, y_t)] \Delta t + o(\Delta t)$
Using Lebesgue's theorem,

\[ m'(t) = E_{\omega} \left[ a \left( X_t, Y_t \right) - a \left( X_t, Y_t^+ \right) \right] \tilde{m} \alpha X_t Y_t \\
+ \left[ a \left( X_t, Y_t^+ \right) - a \left( X_t, Y_t^{++} \right) \right] (1-\tilde{m}) \tilde{m} X_t Y_t \\
+ \left[ a \left( X_t, Y_t^{++} \right) - a \left( X_t, Y_t^{+++} \right) \right] \tilde{m} Y_t \\
- a \left( X_t, Y_t \right) h \left( X_t, Y_t \right) \tilde{e} Z_t \]

\[ m, \] being the derivative on the right of \( m. \)

when \( \Delta t < 0, \) put \( \Delta t' = -\Delta t, \)

\[ t' = t + \Delta t \]

and proceed in the same way using

\[ \frac{m(t+\Delta t) - m(t)}{\Delta t} = \frac{m(t' + \Delta t') - m(t')}{\Delta t'} \]

for \( \Delta t' > 0 \)

As a sample path of the process is a sample function which is continuous in \( t. \) We obtain for the derivative on the left \( m_{-}, \) just the expression we have got for the one on the right.

So \( m'(t) \) exists and if we choose \( a \) and \( h \) such that

\[ [a(i-1,j+1) - a(i,j)] \pi \alpha ij + [a(i-1,j) - a(i,j)] (1-\pi) \alpha ij \]
\[ [a(i,j-1) - a(i,j)] \mathbf{B}_j = a(i,j) \mathbf{h}(i,j) \quad \Rightarrow (5.3.1) \]

\[(i,j) \in \mathcal{D}. \]

\[ a(-1,j) \text{ and } a(i,-1) \text{ having any real value.} \]

Therefore we have \( m'(t) = 0. \)

Therefore \( m(t) = m(t_0). \)

Therefore \( E_{t_0} (a(x_1, y_1) e^{-s}) = E_{t_0} (a(x_1, y_1) e^{-s}) \]

Thus we proved result (ii).

Therefore \( \{V_n, \mathcal{F}_t\} \) is a martingale.

To find the solution of (5.3.1)

Put \( h(i,j) = (A + B)j \)

\[ a(i,j) = c_i \lambda^j. \]

\( A, B, \lambda \) being arbitrary positive constants.

Therefore we have

\[ (c_i, \lambda^{i-1} - c_{i-1}^\lambda) \pi a_{ij} + [c_i, \lambda^j - c_i^\lambda \lambda^j] (1-\pi) a_{ij} + \]

\[ (C_i, \lambda^j - c_i^\lambda \lambda^j) \beta^j = c_i \lambda^{j} (A_i + B)j \]

\[ (\lambda, c_{i-1} - c_i^\lambda \lambda^j \pi) j + (c_i - c_{i+1}) \lambda^j (1-\pi) \alpha_{ij} + c_i \lambda^j (1/\lambda - 1) \beta j \]

Put \( j \to 0 \) we get an identity

For \( j \to 0 \)

\[ (\lambda, c_{i-1} - c_i^\lambda \lambda^j \pi) \alpha_i + (C_i - c_{i+1}) (1-\pi) \alpha_i + c_i \lambda^j (1/\lambda - 1) \beta = c_i (A_i + B) \]
\[ C_i [A^i - B/\lambda + \alpha^i + B - B] = r \left( C_{i-1} \left[ \alpha^i \left( \frac{1}{1-\pi} + \lambda \right) \right] \right) \rightarrow (5.3.2) \]

Put \( \lambda = \lambda_n \) in (5.3.2)

\[ (A + \alpha)^{(i-n)} c_i = c_{(i-n)} \left( 1 - \pi + \lambda \right) \rightarrow (5.3.3) \]

Take \( c_\infty = c_i = c_{n-i} = 0 \)

\[ C_\eta = \frac{\alpha}{A + \alpha} \]

Therefore (5.3.3) becomes

\[ C = \frac{\alpha}{A + \alpha} \]

\[ C_{i-1} = \frac{\alpha}{A + \alpha} \]

Put (5) \[ i = i(i-1) \ldots (n-1) \]

The general form of \( C_i \) is

\[ C_i = \frac{1}{\lambda} \left( \frac{\alpha}{A + \alpha} \right)^i \]

\[ = i(i-1) \ldots (n-1) \left( \frac{\alpha}{A + \alpha} \right)^i \]

\[ = (i)_\eta \left( \frac{\alpha}{A + \alpha} \right)^i \]

5.3.2 Theorem:-

For \( A \) and \( B \) positive constants, \( n \in \mathbb{N} \)

\[ \lambda_n = \beta (A + \alpha)^n + B + B \] and

\[ V_{t, n} = (\lambda_n)^{(i)} \frac{1}{(1 - \pi + \lambda \eta)} \]

then \( V_{t, n} \) is a martingale Also sup \( |V_{t, n}| < \infty \)

\( \rightarrow (5.3.4) \)
Proof:

We have proved that
\[ C_i := (i) \left[ \sum_{n=1}^{\infty} \frac{1}{(1\pi + \pi \lambda n)} \right] \in \mathbb{N} \]
and
\[ Y_{i} = \sum_{n=1}^{\infty} h(x, y) - Z_i \]

\[ Z_i = \int_{0}^{1} h(x, y) \, du. \]

Take \( a_n(x, y) = Cx \lambda n \),

\[ h(x, y) = (Ax + By) y. \]

Therefore
\[ V_{i} = \sum_{n=1}^{\infty} \frac{1}{(1\pi + \pi \lambda n)} \lambda^n e^{-\int_{0}^{r} (Ax + By) y \, du} \]

is a martingale.

Also
\[ \sup_{V_{i}} \| V_{i} \| < + \infty. \]

The Stopping times and the main relation:

Let
\[ T_0 = \inf \{ t : y = r \} \quad \text{for} \quad 0 \leq r < y_0 \]
\[ T_1 = \inf \{ t : x_t + y = r \} \quad \text{for} \quad x_0 \leq r < x_0 + y_0 \]
\[ T_2 = \inf \{ t : 2x_t + y = r \} \quad \text{for} \quad 2x_0 \leq r < 2x_0 + y_0 \]

\[ \alpha \Gamma_{e} = \inf \{ t : e x_t + y = 1 \} \quad \text{for} \quad e x_0 \leq r < e x_0 + y_0 \]

with \( e = 0, 1, 2 \).

Now the classical theorem on stopping times for martingales
\[ \mathbb{E} \left[ \sum_{r \geq e} Y_{r} : \gamma = r \right] = V_{0} \]

\[ \mathbb{E} \left[ \sum_{r \geq e} X_{r} : \gamma = r \right] = V_{0} \]
5.3.3. Theorem:

For \( T_{\xi} = \text{Inf} \{ t : eX_t + Y_t - r < r < eX_t + y_o \} \),

\[
\alpha = 0, 1, 2,
\]

\[
E \left[ (X_{\xi})_n \right] = \left( \frac{1}{\Lambda + \alpha} \right) \int_{\alpha}^{\gamma} (\Lambda \xi + B)y_u\,du
\]

\[
= (X_0) \left( \frac{1}{\Lambda + \alpha} \right) \left[ (1 - \pi + \pi \lambda_n) \right] (\lambda_n) \gamma_o
\]

Where \( \frac{1}{\alpha} \left( \Lambda + \alpha \right) = B + \beta \)

Proof:

\[
E \left[ V_{T_{\xi}, \gamma} \right] = V_0, \gamma
\]

We have

\[
E \left[ X_{\xi} \right] \left( \frac{1}{\Lambda + \alpha} \right) \left( (1 - \pi + \pi \lambda_n) \right) (\lambda_n) \gamma_o
\]

Multiply by \( \lambda_n \)

\[
E \left[ X_{\xi} \right] \left( \frac{1}{\Lambda + \alpha} \right) \left( (1 - \pi + \pi \lambda_n) \lambda_n \right) (\lambda_n) \gamma_o
\]

using \( eX_t + y_o = r \)

The joint distribution of \( X_{\xi} \), \( \int_0^{\xi} x_u \,du \), \( \Delta m \), \( \int_0^{\xi} y_u \,du \).

Let \( U = (u_n) \in \mathbb{R} \) be a sequence of real numbers. For any \( n \in \mathbb{N} \), \( Q_{\gamma} (x ; U) \) will be the unique polynomial in \( x \) of degree \( n \) satisfying for every \( i \in \mathbb{N} \)
\( Q_{n}(u_i, U) = \delta_{n'} \) where \( \delta_{n'} \) - kronecker's function.

\( Q_{n}(x, U) \) can be expressed in the following way.

(i) \( Q_{n}(x, U) = \delta_{n'} = 1 \)

(ii) \( Q_{n}(x, U) = \int_{U_{n-1}} \sum_{\xi_{n}} d\xi_{n} \cdot \cdot \cdot \sum_{\xi_{1}} d\xi_{1} \)

\( Q_{n}(x, U) \) only depends on \( u_0, u_1, \ldots, u_{n-1} \) and not on the whole sequence \( U \).

\[
Q_{n}(x, U) = \int_{U_{0}}^{x} d\xi_{0} (\xi_{0} - u_{1}) = \frac{\xi_{0}^2}{2} - u_{1} \xi_{0} + \sum_{u_{1}}^{x} d\xi_{1}
\]

\[Q_{n}(x) = x^{n'/2} + u_{0}^{n'/2} - u_{0} x = \frac{(x - u_{0})^2}{2}\]

Similarly \( Q_{n}(x) = \frac{(x - u_{0})^n}{n!} \)

5.3.1 Property:

If \( R \) is a polynomial of degree \( n \) then we have Abel's expansion

\[ R(x) = \sum_{j=0}^{n} R^{j}(u_j) Q_{j}(x) \]

Proof:-

\[ R^{j}(u_j) = \sum_{k=0}^{n} \delta_{k,j} b_{k} Q_{k}(u_j) = \sum_{k=0}^{n} b_{k} \delta_{k,j} = b_{j} \]

Therefore \( R(x) = \sum_{j=0}^{n} R^{j}(u_j) Q_{j}(x) \).

Note - if for every \( u_{n'} = u_{0} \) then
(i) \( R(x) = \sum_{j=0}^{n} R^j(u_j) Q^j(x) \)

\[ = R^0(u_0) Q_0 + R^1(u_1) Q_1(x) + R^2(u_2) Q_2(x) + \ldots + R^n(u_n) Q_n(x) \]

\[ = b_0 + R^1(u_1) \frac{(x-u_0)}{1} + R^2(u_2) \frac{(x-u_0)^2}{2} + \ldots + R^n(u_n) \frac{(x-u_0)^n}{n!} \]

Therefore, \( R(x) \) is a Taylor's classical expansion.

(ii) \( f(v) = \sum_{v \in \mathcal{V}} e^{\gamma \sum_{v \in \mathcal{V}}} \)

\[ u_n = \frac{\alpha}{A + \alpha} \frac{1 - \pi + \pi \lambda_n}{\zeta} \]

\[ \chi_{i_E} - \int_{0}^{T_E} (Ax + B) y_{\Lambda} \, du \]

\[ f'(u_n) = \sum_{v \in \mathcal{V}} e^{\gamma \sum_{v \in \mathcal{V}}} \]

\[ f''(u_n) = \sum_{v \in \mathcal{V}} e^{\gamma \sum_{v \in \mathcal{V}}} \]

\[ f''(u_n) = \sum_{v \in \mathcal{V}} e^{\gamma \sum_{v \in \mathcal{V}}} \]

\[ E(\chi_{i_E}) = \begin{pmatrix} \alpha & (1 - \pi + \pi \lambda_n) \frac{1}{\zeta} \\ A + \alpha & \end{pmatrix} \int_{0}^{T_E} (Ax + B) y_{\Lambda} \, du \]

\[ = (x_{\varnothing}) \begin{pmatrix} \alpha & (1 - \pi + \pi \lambda_n) \frac{1}{\zeta} \\ A + \alpha & \end{pmatrix} \]

Therefore, \( \sum_{n=0}^{\infty} f_{\Lambda}^n (u_n) = (x_{\varnothing}) \sum_{n=0}^{\infty} \lambda_n \]

5.3.4 Theorem:

\[ u_n = \frac{\alpha}{A + \alpha} \frac{1 - \pi + \pi \lambda_n}{\zeta} \]
\[ \lambda_{\eta} = \frac{\beta}{(A + \beta)^{n+B+\beta}} \]

\[ E[V] \exp \left\{ - \int_{0}^{\infty} (A_{x} + B) y_{x} \, du \right\} = \sum_{n=0}^{\infty} \left( x_{0} + \text{exp} + y_{0} - t \right) \lambda_{n} u_{n} Q_{n}(v) \]

\[ \rightarrow (3.4.1) \]

**Proof:**

\( f \) is a polynomial of degree \( x_{0} \)

\[ R(x) = \sum R(x_{0}) Q_{x}(x) \]

\[ R(x) = \sum_{n=0}^{x_{0}} \left( x_{0} + \text{exp} + y_{0} - t \right) \lambda_{n} u_{n} Q_{n}(v) \]

\[ \rightarrow (5.3.5) \]

**Simple relations between moments:**

We have

\[ E[V] \exp \left\{ - \int_{0}^{\infty} (A_{x} + B) y_{x} \, du \right\} = \sum_{n=0}^{x_{0}} \left( x_{0} + \text{exp} + y_{0} - t \right) \lambda_{n} u_{n} Q_{n}(v) \]

\[ Q_{\eta} (u_{0}) = Q_{\eta} (u_{0} ; \eta) (u_{n} n \geq 0) = \delta_{u_{0}} \]

put \( v = u_{0} \). \( A = 0 \) and \( B = \eta \) in the above equation.

We have

\[ E[u_{0} \exp \left\{ - \int_{0}^{\infty} (A_{x} \eta) y_{x} \, du \right\}] = \left( x_{0} \right) \lambda_{0} u_{0} Q_{\eta}(u_{0}) \]
\[ Q(\omega) = 1 \quad Q_1(\omega) = \ldots = Q_n(\omega) = 0. \]

\[ Q_0(\omega) = 1 \quad Q_1(\omega) = \ldots = Q_n(\omega) = 0. \]

Where

\[ \lambda_0 = \frac{\beta}{\beta + \eta} \]

\[ \omega = \frac{\alpha}{\alpha + \phi} \]

\[ u = \frac{\alpha}{\alpha + \theta} \left( \begin{array}{c} 1 - \pi + \pi \lambda \alpha \lambda \\ \lambda_0 \end{array} \right) \]

\[ \frac{\alpha}{\alpha + \theta} \left[ \frac{1 - \pi + \pi}{1 + \eta / \beta} \right] = \frac{\alpha}{\alpha + 0} \left[ \frac{(1 - \pi)(1 + \eta / \beta) + \pi(1 + \eta / \beta)}{\epsilon - 1} \right] \]

are function of \( \theta \) and \( \eta \).

Differentiate (5.3.5) with respect to \( \theta \)

and substitute \( \theta = \eta = 0 \) in this result, denote \( (g)_0 = (g)_0^{\theta = \eta = 0} \)

\[ \left( \lambda_0 \right)_0 = \left( \omega \right)_0 = 1 \quad \left( \frac{\partial \lambda_0}{\partial \theta} \right)_0 = 0 \quad \left( \frac{\partial \omega}{\partial \theta} \right)_0 = -1/\alpha \]
We have

\[
E \left[ \frac{\partial u_o}{\partial \theta} \right] - \int_0^{T_o} x_u y_u \, du = \frac{\partial H_o}{\partial \theta} - \chi o \left( \frac{\partial u_o}{\partial \theta} \right)_o
\]

\[
- X_{T_o}
\]

\[
E \left[ \frac{\partial u_o}{\partial \theta} \right] - \int_0^{T_o} x_u y_u \, du = \frac{x o}{\alpha}
\]

\[
E[ \int_0^{T_o} x_u y_u \, du] = \frac{x o}{\alpha} - \frac{1}{\alpha} E[X_{T_o}]
\]

\[
= \frac{x - E[X_{T_o}]}{\alpha} \quad (5.3.7)
\]

Again differentiate partially (5.3.5) with respect to, we get

\[
E \left[ \frac{\partial u_o}{\partial \eta} \right] - \int_0^{T_o} y_u \, du = \left( \frac{\partial H_o}{\partial \eta} \right)_o
\]

\[
\left( \frac{\partial H_o}{\partial \eta} \right)_o = H_o \left[ \frac{\partial \lambda o}{\partial \eta} \right] + \frac{\partial u o}{\partial \eta} + \frac{\partial u o}{\partial \eta} \quad (5.3.8)
\]

Therefore

\[
E[ \int_0^{T_o} y_u \, du] = \frac{\pi \chi o + y_o - (\pi - 1)U(\chi T_o)}{\beta} \quad \rightarrow (5.3.8)
\]
Thus we have

\[ e \langle X \rangle + e \langle \eta \rangle = r \quad \text{and} \]

\[ \frac{e \int_0^{\tau \prod} y \, du}{\beta} = (5.3.9) \]

R.H.S of (5.3.6) and (5.3.9) have simple interpretations because \( E [ x - x_c ] (1 - \pi) \) is the expected number of those susceptibles that have been removed and \( \pi E (x - x_c) \) the expected number of those that have been changed into carriers.

**5.3.5 Theorem:**

\[ E \left[ \int_0^{\tau \prod} x \, du \right] \]

is the expected number of susceptibles involved in the epidemic between times 0 and \( T \) (removed or changed into carriers) and \( E \left[ \int_0^{\tau \prod} y \, du \right] \) the expected number of detected and eliminated carriers during the same period.

**Proof:**

\((5.3.7) \) and \((5.3.9) = \to \) result.

Equation (5.3.7) and (5.3.9) are true if we change \( T \) into

\[ T = \inf \{ t : X_t = 0 \ or \ x + y = 0 \} \]

\[ T = \inf \{ t : \varepsilon x + y = r \} \]

\[ T < T \]

\((5.3.8) \Rightarrow \)

\[ E^* \left[ \int_0^{\tau \prod} y \, du \right] = \frac{\pi [ X_c - E^* (X_c) + \varepsilon \gamma \prod (Y_c)]}{\beta} \]

\( \Rightarrow (5.1.10) \)

where \( E^* = E \left[ .... / T, X_c, Y_c \right] \)
Since \( E = EE^* \)

\[
E \int \frac{\gamma_{T_e} \, du}{\beta} = \pi [x_0 - E(T_e \gamma^*)] + y_0 - E[Y_e \gamma^*] \quad \rightarrow (5.3.11)
\]

Similarly by we can prove that (5.3.7) is true with \( T_e^\dagger \) for \( T \)

5.3.8 Theorem:

If \( T_e^\dagger \) is substituted for \( T \) the above theorem is true.

Proof:

\[
(5.3.9) + (5.3.10) \quad \Rightarrow (5.3.11)
\]

Differentiate (5.3.5) with respect to \( \theta \) and \( \eta \) twice

\[
\frac{\partial \lambda_o}{\partial \theta} = \frac{\partial \lambda}{\partial \eta} \quad \Rightarrow (5.3.12)
\]

\[
\frac{\partial \lambda_o}{\partial \theta} = \frac{\partial \lambda}{\partial \eta} \quad \Rightarrow (5.3.13)
\]

\[
\frac{\partial^2 \lambda_o}{\partial \theta^2} = \frac{\partial^2 \lambda}{\partial \eta^2} \quad \Rightarrow (5.3.14)
\]

\[
\frac{\partial \Pi_o}{\partial \eta} = \frac{(\epsilon x_0 + y_0 - \gamma) (-1/\beta) + x_0 (\epsilon - \pi)}{\beta} \quad \Rightarrow (5.3.15)
\]
Differentiate (5.3.5) twice with respect to \( \eta \) we get

\[
\begin{align*}
\mathbb{E} \left\{ x \gamma_c \left( \frac{\partial u_0}{\partial \eta} \right)^2 + x \gamma_c \frac{\partial^2 u_0}{\partial \eta^2} - 2x \gamma_c \frac{\partial u_0}{\partial \eta} \int_0^{\gamma_c} y_u \, du + \left( \int_0^{\gamma_c} y_u \, du \right)^3 \right\} \\
= \frac{\partial^2 H_0}{\partial \eta^2} \\
= \frac{1}{\beta^2} \left[ (\pi \lambda_\beta + y_\beta - r)^2 + y_\beta - r + (2\pi - \pi^2) \lambda_\beta \right] \\
\rightarrow (5.3.18)
\end{align*}
\]

Again Differentiate (5.3.5) with respect to \( \eta \) and \( \Theta \), we obtain
Differentiate (5.3.5) K times with respect to $\theta$ using Leibnitz's formula, we have

$$\sum_{i=0}^{k} \binom{k}{i} E \left[ \frac{\partial^i u_0}{\partial \theta^i} \right] \left( -\frac{\partial^i X_u Y_u du}{\partial \theta^i} \right) \left( \int_0^\tau_c Y_u du \right) = \frac{\partial^k u_0 x_0}{\partial \theta^k} \left( x_0 \right)$$

but the expansion

$$\left( \frac{u_0}{\theta} \right) \eta_o = \frac{\alpha}{\alpha + \theta} \left( 1 - \pi \right) + \pi$$

$$= \frac{1}{\left( 1 + \theta/\alpha \right)} x_0 = \left( 1 + \theta/\alpha \right)^{-x_0}$$

$$\sum_{k \geq 0} \frac{(-x_0)_k}{\alpha^k} \left( \frac{0^k}{k} \right)$$

$$\left( 1 + \theta/\alpha \right) = 1 + (-x_0)\theta/\alpha + (-x_0) (-x_0-1) (\theta/\alpha)^2 + \ldots$$

gives the value of

$$\frac{\partial^k u_0 x_0}{\partial \theta^k} \left( x_0 \right)$$
Therefore
\[ \sum_{i=0}^{k} \binom{k}{i} \left[ (X_{T_{\alpha}})_{i} - \int_{0}^{T_{\alpha}} X_{u} Y_{u} \, du \right]^{k-i} = \frac{-(X_{0})_{k}}{\alpha^{k}} \]

\[ \sum_{i=0}^{k} \binom{k}{i} \left[ \frac{X_{T_{\alpha}} + i-1}{\alpha^{i}} \left( \int_{0}^{T_{\alpha}} X_{u} Y_{u} \, du \right) \right]^{k-i} = \frac{(X_{0} + k-1)_{k}}{\alpha^{k}} \]

and \( k \leq 2 \), hence  

\[
\begin{align*}
L_{i} &= \left[ \frac{X_{T_{\alpha}}^{2}}{\alpha^{2}} + \int_{0}^{T_{\alpha}} X_{u} Y_{u} \, du \right]^{2} + \frac{2}{\alpha} \int_{0}^{T_{\alpha}} X_{u} Y_{u} \, du \\
&= \frac{(X_{0})_{k} (X_{0}+1)}{\alpha^{2}} \quad \text{(5.3.20)}
\end{align*}
\]

5.3.7 Theorem:-

The joint second order moment of \( X_{T_{\alpha}} \),

\[
\int_{0}^{T_{\alpha}} x_{u} y_{u} \, du \quad \text{and} \quad \int_{0}^{T_{\alpha}} y_{u} \, du
\]
satisfy (5.3.18), (5.3.19) and (5.3.20)

Applications:

\[
\frac{1}{\alpha} = \mathbb{E} \left[ \int_{0}^{T_{\alpha}} x_{u} y_{u} \, du \right]
\]

\[
\frac{E[X_{0} - X_{T_{\alpha}}]}{\beta} = \mathbb{E} \left[ \int_{0}^{T_{\alpha}} y_{u} \, du \right]
\]

\[
\frac{\overline{\tau}_{1} X_{0} + \gamma_{0} - \tau + (6 - \overline{\tau}_{1}) L_{i}}{\alpha^{2}}
\]
are estimators for $1/\alpha$ and $1/\beta$ respectively when $\hat{\eta}$ is already known.

Relations between carrier borne and general epidemic models:

(i) $\eta - \hat{\eta} < 0$

Since $\eta$ it means that $\eta = 0$, $\hat{\eta} > 0$

$$\eta \eta = \frac{\alpha}{\beta} \left( 1 - \hat{\eta} + \hat{\eta} \lambda \right) \left( \frac{\sin \alpha}{\cos \alpha} \right)$$

(ii) $\eta - \hat{\eta} > 0$

Therefore $\eta \eta = \frac{\alpha}{\beta} \left( 1 - \eta \right) \eta + \alpha \left( \frac{\beta + (1 - \eta) \beta}{\beta (\eta + \alpha)} \right)$

These three cases may be summed up in

$$\eta \eta = \frac{\alpha}{\beta} \left( \eta - \hat{\eta} \right) \eta + \alpha \left( \frac{\beta + (\eta - \hat{\eta}) \beta}{\beta (\eta + \alpha)} \right) \delta \eta \eta = 0$$

$\eta \eta$ and $\delta \eta \eta$ do not depend on $\eta$ and $\hat{\eta}$ but only on their difference.
5.3.8 Theorem:-

For > 0, \( G(\xi, 1, x, y) = G(\xi - 1, 0, x, y + x) \)

Proof:-

We have

\[
\text{E} \left[ \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} x^\xi \exp \left( -\sum_{n=0}^{\infty} \lambda_n (x+\lambda_n \beta) \right) \beta^n du \right] = \sum_{n=0}^{\infty} \lambda_n (x+\lambda_n \beta) \beta^n \]

In this result start with \( n = 1, \xi > 0 \)

If we change \( n = n - 1 \)

\( \xi = \xi - 1 \)

\( y = y + x \)

R.H.S of above not vary.

\( G(\xi, n, x, y) \) the joint distribution of \( X_0, X_1, \ldots, X_n \) and

\[
\int_{-\infty}^{\infty} y_n du \text{ with parameter } \xi, \text{ initial data } x, y.
\]

A few additional properties of the polynomials \( Q_n \).

5.3.2 Property:-

For every \( a, b \in \mathbb{R} \),

\[
Q_n \left( (ax+b); (au_i + b)_{i \geq 0} \right) = a^n Q_n \left( x; (u_i)_{i \geq 0} \right)
\]

Proof:-

\[
R(x) = Q_n (ax+b; (au_i + b)_{i \geq 0}) \text{ is a polynomial } \rightarrow (5.3.28)
\]

in \( x \) of degree \( n \) such that,

\[
\frac{d^n}{du_i^n} R(u_i) = A^n Q_n (ax+b; (au_i + b)_{i \geq 0}) = \delta_n^n
\]
For every $j \in \mathbb{N}$,
$$0 \leq j \leq n, \quad Q_{\eta}^j(x; (u_i)i \geq 0) = Q_{n,j}(x, (u_i)i \geq 0) \quad \text{(5.3.29)}$$

$$Q_\eta(x, U) = 0$$

$$Q_\eta(x, U) = \int_{u_0}^{x} \int_{u_1}^{\xi_0} \int_{\xi_1}^{u_2} \ldots \int_{\xi_{n-1}}^{u_{n-1}} d\xi_0 \; d\xi_1 \ldots d\xi_{n-1}, \quad \text{for } n > 0$$

$$Q_\eta(x, U) = d\xi_0 \; d\xi_1 \ldots d\xi_{n-1}$$

$$Q \{x, (u) i > 0\} = \delta_{ij}$$

5.3.1 Corollary:-

For every $j, k \in \mathbb{N}$, $0 \leq k \leq j \leq n$, for every $a \in \mathbb{R}$

$$Q_\eta^j(ax, au) = a^n Q_{n-k}^j(x, (u_i)i \geq k) \quad \rightarrow \text{(5.3.30)}$$

$$Q_\eta^j(ax, au) = Q_\eta^{j-k}(ax, (au_i)i \geq k)$$

$$= a^n Q_{n-j-k}^j(x, (u_i)i \geq j)$$

$$= a^n Q_{n-k-j}(x, (u_i)i \geq k+j-k)$$
If \( U_n \) is a polynomial in \( n \) of degree 1 then

\[
Q_n(x) = \frac{\prod_{i=0}^{n-1} (x-u_i)^{n-i}}{\prod_{i=0}^{n} (n-i)} \quad \Rightarrow (5.3.31)
\]

**Proof:**

When \( n = 0 \)

\[
Q_0(x) = (x-u_0)^{-1}
\]

To show that for \( a > 0 \), R.H.S of property (5.3.2) (i.e) \( Q_n'(x) (u_i) i \geq 0 \) has all the properties of \( Q_n \).

\[
Q_n(x) = \frac{(x-u_0)^n}{\prod_{i=0}^{n} (n-i)}
\]

\[
Q_n'(x) = \frac{n(x-u_0)^{n-1}}{\prod_{i=0}^{n} (n-i)}
\]

\[
Q_n(x) = n(n-1) \ldots \ldots \ldots \frac{n-(n-1)(x-u_0)^{n-1}}{\prod_{i=0}^{n} (n-i)}
\]

(i.e) the \( n^\text{th} \) derivative of \( Q_n(x) = 1 \).

For \( 0 < j < n \), Put \( u_n = a_n + b \) and use Leibniz's formula to find \( j^\text{th} \) derivative of \( Q_n(x) \).

\[
\left[ \frac{\prod_{i=0}^{n-j} (x-u_i)^{n-i}}{\prod_{i=0}^{n} (n-i)} \right]_{x=a_n+b}^j = \frac{n-j}{n-j+1} \cdot \frac{n-(n-j)(x-u_0)^{n-1}}{\prod_{i=0}^{n} (n-i)}
\]
\[ P_{\alpha j}, V - (x - u_{\alpha})^{n-1} = \frac{1}{n!} \left[ \left( u_{j} - u_{\alpha} \right) + j \left( n-1 \right) \right] \]

\[ V = (n-1) (x - u_{\alpha}) \]

\[ = \frac{1}{n!} \left[ \left( u_{j} - u_{\alpha} \right) + j \left( n-1 \right) \left( u_{j} - u_{\alpha} \right) \right] \]

\[ = \frac{1}{n!} \left[ \left( u_{j} - u_{\alpha} \right) + j \left( n-1 \right) \left( u_{j} - u_{\alpha} \right) \right] \]

\[ \text{Proof:} \]

\[ R_{\alpha}^k (x) = \sum_{j=k}^{s} R_{\alpha j}^k (u_{j}) Q_{\alpha j}^k (x), \quad S = k, k+1, \ldots \]

5.3.5 Property: 

Let \( R_{\alpha} \) be a sequence of polynomials \( R_{\alpha} \) - being of degree \( s \), then \( Q_{\alpha j}^k, j = k, k+1, \ldots \) are well determined by the system.

\[ R_{\alpha}^k (x) = \sum_{j=0}^{n} R_{\alpha j}^k \left( u_{j} \right) Q_{\alpha j}^k (x). \]
\[ R_S(x) = \sum_{j=0}^{n} R_x^{(j)}(u_j) Q_j(x). \]

Differentiate \( k \) times,
\[ R_S^{(k)}(x) = \sum_{j=k}^{s} R_x^{(j)}(u_j) Q_j^{(k)}(x), \quad j = k, k+1 \ldots. \]

Since \( R_S^{(k)}(u_j) \neq 0 \), \( Q_j \) is determined for \( j = k, k+1 \ldots. \)

### 5.3.1 Applications:

Put \( \varepsilon = \tau = A = 0 \) in equation (5.3.4)
\[ E[V_{\varepsilon}] = \int_{0}^{\tau} \phi_0 (\Lambda x_\alpha v_\alpha du) + B \gamma_\alpha du] = \int_{0}^{\tau} \phi_0 (\Lambda x_\alpha v_\alpha du) + B \gamma_\alpha du] = \sum_{n=0}^{x_{o}} \sum_{\eta=\eta}^{x_{o} + \eta} Q_{\gamma}(v) \]
\[ E[V_{\varepsilon}] = \int_{0}^{\tau} \phi_0 (\Lambda x_\alpha v_\alpha du) + B \gamma_\alpha du] = \sum_{n=0}^{x_{o}} \sum_{\eta=\eta}^{x_{o} + \eta} Q_{\gamma}(v) \]

Differentiate with respect to \( k \) times and put \( v = 1 \), we have
\[ E\left[ \sum_{\eta=\eta}^{x_{o} + \eta} Q_{\gamma}(v) \right] \]

For calculating it, We need the values of \( Q_{\gamma}^{(k)}(1) \)

Put \( R_S(x) = (1-1/\pi + 1/\pi x) \), \( x = 1 \) \( \Rightarrow (5.3.33) \)

substitute \( R_x^{(k)}(x) = \sum_{j=k}^{s} R_x^{(j)}(u_j) Q_j^{(k)}(x), \quad s = k, k+1 \ldots \)

We get \( (s)_k 1/\pi^k = \sum_{j=k}^{s} (s)_{1/\pi^{j}} (1-1/\pi + 1/\pi u_j) \)

From (5.3.23), \( (1-1/\pi + 1/\pi) \) \( \Rightarrow \lambda_j \) if \( A = 0. \)
These relations determine \( \pi Q(1) \) but they do not depend on \( \pi \).

(b) Suppose \( \pi = 1 \) and put \( k = 0 \) in

\[
\sum_{j=0}^{s} \lambda_j \pi Q(1) = 1
\]

where \( s = 0, 1, 2, \ldots \) \( \rightarrow \) (5.3.35)

(c) To calculate \( P[x_{k_0} = k] \) we and \( Q^k_\gamma(0) \) Put

\[
R_\delta(x) = x^\delta, x = 0 \quad \text{in}
\]

\[
R_\delta^k(x) = \sum_{j=0}^{s} R_\delta^j (u_j) Q_\delta^j (x) \quad s = k, k+1, \ldots
\]

\[
R_\delta^k(x) = x^\delta([s/x]) = x^{\delta-s} \quad s = sx^{\delta-1}
\]

\[
\sum_{j=0}^{s} R_\delta^j (u_j) Q_\delta^j (0) \quad s = k, k+1, \ldots \quad \rightarrow \) (5.3.36)

Which determines \( Q^k_\gamma(0) \).

5.3.2 Application:-

Suppose \( \epsilon - \pi > 0 \), we proved that \( u_\gamma \) is a polynomial in \( n \) degree 1 and \( Q_\gamma \) are given by

\[
Q(x) = \frac{(x-u\omega) (x-u\omega)^{n-1}}{\gamma n} \quad \rightarrow \) (5.3.27)

Therefore Equation (5.3.4) becomes
For instance, \( p^{x \rightarrow \kappa} = \int_{0}^{t} \exp_{x}^{r-y} \left( x_{\omega}^{t} \right) \lambda_{n} \left( \lambda_{n} \right)^{n} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma} \).

\[
\begin{align*}
E[V] &= \left\{ \sum_{k=0}^{\infty} \left( \frac{\lambda_{n}}{\lambda_{n}} \right)^{k} \right\} \left( \frac{\lambda_{n}}{\lambda_{n}} \right)^{n} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma} \\
\end{align*}
\]

For instance,

\[
P[X_{k} = k] = \frac{u_{k} x \exp_{x}^{t-y} \left( x_{\omega}^{t} \right) \lambda_{n} \left( \lambda_{n} \right)^{n} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma}}{\sum_{k=0}^{\infty} \left( \frac{\lambda_{n}}{\lambda_{n}} \right)^{k} \left( \frac{\lambda_{n}}{\lambda_{n}} \right)^{n} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma}}
\]

Relations between Downton's model and the general epidemic \((e = 0)\)

Put \( A = 0 \) in equation (5.3.4) \( e = 0 \).

\[
E \left[ \left( X_{\omega}^{t} \right)^{k} \right] = \left( 0 \right)^{k} \pi \left( 0 \right)^{k} \left( \lambda_{n} \right)^{k} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma} \rightarrow (5.3.38)
\]

If \( \Lambda = \left( \lambda_{i} \right)^{i} \geq 0 \),

\[
Q \left( ax + b ; (n_{i} + b) \right) \geq 0 = \sum_{n=0}^{\infty} \left( \lambda_{n} \right)^{n} Q_{n}^{k} \left( \lambda_{n} \right)^{k} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma} \rightarrow (5.3.39)
\]

If we put \( j = i + n \) and invert the summations,

\[
\begin{align*}
\sum_{n=0}^{\infty} \left( \lambda_{n} \right)^{n} \left( V_{n} \right)^{\delta} \left( \Delta V_{n} \right)^{\gamma} \rightarrow (5.3.39)
\end{align*}
\]
There be the following result

\[ E \left[ \sum_{j=k}^{x_o} \left( \frac{x_o}{d} \right)^j y_{\alpha} \right] = \sum_{j=k}^{x_o} \left( \frac{x_o}{d} \right)^j \frac{1}{\pi} \int_{\alpha}^{\infty} y_{\alpha} du / x_o, y_o, r : \pi \]

\[ \Rightarrow (5.3.40) \]

Generalization of Daniel's formula:-

Daniels gave a formula expressing

\[ P[X_{t_o} = k / x_o, y_o, \rho \pi] \text{ with } P[X_{t_o} = 0 / x_o, y_o, \rho \pi] \text{ for the general epidemic} \]

(\( \rho - \beta \text{ and } \alpha = 1 \)) and Daniels extended it to Downton's model in the form

\[ \frac{1}{\pi'} = \frac{\lambda n(B, \beta)}{\rho a(1-\pi)} \]

Notations.

\[ \lambda n = \lambda n(B, \beta) \]

\[ \lambda n = \lambda n(B', \beta') \]

\[ u_{\alpha} = u_{\alpha}(B, \beta, \pi) \]

\[ u_{\alpha} = u_{\alpha}(B', \beta', \pi') \]

\[ E = E(\ldots, \alpha, \beta, \pi) \]

\[ E^t = E(\ldots, \alpha^t, \beta^t, \pi^t) \]

5.3.9 Theorem:-

For \( 0 \leq k' \leq k \leq x_o \)
\[ \Lambda^t = A, \quad B^t = B - a \cdot (A + \alpha)(k-k') \]

\[ B = B + a \cdot \pi^t = \pi B a - \alpha \pi \quad k = \begin{pmatrix} \alpha^t \\ \pi \\ \beta^t \end{pmatrix} \]

\[ x_\pi^t = x_o - (k-k') \quad y_\alpha^t = y_o + n - e \quad (k-k') \]

We have,

\[ E[(\chi_{\tau_c}) (kv)] \quad \mathbb{E}^t[(\chi_{\tau_c}) (kv)] \]

\[ E^t[(\chi_{\tau_c}) (kv)] \]

\[ \lambda_n = \lambda_n (B, \beta) \]

\[ = \frac{\beta}{n (k-k')} \quad (B', \beta') \]

\[ u_n = u_n (B, \beta', \pi) \quad \rightarrow (5.342) \]

\[ = K u_n (B', \beta', \pi) = K u_n (k-k') \]

Differentiate (5.3.4) k times with respect to \( v \) and substitute kv for \( v \) then

\[ E^t[(\chi_{\tau_c}) (kv)] \quad \mathbb{E}^t[(\chi_{\tau_c}) (kv)] \]

\[ \sum_{n=0}^{\infty} \lambda_n \cdot u_n \cdot Q_n (kv, U) \]

for \( n \leq k-k' \leq k \), \( Q_n (kv, U) = 0 \) therefore the above equation becomes starting the summation at the value \( k-k' \) using.
(5.3.1) \[ \int_0^1 Q_n(xu, au) du = \int_0^1 Q_n(x, (u_i') i \beta k) du \]

\[ \sum_{k, \Delta \beta} \lambda_{n+k} = \frac{1}{K_{\beta n+k}} \]

Put \( S_{l, \beta} \) in theorem (5.3.2)

\[ \int_0^1 (Ax + B) y_{k+1} du \]

\[ \int_0^1 l T \]

Proof:

Put \( v = \epsilon \) in theorem (5.3.9)

5.3.3 Application:

In corollary (5.3.2) put \( A = B = e = r = 0 \)

\[ a = \alpha (k-k') \] we have

\[ E[1(x=1)] = \frac{1}{k} \sum_{k} (x-k)(1-\alpha k') \]

which is Daniel's formula if \( \alpha = 1, \beta = \rho \)

(2) In theorem (5.3.9), Put \( V = 1/k, A = 0, k = k', a = B \) then.
Which shows that the generating function Laplace transform of $X^T_\xi$ and $\int_0^t y \, du$ may be deduced from only the generating function of $X_{I_6}$.

### 5.4 Applications of Martingale theory to some advanced epidemic models:

P. Pietsch [31] considered Weiss's and Downton's models with parameters $\alpha$ and $\gamma$ for the susceptible and carriers, and the martingale approach proved valuable and given explicit results in this case.

**Definition and Notations:**

\[
\alpha = \alpha (i, j) \quad \text{or} \quad \alpha_{ij},
\]

\[
\beta = \beta (i, j) \quad \text{or} \quad \beta_{ij}.
\]

\[
\int_0^t h \, du = \int_0^t (x^\alpha, y^\alpha) \, du
\]

\[
\int_0^t a(x^\alpha, y^\alpha) x^\alpha y^\alpha \, du
\]

$f: N \rightarrow R$ is any function. Let us denote

\[ j f(j) \text{ by } f(j) \text{ and martingale } V_t \text{ instead of } (V_t, \mathcal{F}_t) \quad t \geq 0. \]

**Hypotheses and the key theorem:**

Let us take $h_{ij} = L_i \beta_{ij} \cdot j, \quad L_i \geq 0.$

and suppose that the following hypotheses $H_l$ and $H_r (n)$

\[ n = 0, 1, 2, \ldots. x_0 \text{ are fulfilled.} \]

$H_l$: For any $i$ and $j$, $\alpha_{ij} = \beta_{ij} \eta_i$ with $\eta > 0$ for $i > 0$
II (n) For $i = n, n+1$, $x_i$,

$(\eta_i \eta_{n+1} + 1 \lambda_i - 1_{n+1}) (i-n) \text{ is defined and } f 0$

5.4.1 Theorem:-

For $\lambda_n = (1 + 1_{n+1}) \eta_i$

$$\gamma_n(s) = \gamma n(s+n)$$

and $V(t) = (X_t)^{\gamma} n(s) \lambda_n \exp(-\int_0^t \ln(x_s, y_s) du)$

then $(V(t), n^2)$ $t < 0$ is a martingale

Proof:-

Put $\pi = \pi_i$

$$\alpha \cdot \alpha_j, \beta = \beta_j$$

$$A \cdot B \cdot L_j \cdot B$$

$$\lambda_n = \frac{\beta}{(A+\alpha)n+B+\beta}$$

$$= \frac{\beta_j}{(A+\alpha)n+B+\beta}$$

$$= \frac{\beta_n}{(A+\alpha)n+B+\beta}$$

$$= \frac{\beta_j}{(A+\alpha)n+B+\beta}$$

$$= \frac{\beta_n}{(A+\alpha)n+B+\beta}$$

$\lambda_n \text{ } \frac{1}{\ln + n+1} \rightarrow (5.4.1)$

$$\gamma_n(s) = \gamma n(s+n)$$

$$\gamma_n(i) = (1 - \pi_i) \lambda_n \pi_i \eta_i \eta_{n+1} - L_{n+1} \eta_i (i-n) \eta_i$$

$$V(t) = (X_t)^{\gamma} n^2 (\alpha/A+\alpha) \gamma n (1 - \pi_i) \lambda_n \exp(-\int_0^t \ln(x_s, y_s) du)$$
\[ V_{\xi, n} = (X_0)\pi \gamma_\xi \cdot \exp \left[ \int_0^T h(X_\xi, Y_\xi) \, du \right] \text{ is a martingale.} \]

**5.4.1 Corollary:**

For any stopping time \( T \) and \( n = 0, 1, 2, \ldots \),

\[ E(V_{T, n}) = V_{T, n}. \]

Relation between moments and integral along a trajectory:

Put \( L_\xi = A \cdot \gamma_\xi + B \) where \( A \) and \( B \) are non-negative constants.

\[ \hat{L}_{\xi} = L_\xi - B \]

\[ \gamma_\xi = \frac{1}{1 + L_0 - \hat{L}_0 + \hat{L}_0} \]

\[ \gamma(s) = \left( 1 - \pi s + \frac{\hat{L}_0}{1 + L_0 - \hat{L}_0} \right) \gamma_\xi \]

\[ \gamma_{s,s} = \gamma_{s,s} + A \gamma_{s,s} s \gamma_{s,s} + B = s \gamma_{s,s} (1 + A \gamma_{s,s}) \]

\[ \gamma_{s,s} = \frac{1 + (B(1 - \gamma_{s,s}))}{(1 + B)(1 + A \gamma_{s,s})} \]

Denoting \( V_{s, \hat{\gamma}_{s,s}} \), \( o(A,B) \) for \( V_{s, \hat{\gamma}_{s,s}} \) inorder to point to the dependence upon \( A \) and \( B \)

\[ V_{s, \hat{\gamma}_{s,s}} (A,B) = (X_\xi)_{\pi \gamma_{s,s} (s)\xi} \cdot \exp \left[ \int_0^T h(X_\xi, Y_\xi) \, du \right] \]

\[ = \pi \cdot \frac{1 + (1 - \pi s) B}{1 + (1 + A \gamma_{s,s})} \]

\[ \Rightarrow (1 + B) e^{-\int_0^T h \, du} \]
\[ h_{x_t} y_u = (L x_{x_t} \beta x_{x_t} y_{u_t}) y_{u_t} \]
\[ = (A f x_{x_t} \eta x_{x_t} x_{x_t} + B) \beta x_{x_t} y_{u_t}) y_{u_t} \]
\[ = A f x_{x_t} \eta x_{x_t} \beta x_{x_t} y_{u_t} + B \beta x_{x_t} y_{u_t} \]
\[ = \pi \left( 1 + A f s \right) e^{-A \int_{0}^{t} f(\eta x_{x_t}) \alpha(x_{x_t} x_{x_t}) x_{x_t} y_{u_t} \, du}. \]
\[ = \pi \left( 1 + (1 - \pi) B \right) \left( 1 + B \right)^{\alpha y} \exp(-\delta \int_{0}^{t} f(\eta x_{x_t}) \alpha(x_{x_t} x_{x_t}) x_{x_t} y_{u_t} \, du). \]

**5.4.2 Theorem:**

For any stopping time \( T \) and any non-negative \( A, B \) and

\[ \mathbb{V}_{T, 0}(A, 0) \] and \[ \mathbb{V}_{T, 0}(0, B) \] being uncorrelated.

**Proof:**

\[ \mathbb{V}_{T, 0}(A, 0) \] and \[ \mathbb{V}_{T, 0}(0, B) \] are martingale.

Therefore \( E(\mathbb{V}_{T, 0}(A, 0) \mathbb{V}_{T, 0}(0, B)) = E(\mathbb{V}_{T, 0}(A, B)) = E(\mathbb{V}_{T, 0}(A, B)) \]

\[ = \mathbb{V}_{T, 0}(A, 0) \mathbb{V}_{T, 0}(0, B) \]

\[ = E(\mathbb{V}_{T, 0}(A, 0) E(\mathbb{V}_{T, 0}(0, B))) \]

**5.4.3 Theorem:**

For any \( f_{\alpha} : N \rightarrow R \).

\[ \sum_{\ell \geq 1} \int f(x_{x_t}) \alpha(x_{x_t}, y_{u}) x_{x_t} y_{u} \, du \]

\[ = \sum_{\ell \geq 1} \int f(x_{x_t}) \alpha(x_{x_t}, y_{u}) x_{x_t} y_{u} \, du \]

\[ \rightarrow (5.4.4) \]
are martingales. The first two are uncorrelated with the last two.

**Proof:**

\[
\begin{align*}
V_f (A, 0) &= \pi (1 + A f_s) e^{-A \int_0^\infty f(x) \alpha(x, y) x_{la} y_{la} du} \\
&= \pi (1 + (1 + \pi s)B + \ldots) e^{-A \int_0^\infty f(x) \alpha(x, y) x_{la} y_{la} du} \\
&= \pi (1 - A f_s + A^2 f_s^2 + \ldots) e^{-A \int_0^\infty f(x) \alpha(x, y) x_{la} y_{la} du}
\end{align*}
\]

If we pick up the terms in \( A, A^2, B, B^2 \) from (5.4.5 and 5.4.6) we can get the result.

**5.4.4 Theorem:**

For any \( f^A : N \to \mathbb{R} \) any \( g : N \to \mathbb{R} \) and any stopping time \( T \),

\[
E \int_0^\gamma \alpha(x, y) x_{la} y_{la} du = E \sum_{i=1}^\gamma f_i
\]

**Proof:**

\[
\sum_{i=1}^\gamma f_i = \int_0^\gamma f(x) \alpha(x, y) x_{la} y_{la} du
\]

Taking expectation on both sides.

\[
E \left[ \int_0^\gamma f(x) \alpha(x, y) x_{la} y_{la} du \right] = E \left[ \sum f_i \right]
\]

To get (5.4.8)

\[
\text{Put } g = g_0^\gamma + (g_0 - g_0^\gamma)^\gamma \left[ \frac{f}{f_0} \right]
\]
Using (5.4.7), we get
\[ E \int g(x, y) \beta(x, y) y \, du = -E \gamma \nu \int g(x, y) \beta(x, y) y \, du. \]

Therefore we have
\[ E g(x, y) \beta(x, y) y \, du = -E \gamma \nu \int g(x, y) \beta(x, y) y \, du. \]

5.4.5 Theorem:

For any stopping time \( T \),
\[ E \left[ \int_0^T g(x, y) \beta(x, y) y \, du \right] \]
is the expected number of susceptibles involved in the epidemic between time \( 0 \) and \( T \) and \( E \left[ \int_0^T \beta(x, y) y \, du \right] \) is the expected number of detected and eliminated carriers during the same period.

Proof:

Take \( f = g = 1 \) in theorem (5.4.4)

The Joint Distribution of \( X_T \) and \( \int_0^T h \, du \)

(a) A particular case:

\[ \eta = (\sigma + c_i, 1) \]
\[ h_j = L_j \beta_i \beta_j = (A^T \eta_j + B^T) B_j \]

with \( \pi, \sigma, c_i, A^T, B^T \) are constants.
\[ A^I \geq 0, \quad B^k \geq 0, \quad c^\phi + c^r \cdot i = 0 \quad \text{for} \quad I = 0, 1, 2, \ldots x^r \]

\[ \lambda_n = \frac{1}{1 + L_n + n \gamma_n} \]

\[ L_n = A^I n^\lambda + B^I \]

\[ \lambda_n = \frac{1}{1 + \lambda' [n(c^\phi + c^r n)^i] [B^I + n((c^\phi + c^r n)^I)]} \]

\[ \gamma_n = (1 - \pi^r + \lambda n \pi^r) (\gamma_i n^\lambda + L_n - L_{n}^{-1}) (i-n) \gamma_i \]

\[ = (1 - \pi + \lambda n \pi) (1 + A^I) \]

Therefore \[ \gamma(i) = (1 - \pi + n \lambda n) (1 + A^I) \]

which doesn't depend on \( i \) and will be denoted

5.4.6 Theorem:

\[ U_n = 1 + \frac{c^\phi n^{-1}}{(1 + A^I) (1 - \pi + \lambda n \pi)^{\lambda n \xi}} \]

\[ \lambda_n = \frac{c^\phi n}{(1 + B^I) c^\phi + (1 + A^I) + (1 + B^I) c^r n} \]

\[ \exp \int_0^\tau \alpha(X_{la}, Y_u) \gamma_i Y_u + B^r \Theta_{la} \gamma_i Y_u \, du \]
This theorem enables us to express $E(X^k)$ with $Q_n(k)$ and $P[XT_C = k]$ with $Q_n(k)$.

If $c = \pi = 0$ or

If $c = 2, \pi = 1$, then.

Both these expressions are linear in $n$, hence

$Q^k(v; (u_i)_{i\geq 0}) = Q_{n-k}(v; (u_i)_{i\geq k})$
Therefore $E[(X_k^k)b] = \sum_{n-k}^{x} \frac{e^{r/n} \cdot \lambda^n \cdot Q_k^{n-k} \cdot \sum_{u_n}^{x-n} f_i \cdot e^{y_i} \cdot (1-u_i) \cdot (1-u_i)^{n-k-1}}{n-k}$.

5.5 REPRODUCTION NUMBERS AND THRESHOLDS IN STOCHASTIC EPIDEMIC MODELS HOMOGENEOUS POPULATION.

John A. Jacquez and Philip O'Neill compare threshold results for the deterministic and stochastic versions of the homogeneous SI model with recruitment death due to the disease, a background death rate, and transmission rate.

The basic reproduction number is defined as follows. Let $C$ be the average number of persons contacted per person per unit time, and let $\beta$ be the probability of transmission per contact between a susceptible and an infected. The combined parameter $\lambda = c \cdot \beta$ has units $\text{time}^{-1}$ and is called the number of effective contacts per person per unit time. Let $D$ be the mean duration of the infectious period. Then the number of contacts effective in transmission per infective if all contacts are with susceptibles is $R_0$, the basic or initial reproduction number where $R_0 = cBD$, $R_0$ is a dimensionless number.

Let us examine the deterministic and stochastic formulations for SI, SIS, SIR and SIRS models for homogeneous populations.
The Homogeneous SI Models:

Let \( X, Y \) denote the number of susceptibles and infectives respectively. Both are continuous, non-negative variables. \( U \) denotes a constant rate of \( Yc_1 \) new susceptibles into the population. The rate constant for competing deaths which is same for susceptibles and infecteds is denoted by \( \mu \) and the rate constant for deaths due to the disease is denoted by \( k \).

The Deterministic Equations:

The total number of persons contacted per unit time by all susceptibles is \( CX \). \( \beta \) is the probability of transmission of the disease for a contact between susceptibles and infecteds, the rate at which susceptibles are infected must be

\[
\frac{xy}{x+y-1} \cdot C_\beta = \mu x + U
\]

Thus the differential equations for \( X \) and \( Y \) are

\[
\frac{dx}{dt} = -c\beta \frac{xy}{x+y-1} \quad \mu x + U \quad \Rightarrow (5.5.1)
\]

\[
\frac{dy}{dt} = c\beta \frac{xy}{x+y-1} \quad -(k+\mu)y \quad \Rightarrow (5.5.2)
\]

Put \( x+y \) for \( x+y-1 \) (5.5.2) becomes

\[
\frac{dy}{dt} = -c\beta \frac{xy}{x+y} \quad -(k+\mu)y \quad \Rightarrow (5.5.3)
\]

If \( R_o \) --- <1, the disease free equilibrium is globally stable

\[
\frac{x}{k+\mu} <1, \text{ for } y > 0, \text{ if } c\beta \cdot -(k+\mu) < 0, \text{ the }
\]

\[
\frac{x}{x+y}
\]
derivative of \( y \) is negative except at \( y = 0 \).

\[ x = \frac{U}{\mu}, \quad y = 0 \]

and

\[
\begin{align*}
\frac{x}{x + y - 1} &= \frac{U/\mu}{U/\mu - 1} = \frac{U}{U - \mu} \\
\end{align*}
\]

Therefore \( R_0 = \frac{cB}{k + \mu} < 1 \)

The Stochastic Model :-

\[
\left( \frac{cBxy}{x + y - 1} \right) \Delta t
\]

is the probability that a susceptible is converted to an infected in \( t \), in that transition \( x \) decreases by 1 and \( y \) increases by 1.

This transition probability is zero if \( x = 0 \) and \( y = 0 \). Also if \( x + y < 1 \), there is no transmission

The stochastic Equation :-

\( U \Delta t \) is the probability that \( x \) increases by 1 in \( \Delta t \), by recruitiment.

\( \mu \Delta t \) is the probability of losing one susceptibles to a competing cause of death in \( \Delta t \), \( x \) decreases by 1. The probability of losing one infective to a competing cause of death in \( \Delta t \) is \( \mu y \Delta t \), \( y \) decreases by 1 and the probability of losing one infective due to the disease in \( t \) is \( k y \Delta t \), \( y \) decreases by 1. Let \( P_{x,y}(t) \) be denoted as the probability that the population has \( x \) susceptibles and \( y \) infectives at time \( t \). According to Bailey \[2\] the P differential equation for \( P_{x,y}(t) \) is

\[
P^t_{x,y}(t+\Delta t) = P^t_{x+1,y-1} + \Delta t + P^t_{(k+\mu)(y+1)} \quad \Delta t + \frac{cB(x+1)(y-1)}{x+y-1} \quad \Delta t + \frac{(k+\mu)(y+1)}{y+y+1} \quad \Delta t.
\]
The number of susceptibles is given by a linear death process with immigration whenever the infectives are absent. Let us assume that process is at equilibrium when the infectives are introduced. At that point \( E(x) - m = U/\mu \).

Let us choose that initially there are \( n \) susceptibles and \( m \) infecteds. This gives for initial conditions

\[
P(x,0) = \begin{cases} 1, & x = n \\ 0, & x \neq n \end{cases} \text{ or } y \neq m,
\]

Next let us compare the time courses of the mean number of infecteds from the stochastic model with the time courses of the number of infecteds from the deterministic model.

By definition the expected values are given by

\[
m_y(t) = E(y) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} P_{x,y}(t)
\]

and

\[
m_x(t) = E(x) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} P_{x,y}(t)
\]
\[
\begin{align*}
\frac{\text{dm}_\mathbf{x}}{\text{dt}} &= \mathbf{\Sigma} \mathbf{\Sigma} \quad \frac{\text{dm}_\mathbf{y}}{\text{dt}} = \mathbf{\Sigma} \mathbf{\Sigma} \\
\text{At } x+y=1 \\
\begin{align*}
\frac{\text{dE}(y)}{\text{dt}} &= c\beta \mathbf{\Sigma} \mathbf{\Sigma} \quad \frac{\text{dE}(x)}{\text{dt}} = c\beta \mathbf{\Sigma} \mathbf{\Sigma} \\
\text{Relations between Stochastic means and Deterministic Variables:}
\end{align*}
\end{align*}
\]

Starting with the system in state \((x,y)\) let us calculate the expected value of

\[y(t+\Delta t) - y(t)\]

\[= c\beta \frac{xy}{x+y-1} - (k+\mu) y.\]

\[
\frac{\text{dE}(y)}{\text{dt}} = c\beta \mathbf{\Sigma} \mathbf{\Sigma} = c\beta \mathbf{\Sigma} \mathbf{\Sigma} \\
\text{Hence} \quad \frac{\text{dm}_\mathbf{y}}{\text{dt}} = c\beta \mathbf{\Sigma} \mathbf{\Sigma} \quad \frac{\text{dm}_\mathbf{x}}{\text{dt}} = c\beta \mathbf{\Sigma} \mathbf{\Sigma} .
\]

\[\Rightarrow (5.5.4)\]

Similarly

\[
\frac{\text{dm}_\mathbf{x}}{\text{dt}} = -c\beta \mathbf{\Sigma} \mathbf{\Sigma} \quad \frac{\text{dm}_\mathbf{y}}{\text{dt}} = -c\beta \mathbf{\Sigma} \mathbf{\Sigma} \\
\Rightarrow (5.5.5)
\]

Relations between Stochastic means and Deterministic Variables:
1. Reproduction number and Global stability:

\[ R_o < 1 \]

\[
\frac{dm_y}{dt} = (K + \mu) \left( \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \frac{bc}{k+\mu} \frac{x}{x+y-1} \right) y \frac{P_x}{y} \quad \text{for } y \geq 1 \quad (5.5.6)
\]

Assume that \( R_o - 1 < 0 \), \( y(0) \) is finite.

Next let us prove that the derivative of \( m(t) \) is negative for every \( t \leq \sigma^0 \) and that its asymptotic steady value is zero.

\[ \frac{dm_y}{dt} = 0 \quad \Rightarrow \quad \therefore \text{consider } y \geq 1 \]

For all \( y \geq 1 \) and all \( x > 0 \),

\[ \frac{x}{x+y-1} \leq 1 \]

Hence if \( R_o - 1 < 0 \), co-efficient of \( yP_{x,y} \) in (5.5.5) is negative for \( y \geq 1 \)

Therefore \( \frac{dm_y}{dt} \) is negative, \( m_y \) must always decrease.

Now, let us seek the equilibrium state solution for the equation (5.5.1) \( m_y^e \) for which

\[ \frac{dm_y}{dt} = 0 \quad \text{when } R_o < 1 \]

For the equilibrium state, (5.5.6) can be written

\[ R_o \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \frac{x}{x+y-1} y P_{x,y} = m_e \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} y P_{x,y} \]

The double sum on the L.H.S is written
Hence there exists such that

\[
\sum_{n=0}^{\infty} \sum_{y=1}^{\infty} \xi^{xy} y \, P_{x,y} \leq 1
\]

By hypothesis \( R_0 < 1 \). Therefore \( R_0 e^{-1} \rightarrow \infty \) if \( e^{-1} \rightarrow 0 \).

If \( R_0 < 0 \), the equilibrium state for equation (5.5.4) has for solution \( m_j \rightarrow 0 \).

\[
\alpha n \frac{dm_j}{dt} \text{ is negative}
\]

Since \( y \) is a non-negative variable, if its expected value goes to zero, all probabilities \( P_{x,y} \) for \( y > 0 \) must be zero and hence all higher moments must be zero.

Hence the disease-free equilibrium is globally stable for the stochastic model

The equilibrium state value of \( n_x \)

\[
\frac{dm_x}{dt} = -cB \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \xi^{xy} P_{x,y} - \mu \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x P_{x,y} + \mathcal{U}
\]

\[
\frac{dm_y}{dt} = 0 = \frac{cB \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \xi^{xy} P_{x,y} + \mu \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} x P_{x,y} + \mathcal{U}}
\]

\[
cB \xi^{xy} P_{x,y} = \mathcal{U} \quad \mu \xi^{xy} P_{x,y} = \mathcal{U}
\]

For \( R_0 < 1 \), \( m_j \rightarrow 0 \).

Therefore \( m_x \rightarrow 0 \).

\( P \) will be negligible for large \( y \). Hence for all terms \( P_{x,y} \) for which \( P \) is significant \( P_{x,y} \) will be negligible.
Equation (5.5.4) \[ \Rightarrow \]

\[
\frac{dm}{dt} = c\beta(x+y) - \frac{\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} p_{x,y} = [c\beta(x+y)]m}{x+y-1} \Rightarrow (5.5.8)
\]

If \( R_0 > 1 \), \( m \) grows
\( R_0 < 1 \), \( m \) decreases
\( R_0 = 1 \), \( m \) is stationary

With the approximation of (5.5.7) and (5.5.8) the epidemic becomes a birth and death process.

5.6 A modification of the general Stochastic Epidemic.

Motivated by AIDS Modelling:-

Frank Ball and Philip O'Neill [44] consider a model for the spread of an epidemic in a closed homogeneously mixing population in which new infections occur at rate \( Bxy/x+y \) where \( x \) and \( y \) are numbers of susceptibles and infectious individuals respectively and \( B \) is an infection parameter. But \( Bxy \) is the infection rate in the standard general epidemic.

First let us define the stochastic version of this new model. Consider a closed population consisting initially of \( a \) infectives and \( n \) susceptibles. For \( t \geq 0 \), let \( x(t) \), \( y(t) \), and \( z(t) \) be respectively the numbers of susceptibles, infective and removed individuals at time \( t \). Suppose further that \( x(t) + y(t) + z(t) = n + a \) for \( t \geq 0 \), so the process is completely determined by \( \{ x(t), y(t), z(t) \geq 0 \} \) which we assume is a continuous time Markov chain on the state space

\[ \{ (x,y) \subset z \geq x+y \leq n+a, 0 \leq x \leq n, y \geq 0 \} \]

with transition probabilities.
\[ P_t\{(X(t + \Delta t), y(t + \Delta t))\} = [(x-1, y+1), y(t) = x,y] \]

\[ P_t\{(x(t + \Delta t), y(t + \Delta t))\} = [(x, y-1 / (x(t), y(t) = x,y)] \]

and all \ldots transitions having probability \( O(\Delta t) \)

**Deterministic Model:**

For \( t > 0 \), let \( x(t) \), \( y(t) \) and \( z(t) \) be respectively the numbers of susceptible, infective and removed individuals at time \( t \).

The deterministic model of the modified epidemic is given by

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\beta xy}{x+y} \\
\frac{dy}{dt} &= -\frac{\beta xy}{x+y} \\
\frac{dz}{dt} &= \gamma y
\end{align*}
\]

with initial conditions \( x(0) = n, y(0) = a, z(0) = 0 \).

Where \( \rho : \sqrt{\ell^2} \).

\[
\begin{align*}
\frac{dx}{dt} &= \frac{dz}{dx} \\
&= \frac{1}{\rho (n+a-z)}
\end{align*}
\]

\[
\log x(t) = + \frac{1}{\rho} \log(n+a-z) + k.
\]

\[
t, \log = \frac{1}{\rho} \log(n+a) = k \]

\[
k = \log^n(n+a) \frac{1}{\rho}
\]
\[ x(t) = \left( \frac{n+1-a}{z(t)} \right)^{\frac{1}{\ell \rho}} \]

\[ x(t) = \frac{n(n+1)}{(n+1)^{\frac{1}{\ell \rho}} \left[ 1 - \frac{z(t)}{n+1} \right]^{\frac{1}{\ell \rho}}} \]

\[ = \frac{n(n+1) - z(t)}{(n+1)^{\frac{1}{\ell \rho}}} \]

\[ = \frac{1}{n+1} \]

\[ x(t) + y(t) + z(t) = n+1 \]

\[ \frac{dz}{dt} = \frac{z(t)}{n+1} \]

\[ = \frac{\gamma \left[ (n+1) - z - n \left( 1 - \frac{1}{n+1} \right) \right]}{n+1} \]

\[ = \frac{\gamma \left[ (n+1)^2 - (n+1) - n(n+1) \right]}{n+1} \]

\[ = \frac{n+1}{n+1} \]

\[ = \frac{\gamma \left[ \left( n+1 \right) - az \right]}{n+1} \]

\[ = \frac{\gamma \left[ a - az \right]}{n+1} \]

\[ y(t) = a \frac{e^{-\frac{t}{\gamma \ell \rho}}}{N} \]

\[ z = N \left( 1 - e^{-\frac{t}{\gamma \ell \rho}} \right) \]

\[ y(t) = a \frac{e^{-\frac{t}{\gamma \ell \rho}}}{N} \]

\[ \ell \neq 1 \]

\[ \frac{dz}{dt} = \frac{\gamma \left( n+1 - z - n \left( 1 - \frac{1}{n+1} \right) \right)}{m} \]

\[ \text{put} \quad (1 - Z/n+1) = m \]

\[ m = 1 - Z/n+1 \]

\[ N - Nm = Z \]

\[ \frac{dm}{d\ell} = \frac{\gamma \left( n+1 - az \right)}{n+1} \]

\[ \frac{2 - \ell}{m} \]
\[ \frac{dm}{dt} = \frac{\beta m}{N} - \frac{2 - \ell}{m} \]

Put \( L = m^{\ell - 1} \)
\[ dL = (\ell - 1) m \frac{dm}{dt} \]
\[ \frac{dL}{dt} = -(\ell - 1) \beta L + \frac{\beta n}{N} \]

Solving,
\[ \int_{L}^{n \beta \ell} e^{-(\ell - 1) \beta l} \, dt = \int_{N \gamma^{-\beta}}^{x} e^{-\beta x} \, dx + c \]
\[ \int_{L}^{n \beta \ell} e^{-(\ell - 1) \beta l} \, dt = \frac{n \beta \ell}{N \gamma^{-\beta}} + c \]

Put \( t = 0 \),

We get \( L = 1 \)
\[ n \beta \ell - 1 \beta + \frac{c}{e} = N(\gamma^{-\beta}) \]
\[ C = \frac{a}{N} \]
\[ L = nN + a/N \quad e \]
\[ L = N^{\ell - 1} \left( n + a \right) e^{(\beta - \gamma)l} \]

Therefore \( Z = N \left[ 1 - (N(\beta - \gamma) + \frac{c}{e}) \right] \]

Also
\[ x(t) = n \left[ (n + a) e^{(\beta - \gamma)l} \right] \]
\[ y(t) = a \left[ (n + a) e^{(\beta - \gamma)l} \right] e^{(\beta - \gamma)l} \]

Let \( T = \int_{0}^{\frac{1}{2} \ell} Z(t) \) be the total size of the epidemic then it follows from the above solution.
Threshold behaviour:

If \( P < n \), no true epidemic occurs, since \( \frac{dN}{dt} \leq 0 \).

\[ \frac{dy}{dt} \leq 0 \quad \text{when} \quad \frac{\dot{P}}{x+y} < 1\]

and

\[ \frac{dx}{dt} = \left( \frac{\beta}{x+y} - \gamma \right) > 0, \quad \text{when} \quad \dot{P} < 1\]

We can summarise the threshold behaviour of the modified epidemic as follows:

(i) \( \dot{P} > 1 \), \( \frac{dy}{dt} < 0 \), for all \( t \), \( x(\infty) > 1 \)

(ii) \( \frac{n}{n+a} \leq \dot{P} < 1 \), \( \frac{dy}{dt} < 0 \) for all \( t \),

except \( t = 0 \) if \( \dot{P} = \frac{n}{n+a} \).

(iii) \( \dot{P} < \frac{n}{n+a} \), \( \frac{dy}{dt} > 0 \), \( x(\infty) = 0 \),

\( \frac{dy}{dt} \) at \( t = 0 \).